An Extension of the First Order Abelian Stark Conjecture

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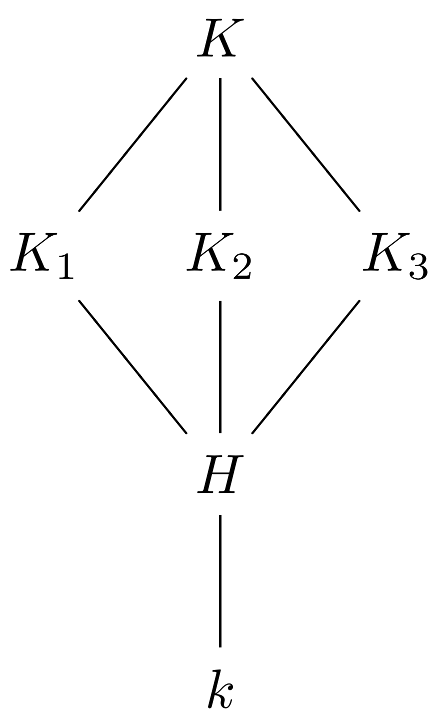
Dummit's Question

Can we extend the Stark Conjectures when S does not contain primes which split completely?

Dummit's Example



k: Totally real cubic field.

H: Hilbert Class Field.

K: Narrow Hilbert class field.

K_i: Decomposition field for $v_\infty^{(i)}$.

Want K/H to be biquadratic with K_i as quadratic subfields.

Suffices that k has a totally positive system of fundamental units.

$v_\infty^{(i)}$ splits in K_i, other two infinite primes ramify.

By the First Order Stark Conjecture, there exists an $\varepsilon_i \in K_i$ which is a $v_\infty^{(i)}$-unit and evaluates all the L-functions of K_i/k.

Using the local conjecture, Dummit & Hayes showed $\varepsilon_i = \eta_i^2$ in K_i.

$\varepsilon = \prod \eta_i$ evaluates all the L-functions of K/k.

The abelian condition does *not* appear to hold for ε.

Preliminaries

Lemma. *The order of vanishing of* $L_S(s,\chi)$ *at* $s=0$ *is equal to* $|S|-1$ *if* χ *is the trivial character, and is equal to the number of primes* v *in* S *for which* χ *is trivial on* G_v*.*

Lemma. *Let* K/k *be a Galois extension of number fields and let* w *be a prime in* K *lying above the prime* v *in* k*. Then for any* $\alpha \in k$*,*

$$\left|\alpha\right|_v = \left|\alpha\right|_w^{1/|G_v|}$$

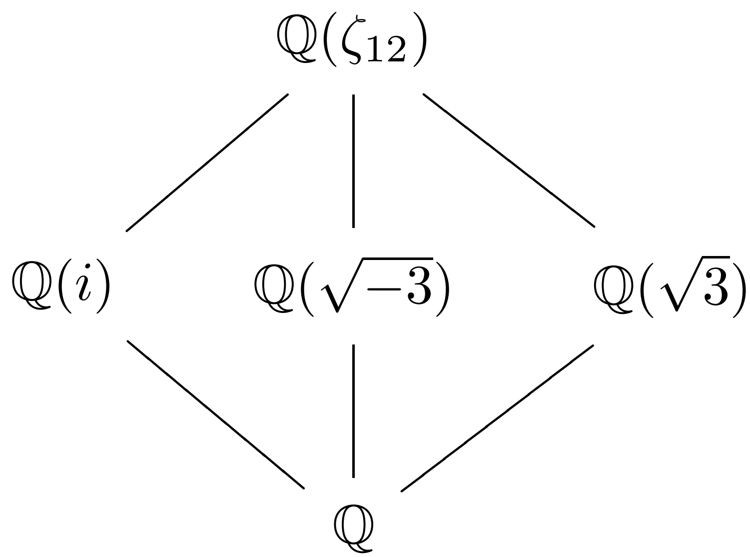
Definitions

1. A *1-covering of* $X \subseteq \widehat{G}$ is a finite set S such that there is a $v \in S$ such that $\chi\big|_{G_v} = 1$ for all χ in X. A *1-subcovering of* S *and* X is a subset S' of S which is a 1-covering of X.
2. $\widehat{G}_{1,S}$ is the subset of characters χ for which $L_S(s, \chi)$ has precisely first order vanishing at $s = 0$.
3. The *minimal 1-subcovering of* $\widehat{G}$, denoted as $S_{\min}$, is the set of primes $v \in S$ such that there is a $\chi \in \widehat{G}_{1,S}$ with $\chi\big|_{G_v} = 1$.

Lemma. *$S_{\min} = \bigcap S'$ where S' runs through all 1-subcoverings of $\widehat{G}$.*

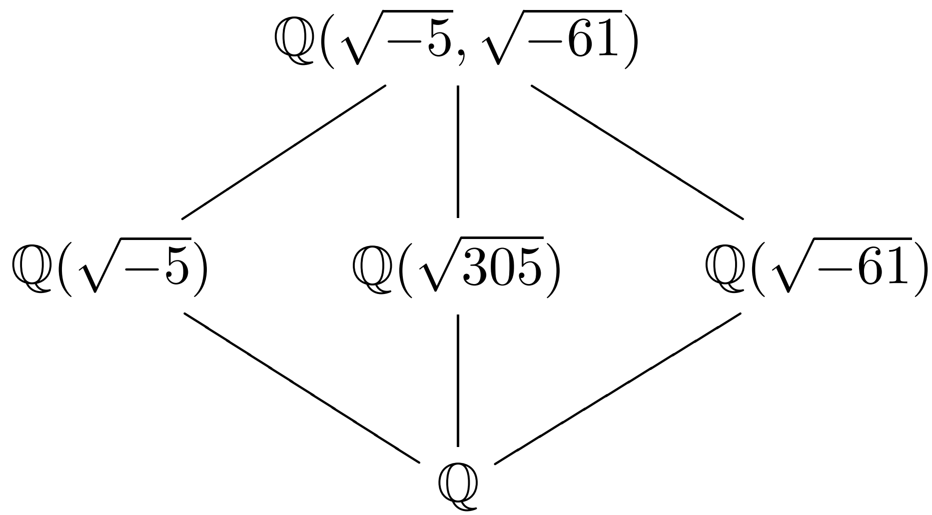
Examples of 1-coverings

$$K/k = \mathbb{Q}(\zeta_{12})/\mathbb{Q},\ G = \mathrm{Gal}(K/k) \cong (\mathbb{Z}/12\mathbb{Z})^\times$$



$$S = \{2, 3, 5, 7, \infty\},\ S_{\min} = \{5, 7, \infty\}$$

$$K/k = \mathbb{Q}(\sqrt{-5}, \sqrt{-61})/\mathbb{Q},\ G = \mathrm{Gal}(K/k) \cong (\mathbb{Z}/2\mathbb{Z})^2$$



$$S = \{2, 5, 61, \infty\},\ S_{\min} = \{5, 61\}.$$

Extended First Order Abelian Stark Question (2001)

Question (StQ($K/k, S$)). *Let S be a 1-covering of $\widehat{G}$ containing all infinite and ramified primes. Let $S_{\min}$ be the minimal 1-subcovering. Assume $|S| \geq |S_{\min}| + 1$. For each $v \in S_{\min}$, fix some w lying above v. Does there exists an $\varepsilon \in K^\times$, unique up to root of unity, with the following properties?*

i. $\left|\varepsilon\right|_w = 1$ *for all w not lying above a prime in S.*
If $|S| = 2$, then for $v' \in S \setminus S_{\min}$, $\left|\varepsilon^\sigma\right|_{w'} = \left|\varepsilon\right|_{w'}$ for all $\sigma \in G$.
If $|S| \geq 3$, then $\left|\varepsilon\right|_w = 1$ for all w not dividing a prime in $S_{\min}$.

ii. *For all $\chi \in \widehat{G}$,*

$$L'_S(0, \chi) = -\frac{1}{W_K} \sum_{\sigma \in G} \chi(\sigma) \log\left(\prod_{v \in S_{\min}} \left|\varepsilon^\sigma\right|_w^{1/|G_v|} \right)$$

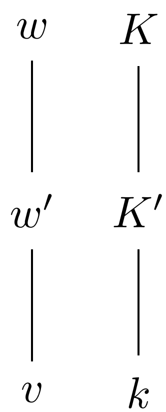
iii. $K(\sqrt[W_K]{\varepsilon})$ *is an abelian extension of k.*

Let $\mathrm{St}(K/k, S)$ denote the original First Order Abelian Stark Conjecture for K/k and S. The following statements are true:

1. If $k \subseteq K' \subseteq K$, then $\mathrm{StQ}(K/k, S)$ implies $\mathrm{StQ}(K'/k, S)$.
2. If $|S| = 2$, then $\mathrm{St}(K/k, S)$ and $\mathrm{StQ}(K/k, S)$ are equivalent.
3. If G is cyclic, then $\mathrm{St}(K/k, S)$ and $\mathrm{StQ}(K/k, S)$ are equivalent.

Hence, it is enough to consider noncyclic extensions and $|S| \geq 3$.

Basic Approach



Suppose $v \in S_{\min}$ and $\chi\big|_{G_v} = 1$. Let $K' = K^{G_v}$.

From $\mathrm{St}(K'/k, S)$, there is an $\varepsilon_v \in K'$ such that

$$L'_S(0,\chi) = -\frac{1}{W_{K'}} \sum_{\sigma \in G/G_v} \chi(\sigma) \log \left|\varepsilon_v^{\sigma}\right|_{w'}$$

Lifting the sum to G and the absolute value to w,

$$L'_S(0,\chi) = -\frac{1}{W_K} \sum_{\sigma \in G} \chi(\sigma) \log \left| \left(\varepsilon_v^{W_K/W_{K'}}\right)^{\sigma} \right|_w^{1/|G_v|^2}$$

By a variety of methods, we can show ε_v is a $|G_v|^{\mathrm{th}}$ power in K'.

The Stark unit for K is $\varepsilon = \prod_{v \in S_{\min}} \eta_v^{W_K/W_{K'}}$, where $\eta_v^{|G_v|} = \varepsilon_v$.

Theorem. *If*

- S *has a 1-subcovering* $S' = \{v_1, v_2, \ldots, v_t\}$ *consisting of only unramified finite primes*
- $\mathrm{St}(K_i/k, S_i)$ *holds for all* $K_i = K^{G_{v_i}}$ *and* $S_i = (S \setminus S') \cup \{v_i\}$

then $\mathrm{StQ}(K/k, S)$ *has an affirmative answer.*

K

|

K_i

|

k

Let ε_i be the Stark unit for $\mathrm{St}(K_i/k, S_i)$.

Then the Stark unit for $\mathrm{StQ}(K/k, S)$ is

$$\varepsilon = \prod_{i=1}^{t} \varepsilon_i^{\frac{W_K}{W_{K_i}} \frac{\rho_i}{|G_i|}}$$

where $\rho_i = \prod_{j \neq i} (1 - \sigma_j^{-1}) \in \mathbb{Z}[G]$ is divisible by $|G_i|$.

Theorem. *If*

- S *has a 1-subcovering* $S' = \{v_0, v_1, \ldots, v_t\}$ *where* v_0 *is a real infinite prime and* v_i *are unramified finite primes for* $1 \leq i \leq t$.
- $\mathrm{St}(K_i/k, S_i)$ *holds for all* K_i *and* $S_i = (S \setminus S') \cup \{v_0, v_i\}$

then $\mathrm{StQ}(K/k, S)$ *has an affirmative answer.*

$\rho_i = \prod_{j \neq i,\, 0} (1 - \sigma_j^{-1})$ is not necessarily divisible by $|G_i|$ for $1 \leq i \leq t$.

After factoring out $1 + \sigma_0$ from ρ_i, the rest is divisible by $|G_i|$.

Since $1 + \sigma_0$ kills ε_i for $1 \leq i \leq t$, the result follows from the previous theorem.

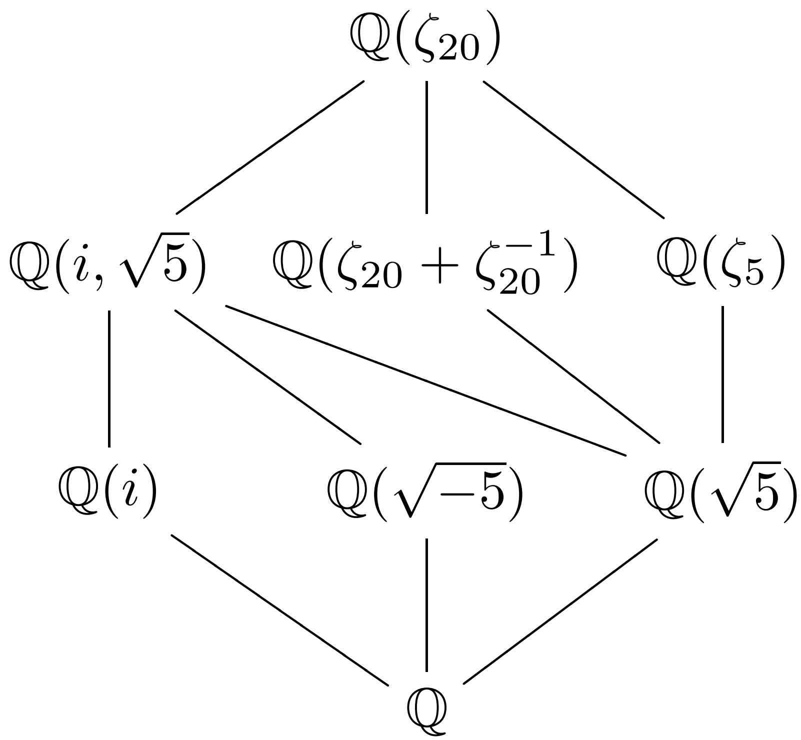
Theorem. *Let K/k be a multiquadratic extension of rank m, i.e., $G = \mathrm{Gal}(K/k) \cong \left(\mathbb{Z}/2\mathbb{Z}\right)^m$. Then $\mathrm{StQ}(K/k, S)$ has an affirmative answer if $|S| > m + 1 - r_k(S)$, where $r_k(S)$ is the 2-rank of the S_{fin}-class group of k.*

Corollary. $\mathrm{StQ}(K/k, S)$ *has an affirmative answer for biquadratic extensions K/k.*

Corollary. $\mathrm{StQ}(K/k, S)$ *has an affirmative answer for multiquadratic extensions when $k = \mathbb{Q}$.*

Cyclotomic Example

$$K/k = \mathbb{Q}(\zeta_{20})/\mathbb{Q},\ G = \mathrm{Gal}(K/k) \cong (\mathbb{Z}/20\mathbb{Z})^{\times}$$



$$S = \{2, 3, 5, 11, \infty\},\ S_{\min} = \{3, 5, 11, \infty\}$$

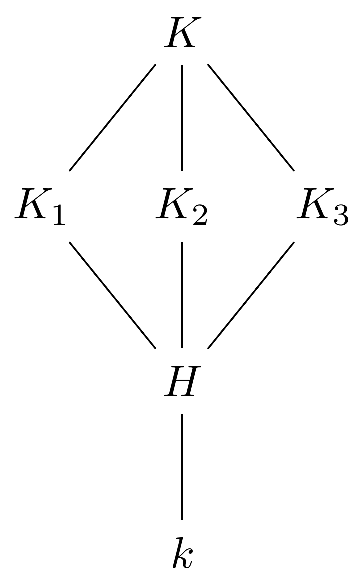
Prime	**Decomp. Field**	S_v	**Stark Unit for** S_v
3	$\mathbb{Q}(\sqrt{-5})$	$\{2,3,5,\infty\}$	$\varepsilon_3 = \left(\frac{1+\sqrt{-5}}{1-\sqrt{-5}}\right)^2$
5	$\mathbb{Q}(i)$	$\{2,5,\infty\}$	$\varepsilon_5 = \frac{2+i}{2-i}$
11	$\mathbb{Q}(\zeta_5)$	$\{5,11,\infty\}$	$\varepsilon_{11} = \frac{(2+\zeta_5)^3(2+\zeta_5^{-2})}{(2+\zeta_5^{-1})^3(2+\zeta_5^2)}$
∞	$\mathbb{Q}(\zeta_{20}+\zeta_{20}^{-1})$	$\{2,5,\infty\}$	$\varepsilon_\infty = (1-\zeta_{20})(1-\zeta_{20}^{-1})$

When S_v is enlarged to S, the necessary powers of $|G_v|$ arise after removing a factor of $\mathbf{N}(G/G_v)$ from $\rho_v = \prod_{v' \in S \setminus S_v} (1 - \sigma_{v'}^{-1})$.

Dummit's Example Revisited

$$k = \mathbb{Q}(\alpha),\ \alpha^3 - 22\alpha - 25 = 0$$

K is the narrow Hilbert class field of k.



$$G \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z}),\ S = S_{\min} = \{v_\infty^{(1)}, v_\infty^{(2)}, v_\infty^{(3)}\}$$

To check the abelian condition, we tested Coates' criterion:

$$\varepsilon^{\sigma-1} = \alpha_\sigma^2 \text{ and } \alpha_\sigma^{\tau-1} = \alpha_\tau^{\sigma-1} \text{ for all } \sigma, \tau \in G.$$

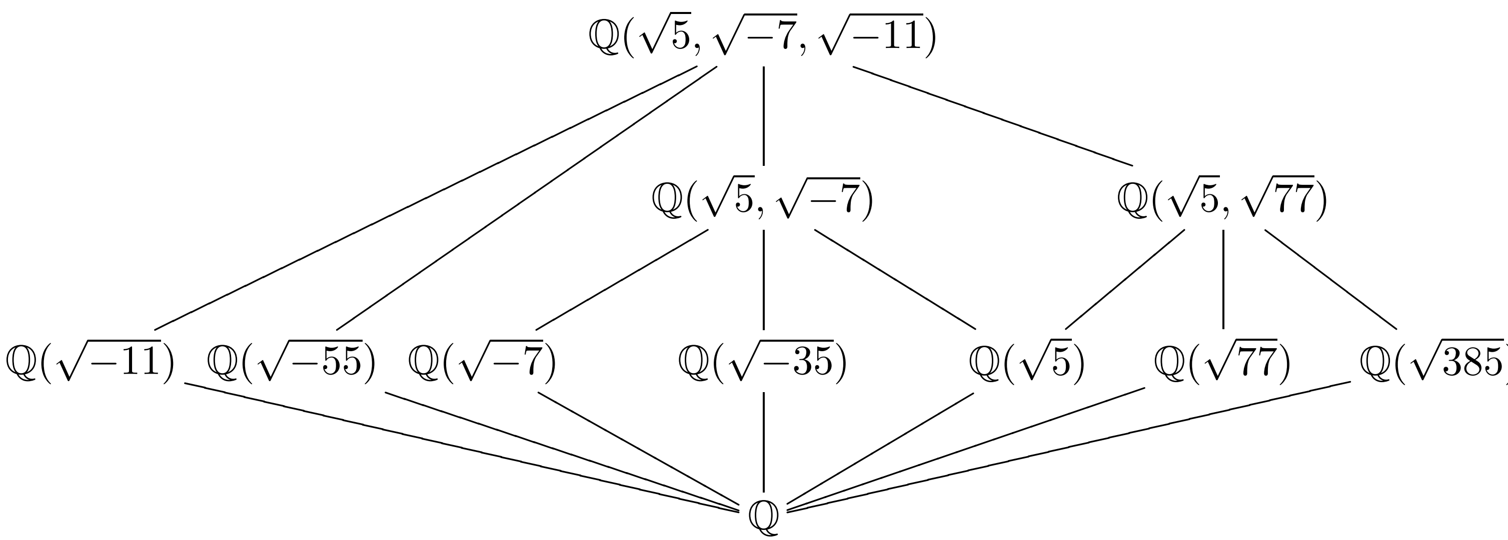
$\varepsilon^{\sigma-1}$ is a square for $\sigma \in G$ of order 1 and 2, but not for order 4.

Adding any other prime to S restores the abelian condition, supporting $|S| \geq |S_{\min}| + 1$ as a necessary assumption.

We checked all 16 possible ways to combine conjugates of the η_i, corresponding to the 16 choices of primes lying above $v_\infty^{(i)}$.

Multiquadratic Example

$$K/k = \mathbb{Q}(\sqrt{5}, \sqrt{-7}, \sqrt{-11})/\mathbb{Q} \qquad S = S_{\min} = \{\,5, 7, 11, \infty\,\}$$



Field	Split	Inert	Ramified	Stark Unit
$\mathbb{Q}(\sqrt{-11})$	5	7	11, ∞	$\left(\frac{3+\sqrt{-11}}{3-\sqrt{-11}}\right)^2$
$\mathbb{Q}(\sqrt{-55})$	7		5, 11, ∞	$\left(\frac{39+4\sqrt{-55}}{49}\right)^2$
$\mathbb{Q}(\sqrt{-7})$	11	5	7, ∞	$\left(\frac{2+\sqrt{-7}}{2-\sqrt{-7}}\right)^2$
$\mathbb{Q}(\sqrt{-35})$	11		5, 7, ∞	$\left(\frac{3+\sqrt{-35}}{3-\sqrt{-35}}\right)^2$
$\mathbb{Q}(\sqrt{5})$	11, ∞	7	5	1
$\mathbb{Q}(\sqrt{77})$	∞	5	7, 11	$\left(\frac{9+\sqrt{77}}{2}\right)^2$
$\mathbb{Q}(\sqrt{385})$	∞		5, 7, 11	$\left(95831+4884\sqrt{385}\right)^2$

Stark units are squares, but not fourth powers, in their respective fields. However, they are fourth powers in the top field K.

$$\varepsilon_5 \cdot \varepsilon_7 = \left(\frac{3+\sqrt{-11}}{3-\sqrt{-11}}\right)^2 \left(\frac{39+4\sqrt{-55}}{49}\right)^2 = \left(\frac{(3+\sqrt{-11})(5-2\sqrt{-55})}{2\cdot 5\cdot 7}\right)^4$$

$$\varepsilon_{11,v} \cdot \varepsilon_{11,v'} = \left(\frac{2+\sqrt{-7}}{2-\sqrt{-7}}\right)^2 \left(\frac{3+\sqrt{-35}}{3-\sqrt{-35}}\right)^2 = \left(\frac{(2+\sqrt{-7})(3+\sqrt{-35})}{2\cdot 11}\right)^4$$

$$\varepsilon_{77} \cdot \varepsilon_{385} = \left(\frac{9+\sqrt{77}}{2}\right)^2 \left(95831+4884\sqrt{385}\right)^2 = \left(363+\tfrac{259}{2}\sqrt{5}+33\sqrt{77}+\tfrac{37}{2}\sqrt{385}\right)^4$$

The Stark unit which satisfies $St'(K/\mathbb{Q}, S)$ is

$$\varepsilon = \frac{(3+\sqrt{-11})(5-2\sqrt{-55})(2+\sqrt{-7})(3+\sqrt{-35})}{4\cdot 5\cdot 7\cdot 11} \cdot \left(363+\tfrac{259}{2}\sqrt{5}+33\sqrt{77}+\tfrac{37}{2}\sqrt{385}\right)$$

The abelian condition is satisfied by verifying Coates' criterion.

Conclusions

- Arbitrary extensions over $\mathbb{Q}$.
- Extension for r^{th} order vanishing.
- Function field analogues.
- p-adic refinements.