Cabin Activities

Mathematical Explorations – Math 110

Block 2, Fall 2007

Zome

1. Play with the Zome pieces and make as many interesting shapes as you can. Can you make a cube? A tetrahedron (three-sided pyramid)? An octohedron (two square pyramids attached at the base)? A dodecahedron (12 pentagonal faces)? An icosahedron (20 triangular faces)?

2. Use your tetrahedron and cube to find all possible rotations of these objects. Come up with your own way of describing a rotation. Show how one rotation followed by a second rotation is simply a third rotation in disguise. See if you can find any patterns.

3. Make lots of different polyhedra and count the number of faces, the number of edges, and the number of vertices. Do you notice any patterns? If you add another face, what happens to the number of edges and vertices? Can you come up with a formula in terms of F, E, and V which is true for every polyhedron?

4. Most of the polyhedra you make will be convex (all angles are less than 180 degrees). Can you construct a non-convex polyhedron? What properties will these nonconvex polyhedron have? Describe the object you constructed in detail.

5. Dip your polyhedra in soap bubbles. What shapes can you get? Try making a "saddle" out of four white bulbs and four small orange pieces (not in a diamond shape). Try making structures with one object inside another (like a cube within a cube, a "starburst" inside a icosahedron, etc.) and dipping them in soap. Why do you think the soap bubbles make the surfaces that they do?

Geometry in Higher Dimensions

1. One can think about the fourth dimension as three dimensional space plus time as the fourth dimesion. Each three dimensional instant in time is a cross section of 4D space-time. Another analogy would be what a 3D sphere looks like to a 2D creature as it passes through its 2D universe. At first, the sphere would appear as a single point. As it passes through, the 2D creature would see a circle growing larger and larger, until it reaches the equator. After that, the circle would grow smaller and smaller, eventually reaching a point and disappearing. What would a 4D hypersphere look like as it passes through our 3D universe? What about other 4D hyper-polyhedrons?

2. A 2D triangle has one face, three edges, and three vertices. How many solids, faces, edges and vertices does a 3D tetrahedron have? What do you think the number of hypersolids, solids, faces, edge, and vertices a 4D hypertetrahedron would have? Can you explain why? Will this pattern continue for higher dimensions? Do the same for a 2D square, a 3D cube, and a 4D hypercube.

Packings and Tilings

1. Find the "best" configuration for having two balls around a common ball. What about three, four, five, six, etc.? What's the most number of balls of the same size that will fit around a common ball? What's the "best" packing of spheres of the same size? How much space is wasted?

2. Periodic tilings are those which can be picked up and translated in one or more directions, and the tilings fit exactly. Find all possible ways of tiling the plane with periodic tilings. Try it first with simple regular polygons first such as equilateral triangles, squares, etc. Then try it with more than one type or size of regular polygons.

3. In <u>Archimedes' Revenge</u>, they describe an aperiodic tiling discovered by Roger Penrose. Play around with the Penrose tiles in the white envelopes. See what cool patterns and shapes you can make with Penrose tiles. Study some of the properties that are mentioned in the book, like finding an exact copy of your neighborhood not more than two diameters away. Is this really always true?

4. Conway's Game of Life is an example of complex behaviour out of simple systems. The Game of Life takes place on a square grid, and any square which directly adjacent or diagonal to a square is considered a "neighbor." The game takes place in stages. The rules to get from one stage to the next are the following.

   I. A cell with 0 or 1 neighbors dies from loneliness.

   II. A cell with 4 or more neighbors dies from overcrowding.

   III. A cell with 2 or 3 neighbors lives happily to the next stage.

   IV. An empty cell with exactly three neighbors spontaneously generates a new living cell.

   Play around with different starting configurations on the Go board, starting with simple patterns with only a few stones. I would recommend using black stones for the initial configuration in one stage, placing the white stones for new life, then removing the black stones from the previous stage, and finally swapping in black stones for the white stones. Record each stage of life on a separate piece of paper to see how life "evolves." Are there any stable positions? Oscilating positions? Configurations which "move" along the grid? Die off or expand infinitely?

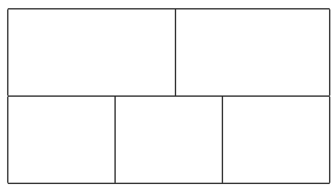
Knot Theory

1. In mathematics, a knot is a 1-dimensional loop embedded in 3-dimensions. The loop may be stretched, folded, looped, etc. but not cut and reassembled. Take several pieces of string, about a foot long. Try arranging the knots in different configurations, starting with as simple as you can. Figure out a way to represent your knots by a drawing. How can you tell if two knots are different, or if one can be rearranged into another from your drawings?

Combinatorics

1. Find the number of ways to tile a 2×1, 2×2, 2×3, 2×4, 2×5, and 2×6 board with 1×2 dominoes. What do you think the pattern is? Now find the number of ways to tile a 3×2, 3×4, 3×6, and 3×8 board with 1×2 dominoes (why only evens?). Is there any pattern? Try the same 4×1, 4×2, 4×3, 4×4, 4×5, and 4×6 boards. Is there a general formula for the number of ways to tile $m \times n$ board with 1×2 dominoes?

2. Euler is famous for solving the "Konigsburg Bridge Problem." There were seven bridges in Konigsburg which people would cross on Sunday afternoon strolls. Some people asked whether it was possible to go over each bridge exactly once. Another variation of this puzzle is to go over every edge exactly once of the following rectangle without picking up your pencil (including the outside edges of the box):



   Either find a way of doing it, or explain why it can't be done.

3. Draw several different "maps" with countries in various configurations. Try to color the maps with the fewest possible number of colors so that any countries sharing a common border have different colors. What's the most number of colors that you find necessary?

4. Any straight line intersecting the middle of a cirlce will divide the circle into two regions. Any two straight lines will cut a circle into three or four regions. What's the maximum number of region that a circle can be cut into by 3 straight lines? By 4? By 5? By n lines?
   Also, try the same coloring game as in #3. What the most number of colors necessary so that any two touching regions have different colors?

5. For $n = 3, 4, 5$, place n points around a circle and connect every two points with a line, trying to maximize the number of regions obtained. Count the maximum number of regions for $n = 3, 4, 5$. What is your guess for $n = 6$? Now find an arrangement for 6 points around a circle which maximizes the number of regions and count them.

Number Theory

1. Start with any positive integer x_0. If x_n is even, then $x_{n+1} = \frac{x_n}{2}$. If x_n is odd, then $x_{n+1} = 3x_n + 1$. Write down the first 15-20 numbers in sequence for all the starting values from $x_0 = 1$ up to $x_0 = 16$. What patterns do you notice? Do you think this will always happen?

2. Start with any positive integer (say 37). Reverse the digits and add it to the original number (37 + 73 = 110). Keep doing this until you get a palindrome, a number which is the same written forwards and backwards (110 + 011 = 121 is a palindrome). Try this for several different starting values. What do you notice? If you're brave, try starting with 89 (you'll have to be very patient and have a big calculator). If you're very brave, try 196. No one knows if the sequence will terminate starting with 196. It is believed that 196 is the *only* number that will not terminate.