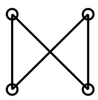
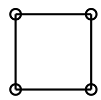
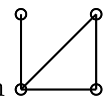
Graph Theory

Mathematical Explorations – Math 110
Block 2, Fall 2007

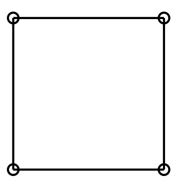
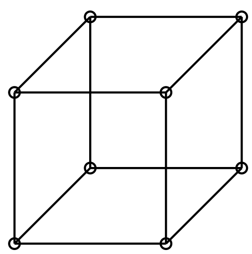
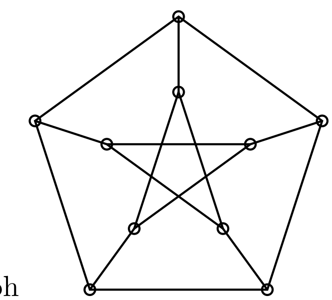
A *graph* consists of a finite number of vertices and edges between pairs of vertices. Two graph are the same if they have the same number of vertices, the same number of edges, and one graph can be changed into the other without changing which vertices are attached to each other.

For example,  is the same as  but different from 

A graph is *directed* if each edge has a direction on it; otherwise, the graph is *undirected*. A graph is *simple* if it contains no loops or double edges.

The *complete graph* on n vertices is found by attaching every vertex to every other vertex with an edge. For example, the triangle is the complete graph on 3 vertices.

Some more examples of graphs:

Square  Cube  Petersen Graph 

1. Draw all simple, undirected graphs with exactly three vertices which are different from each other. Then draw all distinct simple, undirected graphs with exactly four vertices. (Hint for four vertices: There is 1 graph with 0 edges, 1 graph with 1 edge, 2 with 2 edges, 3 with 3 edges, 2 with 4 edges, 1 with 5 edges, and 1 with 6 edges.)

2. A graph is *Eulerian* if it is possible to form a path which goes over each edge exactly once. For example, a square is Eulerian. Explain why a cube is **not** Eulerian.

3. A graph is *Hamiltonian* if it is possible to form a path which visits each vertex exactly once. For example, a square is Hamiltonian. Show that a cube is Hamiltonian by drawing a Hamiltonian path on the cube.

4. The *chromatic number* of a graph is the minimum number of colors needed to color the vertices so that no two adjacent vertices have the same color. For example, the square has chromatic number 2. Show that the cube also has chromatic number 2, but the Petersen graph has chromatic number 3.

5. Six is the smallest number of people necessary so that three people all know each other or three people all don't know each other. This situation can be translated into a graph theory problem. Draw a vertex for each person and draw an edge between each vertex. Color an edge red if two vertices know each other, and color the edge blue if they do not know each other. Show that it is possible to color the complete graph on five vertices without a red triangle (3 mutual friends) or a blue triangle (3 mutual strangers). Then show that it is impossible to do the same on the complete graph on six vertices.