

Limits and Infinity

Mathematical Explorations – Math 110

Block 2, Fall 2007

1. Find the following limits:

$$\lim_{n \rightarrow \infty} \frac{n}{2n+1} = \qquad 2 \cdot \lim_{n \rightarrow \infty} \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdots (2n) \cdot (2n)}{1 \cdot 3 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} =$$
$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \qquad 4 \cdot \lim_{n \rightarrow \infty} \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^n \frac{1}{2n+1}\right) =$$

2. At the Cantor Hotel, there are infinitely many rooms. The rooms are labeled Room 1, 2, 3, 4, . . .
- (a) A family comes in around midnight and asks for a room. The front desk says, “I’m sorry, but every room is filled tonight.” The mother says, “That’s no problem. Here’s a way to get an empty room for us.” What was the mother’s plan?
- (b) An infinitely long bus shows up an hour later with infinitely many tourists. Each needs their own room. Again, the front desk says, “There’s no way we can fit all of you!” The tour guide says, “Of course you can! Here’s the plan.” What did the tour guide suggest?
3. Find a way to count the integers (both positive and negative) exactly once. (Hint: Think leapfrog.)
4. Set up a grid of points (m, n) with m and n being natural numbers. Find a way to count each point exactly once. (Hint: Think diagonally.) What does this say about the set of all fractions $\frac{m}{n}$? Are there more, less, or the same number of fractions as the natural numbers?
5. Suppose that $\{x_1, x_2, x_3, \dots\}$ is a list of all real numbers between 0 and 1. Write the decimal expansion of each real number as follows:

$$\begin{aligned}x_1 &= 0.d_{11}d_{12}d_{13} \dots \\x_2 &= 0.d_{21}d_{22}d_{23} \dots \\x_3 &= 0.d_{31}d_{32}d_{33} \dots \\&\vdots\end{aligned}$$

Create a new number $X = 0.D_1D_2D_3 \dots$ where $D_i = 7$ if $d_{ii} \neq 7$ and $D_i = 4$ if $d_{ii} = 7$. Is X in the list of $\{x_1, x_2, x_3, \dots\}$? Why or why not? What does this say about the set of real numbers between 0 and 1 in terms of the natural numbers? Are there more, less, or the same number of real numbers between 0 and 1 as the natural numbers?