# Math 120A

Lecture Notes for Monday, August 21, 2000

## Review

| Functions | Properties |
|---|---|
| $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$ | $e^z \neq 0$ for all $z$ <br> $e^{z+w} = e^z e^w$ <br> $\frac{d}{dz} e^z = e^z$ <br> $\lvert e^z \rvert = e^x$, $\arg e^z = y$ |
| $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ <br> $\cos z = \frac{e^{iz} + e^{-iz}}{2}$ | $\frac{d}{dz} \sin z = \cos z$ <br> $\frac{d}{dz} \cos z = -\sin z$ <br> All regular trigonometric identies hold <br> $\lvert \sin z \rvert^2 = \sin^2 x + \sinh^2 y$ <br> $\lvert \cos z \rvert^2 = \cos^2 x + \sinh^2 y$ |
| $\log z = \ln \lvert z \rvert + i \arg z$ <br> $\log re^{i\theta} = \ln r + i\theta \quad (z \neq 0)$ <br> $\operatorname{Log} z = \ln \lvert z \rvert + i \operatorname{Arg} z$ | $e^{\log z} = z$ <br> $\log(e^z) = z + 2k\pi i$ <br> $\frac{d}{dz} \log z = \frac{1}{z}$ <br> $\log(z_1 z_2) = \log z_1 + \log z_2$ |
| $z^c = e^{c \log z}$ <br> $c^z = e^{z \log c}$ | $z^{c+d} = z^c z^d$ <br> $\frac{d}{dz} z^c = c z^{c-1}$ <br> $\frac{d}{dz} c^z = (\log c)\, c^z$ |

Say that we want to find an inverse to $\sin z$ : $w = \sin^{-1} z$

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i} \Rightarrow (e^{iw})^2 - 2zi\, e^{iw} - 1 = 0$$

$$e^{iw} = \frac{2iz + (-4z^2 + 4)^{\frac{1}{2}}}{2} = iz + (1 - z^2)^{\frac{1}{2}} \Rightarrow w = -i \log\left[iz + (1-z^2)^{\frac{1}{2}}\right]$$

Note: $\sin^{-1} z$ is multiple-valued (it has a square root and log)

Likewise for $\cos^{-1} z$ :
$$z = \cos w = \frac{e^{iw} + e^{-iw}}{2} \Rightarrow (e^{iw})^2 - 2z e^{iw} + 1 = 0$$

$$e^{iw} = z + (z^2 - 1)^{\frac{1}{2}} \Rightarrow w = \cos^{-1} z = -i \log\left[z + (z^2-1)^{\frac{1}{2}}\right]$$

There are formulas for $\tan^{-1} z = \frac{i}{2} \log \frac{i+z}{i-z}$ and for the derivatives, which are the same as they were for real-valued $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$.

Example Find all complex $z$ such that $\cos z = \frac{1}{2}$.

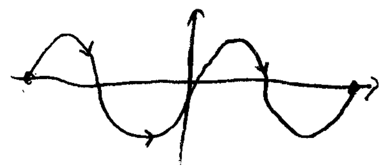
$$z = \cos^{-1} \tfrac{1}{2} = -i \log\left[\tfrac{1}{2} + \left((\tfrac{1}{2})^2 - 1\right)^{\frac{1}{2}}\right] = -i \log\left[\tfrac{1}{2} \pm \tfrac{\sqrt{3}}{2} i\right]$$

$$= -i\left[\ln 1 + i\left(\pm\tfrac{\pi}{3} + 2k\pi\right)\right] = \pm \tfrac{\pi}{3} + 2k\pi \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

We've already discussed derivatives in depth; we would now like to extend what we know about integrals of real-valued functions to complex-valued

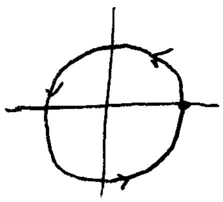
Before doing $f: \mathbb{C} \to \mathbb{C}$, we start with $w: \mathbb{R} \to \mathbb{C}$. $w(t) = u(t) + i\, v(t)$, where $u(t)$ and $v(t)$ are real-valued

Examples $w(t) = t + i \sin t$



$-2\pi \le t \le 2\pi$

$w(t) = \cos t + i \sin t$
$= e^{it}$



$0 \le t \le 2\pi$

We'll define $\underline{w'(t) = u'(t) + i\, v'(t)}$. Notice that this is the same definition as for $f: \mathbb{R} \to \mathbb{R}^2$ (an arc in two dimensions).

Now we can define definite integrals on $w(t)$ in the following way:

$\underline{\int_a^b w(t)\, dt = \int_a^b u(t)\, dt + i \int_a^b v(t)\, dt}$. From this definition, it is apparent that

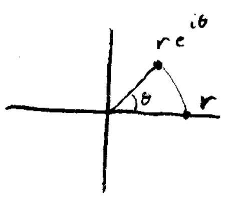
$\mathrm{Re} \int_a^b w(t)\, dt = \int_a^b \mathrm{Re}[w(t)]\, dt$ (the real part of an integral is the integral of the real part)

$\mathrm{Im} \int_a^b w(t)\, dt = \int_a^b \mathrm{Im}[w(t)]\, dt$ ( im " " " im )

Example $\int_0^\pi [\cos t + i \sin t]\, dt = \int_0^\pi \cos t\, dt + i \int_0^\pi \sin t\, dt = \sin t \Big|_0^\pi + i(-\cos t)\Big|_0^\pi = 2i$

An important property of the integral is $\left| \int_a^b w(t)\, dt \right| \le \int_a^b |w(t)|\, dt$ (look at previous example)

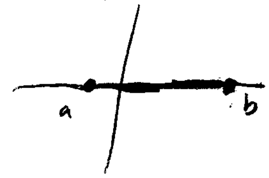
Proof $\int_a^b w(t)\, dt = r e^{i\theta}$ for some $r$ and $\theta$ $\Rightarrow \int_a^b e^{-i\theta} w(t)\, dt = r$, since $e^{-i\theta}$ is a constant



But $\mathrm{Re}\, r = r \Rightarrow r = \int_a^b e^{-i\theta} w(t)\, dt = \mathrm{Re} \int_a^b e^{-i\theta} w(t)\, dt = \int_a^b \mathrm{Re}[e^{-i\theta} w(t)]\, dt$

Finally, $\mathrm{Re}[e^{-i\theta} w(t)] \le |e^{-i\theta} w(t)| = |e^{-i\theta}|\, |w(t)| = |w(t)| \Rightarrow r \le \int_a^b |w(t)|\, dt$

So far, we have only defined integrals on the real line:

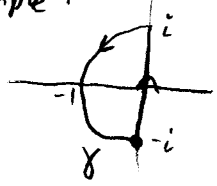



But we'd like to define integrals on more general curves.

**Def** An <u>arc</u> is a set of points $z = (x, y)$ in the complex plane such that $x = x(t)$ and $y = y(t)$ $(a \le t \le b)$ are continuous functions (basically, it is anything that can be drawn without lifting the pencil)

$z = z(t) = (x(t), y(t))$ is called a <u>parametrization</u> of the arc.

Example:




How can we parametrize $\gamma$? Here's an example:

$$\gamma(t) = \begin{cases} it & -1 \le t \le 1 \\ e^{i\frac{\pi}{2}t} = [\cos \frac{\pi}{2}t + i \sin \frac{\pi}{2}t] & 1 < t \le 3 \end{cases}$$

Given a parametrization $z = z(t)$ $(a \le t \le b)$, we can go twice as fast:

$z = z(2t)$ $\left(\frac{a}{2} \le t \le \frac{b}{2}\right)$ or backwards: $z = z\big((a+b) - t\big)$ $(a \le t \le b)$

Also, we can change $a \le t \le b$ into $0 \le s \le 1$ by letting $s = \frac{t-a}{b-a}$

so $z = z\big(a + (b-a)s\big)$ $0 \le s \le 1$ is a parametrization for $\gamma$.

Now if $x(t)$ and $y(t)$ are differentiable, then $z'(t) = x'(t) + i y'(t)$; this complex number represents the tangent vector to the arc; $|z'(t)|$ represents speed

Length of the arc $= \int_a^b |z'(t)|\, dt$

**Def** A <u>smooth</u> arc is a differentiable arc such that $z'(t) \neq 0$ for $a < t < b$

A <u>contour</u> is a sequence of connected smooth arcs

not simple: $\infty$

A <u>simple</u> contour is one that doesn't overlap (except maybe at the end points)

A <u>closed</u> contour is one whose end points <u>are</u> the same $\big(z(a) = z(b)\big)$

# Contour Integrals

Given a contour $C$ and a parametrization $z(t)$ $(a \le t \le b)$, we define

$$\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\, dt$$

**Example** $\displaystyle\int_{|z|=1} \frac{1}{z}\,dz$ In this case, $C$ is the unit circle, so let $z(t) = e^{it}$ $(0 \le t \le 2\pi)$

$$z'(t) = ie^{it}$$

$$\int_C \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{e^{it}}\left(ie^{it}\right)dt = \int_0^{2\pi} i\,dt = it\Big|_0^{2\pi} = 2\pi i$$

Contour integrals have all the properties that we would expect:

$$\int_C k\, f(z)\,dz = k\int_C f(z)\,dz$$

$$\int_C [f(z) + g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$$

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$$

$$\int_{C_1 + C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

$z_1(t) = t - i$ $\quad z_2(t) = 1 + it$ $\quad z_3(t) = -t + i$ $\quad z_4(t) = -1 - it$

**Example** $$\int_{\substack{-1+i \quad 1+i \\ -1-i \quad 1-i}} z^2\,dz = \int_{-1-i \to 1-i} z^2\,dz + \int_{1-i \to 1+i} z^2\,dz + \int_{1+i \to -1+i} z^2\,dz + \int_{-1+i \to -1-i} z^2\,dz$$

$$= \int_{-1}^{1} (t-i)^2 (1)\,dt + \int_{-1}^{1} (1+it)^2 (i)\,dt + \int_{-1}^{1} (-t+i)^2 (-1)\,dt + \int_{-1}^{1} (-1-it)^2 (-i)\,dt$$

$$= \int_{-1}^{1} (t^2 - 2it - 1)\,dt + \int_{-1}^{1} (i - 2t - it^2)\,dt + \int_{-1}^{1} (-t^2 + 2it + 1)\,dt + \int_{-1}^{1} (-i + 2t + it^2)\,dt$$

$$= \left[\frac{t^3}{3} - it^2 - t\right]\Bigg|_{-1}^{1} + \left[it - t^2 - \frac{it^3}{3}\right]\Bigg|_{-1}^{1} + \left[-\frac{t^3}{3} + it^2 + t\right]\Bigg|_{-1}^{1} + \left[-it + t^2 + \frac{it^3}{3}\right]\Bigg|_{-1}^{1}$$

$$= -\frac{4}{3} + \frac{4}{3}i + \frac{4}{3} - \frac{4}{3}i = \boxed{0}$$