Math 220A (Monday, December 2, 2002)

Today we want to prove the generalized residue theorem, namely, $\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{m} n(\gamma, a_k)\cdot \underset{z=a_k}{\mathrm{Res}}\, f(z)$, where a_k runs through all the singularities inside γ.

Recall that the residue of $f(z)$ at z_0 is the coefficient a_{-1} of the $\frac{1}{z-z_0}$ term of the Laurent series of $f(z)$ expanded in a ~~deleted~~ punctured neighborhood of z_0. This last part is important because $f(z)$ has different Laurent series which converge in different regions. For example, $\frac{1}{z(z-1)(z-2)}$ has three different Laurent series about $z=0$. The one we're interested in is

$$\frac{1}{z}\cdot\left[\frac{1}{z-2}-\frac{1}{z-1}\right] = \frac{1}{z}\left[-\frac{1}{2}\cdot\frac{1}{1-\left(\frac{z}{2}\right)}+\frac{1}{1-z}\right] = \frac{1}{z}\cdot\left[-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n+\sum_{n=0}^{\infty} z^n\right]$$

$$= \frac{1}{z}\cdot\left[\sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)z^n\right] = \sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)z^{n-1} = \underline{\sum_{n=-1}^{\infty}\left(1-\frac{1}{2^{n+2}}\right)z^n}$$

This converges on $0<|z|<1$, a punctured neighborhood of $z_0=0$.

The idea behind the residue theorem is that if $f(z)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$, and γ is a curve in the punctured nbd. which this Laurent series converges, then

$$\int_\gamma f(z)\,dz = \int_\gamma\left[\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n\right]dz = \sum_{n=-\infty}^{\infty} a_n \int_\gamma (z-z_0)^n\,dz = a_{-1}\cdot 2\pi i\cdot n(\gamma, z_0)$$

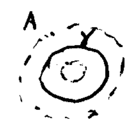
since $\int_\gamma (z-z_0)^n\,dz = \begin{cases} 0 & \text{when } n\ge 0 \quad \text{(Cauchy)} \\ 2\pi i & \text{when } n=-1 \\ 0 & \text{when } n\le -2 \end{cases}$ (Cauchy Integral Formula).

However, there's a step that needs justifying, namely switching the sum and integral signs. If it were a finite sum, no problem. But we must be cautious with infinite sums.

I'm not sure how far he got last time in proving the residue thm for just one pole, or which proof he used. But I'll assume you have the following lemma:

If γ is a closed curve in $\mathbb{C}-\{z_0\}$, $f(z)$ is analytic in $\mathbb{C}-\{z_0\}$ then $\int_\gamma f(z)\,dz = 2\pi i \; n(\gamma, z_0)\cdot \operatorname{Res}_{z=z_0} f(z)$.

From this we'll prove the general residue theorem:

The Residue Theorem Let f be analytic on a region A except for isolated singularities. Let γ be a closed curve in A which is homotopic to a point in A. (i.e., not like 

) and which does not pass through any singularities of $f(z)$. Then $\int_\gamma f(z)\,dz = 2\pi i \sum_{a\in A} n(\gamma, a) \operatorname{Res}_{z=a} f(z)$ and the sum is finite in the sense that only finitely many terms are nonzero.

Maybe a slightly better way of phrasing this is the following:
If γ is a closed curve in $\mathbb{C}$, then there is exactly one unbounded component of $\mathbb{C}-\gamma$. For all singularities of a function $f(z)$ in that unbounded component, $n(\gamma, a_k)=0$. So the only singularities that might possibly contribute are those in the bounded components of $\mathbb{C}-\gamma$, which by compactness implies there are only a finite number of isolated singularities inside γ. If $\{a_1, \dots, a_m\}$ are a list of those singularities, then

$$\int_\gamma f(z)\,dz = 2\pi i \cdot \sum_{k=1}^{m} n(\gamma, a_k)\cdot \operatorname{Res}_{z=a_k} f(z)$$

Proof of the Residue Theorem. The homotopy $\varphi(t,u)$ of γ to a point in A is used to ~~…~~ reduce the sum to a finite sum in just the same way as previously described. Let $B = \gamma([a,b] \times [0,1])$, the image of shrinking φ to a point. B is compact (why?) and therefore only contains a finite number of the isolated singularities of $f(z)$ inside B. For singularities $\alpha \in A$ outside of B, the homotopy to a point occurs in $A - \{\alpha\}$, and so $n(\gamma, \alpha) = 0$.

Let $\{\alpha_1, \dots, \alpha_m\}$ be the finite number of singularities inside of B.

We now want to show $\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{m} n(\gamma, \alpha_j) \cdot \underset{z=\alpha_j}{\operatorname{Res}} f(z)$

Recall that at each α_j, we can write $f(z) = f_{j1}(z) + f_{j2}(z)$, where $f_{j1}(z)$ is analytic in some nbd. of α_j and $f_{j2}(z)$ is analytic in $\mathbb{C} - \{\alpha_j\}$ (if $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-\alpha_j)^n$, then $f_{j1}(z) = \sum_{n=0}^{\infty} a_n (z-\alpha_j)^n$ and $f_{j2}(z) = \sum_{n=-1}^{-\infty} a_n (z-\alpha_j)^n = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-\alpha_j)^n}$)

Consider $f(z) - \sum_{j=1}^{m} f_{j2}(z)$. This is clearly analytic on $A' = A - \left\{\begin{smallmatrix}\text{all isolated sing.} \\ \text{of } f(z) \text{ in } A-\gamma\end{smallmatrix}\right\}$

Furthermore, it is also analytic at each α_j inside γ, since

$$f(z) - \sum_{j=1}^{m} f_{j2}(z) = f_{J1}(z) - \sum_{\substack{j=1 \\ j \neq J}}^{m} f_{j2}(z) \text{ is analytic in some nbd of } \alpha_J.$$

Finally, note that each $f_{j2}(z)$ satisfies the Lemma for one singularity.

$$\text{Hence, } \int_\gamma f(z)\,dz = \int_\gamma \Big[f(z) - \sum_{j=1}^{m} f_{j2}(z)\Big]dz + \sum_{j=1}^{m} \int f_{j2}(z)\,dz$$

$$= 0 + \sum_{j=1}^{m} 2\pi i \cdot n(\gamma, \alpha_j) \cdot \underset{z=\alpha_j}{\operatorname{Res}} f(z)$$

$$= 2\pi i \sum_{j=1}^{m} n(\gamma, \alpha_j) \underset{z=\alpha_j}{\operatorname{Res}} f(z).$$

QED

Now for some examples.

① $f(z) = \dfrac{1}{z(z-1)(z-2)} = \dfrac{1}{z}\left[\displaystyle\sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)z^n\right] = \displaystyle\sum_{n=-1}^{\infty}\left(1-\frac{1}{2^{n+2}}\right)z^n$

$\operatorname*{Res}_{z=0} f(z) = a_{-1}$ in the Laurent series above, which is $1-\frac{1}{2^{-1+2}} = \frac{1}{2}$.

Hence $\displaystyle\int_{|z|=\frac{1}{2}} \frac{dz}{z(z-1)(z-2)} = 2\pi i \cdot \operatorname*{Res}_{z=0} f(z) = \boxed{\pi i}$

Right now you're saying: "Big deal. I could have done this with Cauchy Integral Form

Yes, but what if I change the problem slightly?

①' $f(z) = \dfrac{1}{z^k(z-1)(z-2)} = \dfrac{1}{z^k}\cdot\left[\displaystyle\sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)z^n\right] = \displaystyle\sum_{n=-k}^{\infty}\left(1-\frac{1}{2^{n+k+1}}\right)z^n.$

Then $\operatorname*{Res}_{z=0} f(z) = 1 - \dfrac{1}{2^k} = \dfrac{2^k-1}{2^k}$

Hence $\displaystyle\int_{|z|=\frac{1}{2}} \frac{dz}{z^k(z-1)(z-2)} = 2\pi i \cdot \operatorname*{Res}_{z=0} f(z) = \boxed{\frac{(2^k-1)\pi i}{2^{k-1}}}$

Without the residue theorem (and some fancy series manipulation), you'd end up having to figure out the $(k-1)$st derivative of $\frac{1}{(z-1)(z-2)}$ at $z=0$

In fact, this goes both ways. Knowing the residues allows you to calculate derivativ

Before going on with more examples, I want to give an ~~incredibly~~ useful ~~special~~ way ~~case~~ ~~to~~ of computing residues. In general, if $f(z)$ has a pole of order n at z_0, then

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!}\lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n \cdot f(z)\right]}$$

This is essentially taking the Laurent series, making it a power series (by multiplying by $(z-z_0)^n$), then taking derivatives to find the coefficient that used to be the $n=-1$ term.

An incredibly useful shortcut for computing residues in the case of a simple pole is this:

Thm Let $f(z), g(z)$ be analytic function in a nbd. of z_0, s.t. $f(z_0) \neq 0$ and $g(z)$ has a simple zero at z_0 (ie., $g(z_0)=0$ but $g'(z_0) \neq 0$).

Then $\displaystyle\operatorname*{Res}_{z=z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g'(z_0)}$.

Proof $\displaystyle\operatorname*{Res}_{z=z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} (z-z_0)\cdot\frac{f(z)}{g(z)} = \lim_{z\to z_0} f(z) \cdot \lim_{z\to z_0} \frac{z-z_0}{g(z)-g(z_0)} = f(z_0)\cdot\frac{1}{g'(z_0)}$. ∎

Let's use this to compute some residues:

Example 2

$$\int_{|z|=2} \frac{dz}{z^2+1} = 2\pi i \cdot \left[\operatorname*{Res}_{z=i} \frac{1}{z^2+1} + \operatorname*{Res}_{z=-i} \frac{1}{z^2+1}\right]$$

$$= 2\pi i \left[\frac{1}{2i} + \frac{1}{2(-i)}\right] = \boxed{0}$$

Alternatively, we could use expanding circles to prove this:

$$\left|\int_{|z|=2} \frac{dz}{z^2+1}\right| = \left|\int_{|z|=R} \frac{dz}{z^2+1}\right| \leq \frac{2\pi R}{R^2-1} \to 0 \quad \text{as } R\to\infty$$

More examples of series manipulation:

(3) $\displaystyle\int_{|z|=1} \frac{dz}{e^z - 1 - z}$.

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots$$

$$e^z - 1 - z = \frac{z^2}{2} + \frac{z^3}{6} + \dots = z^2\left(\frac{1}{2} + \frac{z}{6} + \dots\right).$$

$$\frac{1}{e^z - 1 - z} = \frac{1}{z^2}\cdot\left(\frac{1}{2} + \frac{z}{6} + \dots\right)^{-1}$$

Let $\sum_{n=0}^{\infty} a_n z^n$ be the inverse power series of $\sum_{n=0}^{\infty} \frac{z^n}{(n+2)!}$.

i.e., $(a_0 + a_1 z + a_2 z^2 + \dots)\cdot\left(\frac{1}{2} + \frac{z}{6} + \frac{z^2}{24} + \dots\right) = 1 + 0z + 0z^2 + \dots$

$$\frac{1}{2} a_0 = 1 \Rightarrow \boxed{a_0 = 2}$$

$$\frac{1}{6} a_0 + \frac{1}{2} a_1 = 0 \Rightarrow \boxed{a_1 = -\frac{2}{3}}$$

$$\frac{1}{24} a_0 + \frac{1}{6} a_1 + \frac{1}{2} a_2 = 0, \Rightarrow a_2 = 2\left(\frac{1}{9} - \frac{1}{12}\right) = \boxed{\frac{1}{18}}$$

Therefore, $\displaystyle\frac{1}{e^z - 1 - z} = \frac{1}{z^2}\cdot\left(2 - \frac{2}{3}z + \frac{1}{18}z^2 + \dots\right)$

$$= 2\cdot\frac{1}{z^2} - \frac{2}{3}\cdot\frac{1}{z} + \frac{1}{18} + \dots$$

and hence $\displaystyle\operatorname*{Res}_{z=0} \frac{1}{e^z - 1 - z} = -\frac{2}{3}$ and $\displaystyle\int_{|z|=1} \frac{dz}{e^z - 1 - z} = 2\pi i \cdot \operatorname*{Res}_{z=0} f(z)$

$$= \boxed{-\frac{4\pi i}{3}}$$

For practice, work out Laurent series for $\frac{1}{\sin x - x}$.