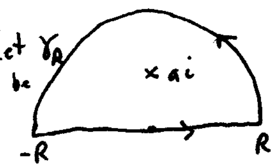
Let's start with a very straight forward application of residues.

$$\int_{-\infty}^{\infty} \frac{1}{x^2+a^2}\,dx \qquad \text{(I know we can do this without residues)}$$

We first have to set up a contour that will give us the desired result.

In this case, Let γ_R be 

so that $R > a$. The line on the real axis will gives us what we want in the limit as $R \to \infty$, so we need to show the semicircle part goes to 0 as $R \to \infty$.

$$\int_{\gamma_R} \frac{1}{z^2+a^2}\,dz = 2\pi i \cdot \operatorname*{Res}_{z=ai}\left[\frac{1}{z^2+a^2}\right] = 2\pi i \cdot \frac{1}{2ai} = \boxed{\frac{\pi}{a}} \qquad \text{for } R > a.$$

Now $$\int_{\gamma_R} \frac{1}{z^2+a^2}\,dz = \int_{\underset{-R\quad R}{\longrightarrow}} \frac{1}{z^2+a^2}\,dz + \int_{\underset{-R\quad R}{\curvearrowleft}} \frac{1}{z^2+a^2}\,dz.$$

$$\left|\int_{\underset{-R\quad R}{\curvearrowleft}} \frac{1}{z^2+a^2}\,dz\right| \le \frac{1}{R^2-a^2}\cdot \pi R = \frac{\pi R}{R^2-a^2} \longrightarrow 0 \quad \text{as } R \to \infty.$$

Hence, $$\frac{\pi}{a} = \lim_{R\to\infty} \int_{\gamma_R} \frac{1}{z^2+a^2}\,dz = \lim_{R\to\infty} \int_{\underset{-R\quad R}{\longrightarrow}} \frac{1}{z^2+a^2}\,dz + \lim_{R\to\infty} \int_{\underset{-R\quad R}{\curvearrowleft}} \frac{1}{z^2+a^2}\,dz$$

and so $$\underline{\frac{\pi}{a} = \int_{-\infty}^{\infty} \frac{1}{x^2+a^2}\,dx + 0}$$

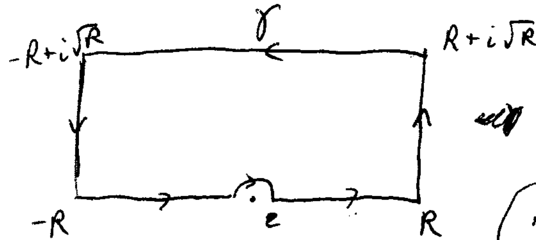
From this, we can easily deduce $\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}\,dx$ by taking a derivative with respect to a:

$$-\frac{\pi}{a^2} = \int_{-\infty}^{\infty} -\frac{2a}{(x^2+a^2)^2}\,dx$$

$$\frac{\pi}{2a^3} = \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}\,dx$$

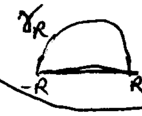
$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$$

$$= \lim_{R\to\infty} \operatorname{Im} \int_{-R}^{R} \frac{e^{iz}}{z}\,dz$$



Note this problem is only hard because w[e] only have a $\frac{1}{z}$.

In case of $\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2}\,dx$,



will work for contou[r]

$$\left|\int_{-R+i\sqrt{R} \leftarrow R+i\sqrt{R}} \frac{e^{iz}}{z}\,dz\right| = \left|\int_{R}^{-R} \frac{e^{i(x+i\sqrt{R})}}{x+i\sqrt{R}}\,dx\right| = \left|\int_{R}^{-R} \frac{e^{-\sqrt{R}}\cdot e^{ix}}{x+i\sqrt{R}}\,dx\right| \le \frac{e^{-\sqrt{R}}}{\sqrt{R}}\cdot 2R = 2\sqrt{R}e^{-\sqrt{R}}$$

$\to 0$ as $R\to\infty$

$$\boxed{\begin{aligned} z &= x+i\sqrt{R} \\ dz &= dx \end{aligned}}$$

$$\left|\int_{R \uparrow R+i\sqrt{R}} \frac{e^{iz}}{z}\,dz\right| = \left|\int_0^{\sqrt{R}} \frac{e^{i(R+iy)}}{R+iy}\,i\,dy\right| \le \int_0^{\sqrt{R}} \frac{e^{-y}}{|R+iy|}\,dy < \int_0^{\sqrt{R}} \frac{e^{-y}}{R}\,dy = \frac{1}{R}\left[-e^{-\sqrt{R}}+e^{0}\right] < \frac{1}{R}$$

$z = R+iy$
$dz = i\,dy$

Likewise for $\downarrow_{-R}^{-R+i\sqrt{R}}$.

Hence, need to ~~show~~ find $\lim_{\varepsilon\to 0}\int_{\frown_\varepsilon} \frac{e^{iz}}{z}\,dz$. For this, we turn to Conway:

$\frac{e^{iz}}{z} - \frac{1}{z} = \frac{e^{iz}-1}{z}$ ~~has~~ a removable discontinuity at $z=0$,

so it is bounded around $z=0$. Hence, $\left|\int_{\frown_\varepsilon} \frac{e^{iz}-1}{z}\right| \le M\cdot\pi\varepsilon \to 0$ as $\varepsilon\to 0$.

and so $\lim_{\varepsilon\to 0}\int_{\frown_\varepsilon} \frac{e^{iz}}{z}\,dz = \lim_{\varepsilon\to 0}\int_{\frown} \frac{1}{z}\,dz = -\pi i$

Since $\int_{\sqcap} \frac{e^{iz}}{z}\,dz = 0$, and $\int_{\downarrow \leftarrow \uparrow} \frac{e^{iz}}{z}\,dz \to 0$ as $R\to\infty$,

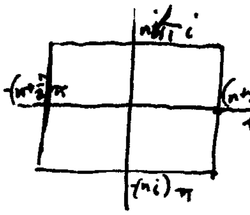
$\int_{\frown} \frac{e^{iz}}{z}\,dz \to -\pi i$ as $\varepsilon\to\infty$,

We have $\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx - \pi i = 0$, or $\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = \pi i$

and hence $\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \operatorname{Im}\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = \boxed{\pi}$

Here's one more example of the power of residues. (from Conway)

Let γ_n be the rectangular path between the points $\pm(n+\frac{1}{2})\pi \pm n\pi i$



Consider $\int_{\gamma_n} \frac{\cot z}{z^2-a^2}\,dz$ where a is not an integer multiple of π.

This is an example of a function that has an infinite number of isolated singularities.

Choose n larger than $|a|$. Then there are simple poles at $\pm a$ and $k\pi$, $-n \le k \le n$

By residue theorem, $\int_{\gamma_n} \frac{\cot z}{z^2-a^2} = 2\pi i \cdot \left[\sum_{k=-n}^{n} \operatorname*{Res}_{z=k\pi} f(z) + \operatorname*{Res}_{z=\pm a} f(z)\right]$

$$\operatorname*{Res}_{z=a} \frac{\cot z}{z^2-a^2} = \frac{\cot a}{2a} \qquad \operatorname*{Res}_{z=-a} \frac{\cot z}{z^2-a^2} = \frac{\cot(-a)}{-2a} = \frac{\cot a}{2a}$$

$$\operatorname*{Res}_{z=k\pi} \frac{\cot z}{z^2-a^2} = \frac{\cos(k\pi)}{(k\pi)^2-a^2} \lim_{z\to k\pi} \frac{z-k\pi}{\sin z} = \frac{(-1)^k}{(k\pi)^2-a^2} \cdot \lim_{z\to k\pi} \frac{1}{\cos z} = \frac{1}{(k\pi)^2-a^2}$$

So $\int_{\gamma_n} = 2\pi i \left[\sum_{k=-n}^{n} \frac{1}{(k\pi)^2-a^2} + \frac{\cot a}{a}\right]$

Now we show the limit tends to 0 as $n\to\infty$. First, $|\cot z| = \frac{\cos^2 x + \sinh^2 y}{\sin^2 x + \sinh^2 y}$

so without much trouble, it's not hard to show that on γ_n, $|\cot z| \le 2 \ \forall z \in \gamma_n$.

Then $\left| \int_{\gamma_n} \frac{\cot z}{z^2-a^2}\,dz \right| \le \frac{2}{(n\pi)^2-a^2} \cdot 4 \cdot 2(n+\frac{1}{2})\pi$, which tends to 0 as $n\to\infty$.

Hence $\lim_{n\to\infty} \int_{\gamma_n} \frac{\cot z}{z^2-a^2}\,dz = 0 = \lim_{n\to\infty} 2\pi i \left[\sum_{k=-n}^{n} \frac{1}{(k\pi)^2-a^2} + \frac{\cot a}{a}\right]$

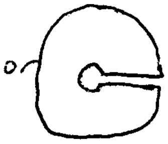
and we conclude that $\frac{\cot a}{a} = \sum_{k=-\infty}^{\infty} \frac{1}{a^2-(k\pi)^2} = \frac{1}{a^2} + 2 \cdot \sum_{k=1}^{\infty} \frac{1}{a^2-(k\pi)^2}$

In fact, one can show $\pi\cot(\pi x) = \sum_{n\in\mathbb{Z}} \frac{1}{x+n}$

Kind of contours that might need to use:

① $\int_{-\infty}^{\infty}$ or $\int_{0}^{\infty}$ of some function (usually even) with no poles or branch cuts on real axis. Then use $\gamma_R =$ 

and try to show semicircle tends to 0 as $R \to \infty$.

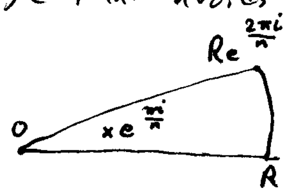
② $\int_{-\infty}^{\infty}$ or $\int_{0}^{\infty}$ where there is a pole on real axis, or there is a branch cut to worry about like $\log z$ or z^{-c}. Then Want to avoid branch or poles with a contour like  or 

③ Finite integrals involving trigonometric functions, such as $\int_0^{\pi} \frac{d\theta}{a+\cos\theta}$, $a>1$

Make integral into $\int_0^{2\pi}$ and substitute $z = e^{i\theta}$, $\frac{dz}{iz} = d\theta$ γ is $|z|=1$

Substitute $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ and get a polynomial to integ.

④ Other tricks: - For infinite integrals involving $\sin x$ or $\cos x$, change it to e^{ix} and take real and imaginary parts later.

- For infinite sums, try taking big rectangle that avoids the poles.
- In the case of $\int_0^{\infty} \frac{1}{x^n+1}\,dx$, take γ_R 

- If you have a result that depends on a parameter a, then you can sometimes take a derivative with respect to a to get other results.