Open problems of Paul Erdős in graph theory*

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The main treasure that Paul Erdős has left us is his collection of problems, most of which are still open today. These problems are seeds that Paul sowed and watered by giving numerous talks at meetings big and small, near and far. In the past, his problems have spawned many areas in graph theory and beyond (e.g., in number theory, probability, geometry, algorithms and complexity theory). Solutions or partial solutions to Erdős problems usually lead to further questions, often in new directions. These problems provide inspiration and serve as a common focus for all graph theorists. Through the problems, the legacy of Paul Erdős continues (particularly if solving one of these problems results in creating three new problems, for example.)

There is a huge literature of almost 1500 papers written by Erdős and his (more than 460) collaborators. Paul wrote many problem papers, some of which appeared in various (really hard-to-find) proceedings. Here is an attempt to collect and organize these problems in the area of graph theory. The list here is by no means complete or exhaustive. Our goal is to state the problems, locate the sources, and provide the references related to these problems. We will include the earliest and latest known references without covering the entire history of the problems because of space limitations. (The most up-to-date list of Erdős' papers can be found in [65]; an electronic file is maintained by Jerry Grossman at grossman@oakland.edu.) There are many survey papers on the impact of Paul's work, e.g., see those in the books: *"A Tribute to Paul Erdős"* [84], *"Combinatorics, Paul Erdős is Eighty"*, Volumes 1 and 2 [83], and *"The Mathematics of Paul Erdős"*, Volumes I and II [81].

To honestly follow the unique style of Paul Erdős, we will mention the fact that Erdős often offered monetary awards for solutions to a number of his favorite problems. In November 1996, a committee of Erdős' friends decided no more such awards will be given in Erdős' name. However, the author, with the help of Ron Graham, will honor future claims on the problems in this paper, provided the requirements previously set by Paul are satisfied (e.g., proofs have been verified and published in recognized journals).

Throughout this paper, the constants $c, c', c_1, c_2, \ldots$ and extremal functions $f(n), f(n,k), f(n,k,r,t), g(n), \ldots$ are used extensively, although within the context of each problem, the notation is consistent. We interpret graph theory in the broad sense, for example, including hypergraphs and infinite graphs.

Ramsey theory

For two graphs G and H, let $r(G, H)$ denote the smallest integer m satisfying the property that if the edges of the complete graph K_m are colored in blue and red, then there is either a subgraph isomorphic to G with all blue edges or a subgraph isomorphic to H with all red edges [1]. The classical Ramsey numbers are those for the complete graphs and are denoted by $r(s, t) = r(K_s, K_t)$.

Classical Ramsey numbers

In 1935, Erdős and Szekeres [135] gave an upper bound for the Ramsey number $r(s, t)$. In 1947, Erdős [90] used probabilistic methods to establish a lower bound for $r(n, n)$. The following results play an essential role in laying the foundations for both Ramsey theory and combinatorial probabilistic methods:

$$(1 + o(1)) \frac{1}{e\sqrt{2}} n 2^{n/2} < r(n, n) \leq \binom{2n-2}{n-1} \tag{1}$$

In the past fifty years, relatively little progress has been made. The current best lower bound and upper bound are due to Spencer [209] and Thomason [211], respectively.

$$(1 + o(1)) \frac{\sqrt{2}}{e} n 2^{n/2} < r(n, n) < n^{-1/2 + c/\sqrt{\log n}} \binom{2n-2}{n-1}$$

(1) *Conjecture, 1947* ($100)
   The following limit exists:

$$\lim_{n \to \infty} r(n, n)^{1/n} = c$$

(2) *Problem, 1947* ($250)
   Determine the value of the limit c above (if it exists).

   By (1), the limit, if it exists, is between $\sqrt{2}$ and 4. The proof for the lower bound in (1) is by the probabilistic method.

(3) *A problem on explicit constructions* ($100)
   Give a constructive proof for

$$r(n, n) > (1 + c)^n$$

   for some constant $c > 0$. The best known constructive lower bound $n^{c \log n / \log \log n}$ is due to Frankl and Wilson [144].

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[1] In this paper, by a graph we mean a simple loopless graph, unless otherwise specified.

(4) *Conjecture, 1947*
   For fixed l,
$$r(k, l) > \frac{k^{l-1}}{(\log k)^{c_l}}$$

   for a suitable constant $c_l > 0$ and k sufficiently large.

   For $r(3, n)$, Kim [163] recently used a complicated probabilistic argument to prove the following lower bound for $r(3, n)$ which is of the same order as the upper bound previously established by Ajtai, Komlós and Szemerédi [2]:
$$\frac{c_1 n^2}{\log n} < r(3, n) < \frac{c_2 n^2}{\log n}.$$

(5) It would be of interest to have an asymptotic formula for $r(3, n)$.

   For $r(4, n)$ the best lower bound is $c(n/\log n)^{5/2}$ due to Spencer [207] and the upper bound is $c'n^3/\log^2 n$, proved by Ajtai, Komlós and Szemerédi [2].

(6) *Problem* \$250 (see [78])
   Prove or disprove that
$$r(4, n) > \frac{n^3}{\log^c n}$$

   for n sufficiently large.

(7) *Conjecture* (proposed by Burr and Erdős [80])

$$r(n + 1, n) > (1 + c)r(n, n)$$

   for some fixed $c > 0$.

(8) *Conjecture* (proposed by Erdős and Sós [80])

$$r(n + 1, 3) - r(n, 3) \to \infty$$

   In particular, prove or disprove

(9)

$$\frac{r(n + 1, 3) - r(n, 3)}{n} \to 0.$$

Graph Ramsey numbers

(10) *A conjecture on Ramsey number for bounded degree graphs*
   (proposed by Burr and Erdős [35])

   For every graph H on n vertices in which every subgraph has minimum degree $\leq c$,
$$r(H, H) \leq c'n$$

where the constant c' depends only on c.

We remark that Chvátal, Rödl, Szemerédi and Trotter [57] proved that for graphs with bounded maximum degree, the Ramsey number grows linearly with the size of the graph. An extension of this result was obtained by Chen and Schelp [44] by showing that the bounded degree condition can be replaced by a somewhat weaker requirement. In particular, they proved that the Ramsey number for a planar graph on n vertices is bounded above by cn. Rödl and Thomas [194], generalizing results in [44], showed that graphs with bounded genus have linear Ramsey numbers.

(11) *Conjecture* (proposed by Erdős and Graham [102])
If G has $\binom{n}{2}$ edges for $n \geq 4$, then

$$r(G, G) \leq r(n, n).$$

(12) More generally, if G has $\binom{n}{2} + t$ edges, then

$$r(G, G) \leq r(H, H)$$

where H denotes the graph formed by connecting a new vertex to t of the vertices of a K_n and $t \leq n$.

(13) *Problem* [72]
Is it true that if a graph G has e edges, then

$$r(G, G) < 2^{ce^{1/2}}$$

for some constant c?

(14) *A problem on n-chromatic graphs* [82]
Let H denote an n-chromatic graph. Is it true that

$$r(H, H) > (1 - \epsilon)^n r(n, n)$$

holds for any $0 < \epsilon < 1$ provided n is large enough?

(15) *Problem* [82]
Prove that there is some $\epsilon > 0$ so that for all n and all H of chromatic number n,

$$\frac{r(H, H)}{r(n, n)} > \epsilon.$$

This is a modified version of an old conjecture $r(H, H) \geq r(n, n)$ (see [28]) which, however, has a counterexample for $n = 4$ given by Faudree and McKay [138] who showed $r(W, W) = 17$ for the pentagonal wheel W.

(16) *Conjecture* [63]
For some $\epsilon > 0$,

$$r(C_4, K_n) = o(n^{2-\epsilon}).$$

It is known that

$$c(\frac{n}{\log n})^2 > r(C_4, K_n) > c(\frac{n}{\log n})^{3/2}$$

where the lower bound is proved by probabilistic methods [207] and the upper bound is due to Szemerédi (unpublished, also see [95]).

(17) *Problem* (proposed by Erdős, Faudree, Rousseau and Schelp [95])
Is it true that
$$r(C_m, K_n) = (m-1)(n-1) + 1$$

if $m \geq n > 3$?

The answer is affirmative if $m \geq n^2 - 2$ (see [27, 139]).

(18) *Problem* (proposed by Burr, Erdős, Faudree, Rousseau and Schelp [37])
Determine $r(C_4, K_{1,n})$.

It is known that

$$n + \lceil \sqrt{n} \rceil + 1 \geq r(C_4, K_{1,n}) \geq n + \sqrt{n} - 6n^{11/40}$$

where the upper bound can be easily derived from the Turán number of C_4 and the lower bound can be found in [37]. Füredi can show (unpublished) that $r(C_4, K_{1,n}) = n + \lceil \sqrt{n} \rceil$ holds infinitely often.

(19) *Conjecture* (proposed by Burr, Erdős, Faudree, Rousseau and Schelp [37])
If G is fixed and n is sufficiently large, then $r(G, T) \leq r(G, K_{1,n-1})$ for every tree T on n vertices.

(20) *A Ramsey problem for n-cubes* (proposed by Burr and Erdős [35])
Let Q_n denote the n-cube on 2^n vertices and $n2^{n-1}$ edges. Prove that

$$r(Q_n, Q_n) \leq c2^n.$$

Beck [18] showed that $r(Q_n, Q_n) \leq c2^{n^2}$.

(21) *Linear Ramsey bounds*
(proposed by Burr, Erdős, Faudree, Rousseau and Schelp [37])
Suppose a graph G satisfies the property that every subgraph of G on p vertices has at most $2p - 3$ edges. Is it true that

$$r(G, G) \leq cn?$$

In general, the problem of interest is to characterize graphs whose Ramsey number $r(G, G)$ is linear.

Alon showed [6] that if no two vertices of degree exceeding two are adjacent in a graph G, then the Ramsey number of G is linear.

(22) *Graphs with linear Ramsey bounds*
(proposed by Burr, Erdős, Faudree, Rousseau and Schelp [37])
For a graph G, where G is Q_3, $K_{3,3}$ or H_5 (formed by adding two vertex-disjoint chords to C_5), is it true that

$$r(G, H) \leq cn$$

for any graph H with n vertices?

Multi-colored Ramsey numbers

For graphs G_i, $i = 1, \ldots, k$, let $r(G_1, \ldots, G_k)$ denote the smallest integer m satisfying the property that if the edges of the complete graph K_m are colored in k colors, then for some i, $1 \leq i \leq k$, there is a subgraph isomorphic to G_i with all edges in the i-th color. We denote $r(n_1, \ldots, n_k) = r(K_{n_1}, \ldots, K_{n_k})$.

(23) *Conjecture* ($250, a very old problem of Erdős')
Determine
$$\lim_{k \to \infty} \left(r(\underbrace{3, \ldots, 3}_{k}) \right)^{1/k}.$$

This problem goes back essentially to Schur [200] who proved

$$r(\underbrace{3, \ldots, 3}_{k}) < e\ k!$$

It is known [45] that $r(\underbrace{3, \ldots, 3}_{k})$ is supermultiplicative so that the above

limit exists.

(24) *Problem* ($100) Is the limit above finite or not?

Any improvement for small values of k will give a better general lower bound. The current best lower bound is $(321)^{1/5}$ using the 5-colored construction given by Exoo [134].

(25) *A coloring problem for cycles* (proposed by Erdős and Graham [80])
Show that
$$\lim_{k \to \infty} \frac{r(\overbrace{C_{2n+1}, \ldots, C_{2n+1}}^{k})}{r(\underbrace{3, \ldots, 3}_{k})} = 0$$

This problem is open even for $n = 2$.

(26) *A problem on three cycles* (proposed by Bondy and Erdős [80])
$$r(C_n, C_n, C_n) \leq 4n - 3.$$

For odd n, if the above inequality is true, it is the best possible. Recently, Łuczak (personal communication) showed that $r(C_n, C_n, C_n) \leq 4n + o(n)$.

Size Ramsey numbers

The *size Ramsey number* $\hat{r}(G)$ is the least integer m for which there exists a graph H with m edges so that in any 2-coloring of the edges of H, there is always a monochromatic copy of G in H.

(27) *A size Ramsey problem for bounded degree graphs*
(proposed by Beck and Erdős [189])
For a graph G on n vertices with bounded degree, prove or disprove that

$$\hat{r}(G) \le cn.$$

The case for paths was proved by Beck [19] and the case for cycles was proved by Haxell, Kohayakawa, and Łuczak [162]. Friedman and Pippenger [145] settled the case for any bounded degree tree.

(28) *A size Ramsey problem*
(proposed by Burr, Erdős, Faudree, Rousseau and Schelp [36])
For $F_1 = \cup_{i=1}^{s} K_{1,n_i}$ and $F_2 = \cup_{i=1}^{t} K_{1,m_i}$, prove that

$$\hat{r}(F_1, F_2) = \sum_{k=2}^{s+t} l_k$$

where $l_k = \max\{n_i + m_j - 1 : \ i + j = k\}$.

It was proved in [36] that

$$\hat{r}(sK_{1,n}, tK_{1,m}) = (m + n - 1)(s + t - 1).$$

Induced Ramsey theory

The *induced Ramsey number* $r^*(G)$ is the least integer m for which there exists a graph H with m vertices so that in any 2-coloring of the edges of H, there is always an induced monochromatic copy of G in H. The existence of $r^*(G)$ was proved independently by Deuber [60], Erdős, Hajnal and Pósa [111], and Rödl [193].

(29) *Problem* (proposed by Erdős and Rödl [77])
If G has n vertices, is it true that

$$r^*(G) < c^n$$

for some absolute constant c?

This holds for a bipartite graph [193]. Łuczak and Rödl [183] showed that a graph on n vertices with bounded degree has its induced Ramsey number bounded by a polynomial in n, confirming a conjecture of Trotter. Suppose G has k vertices and H has $t \ge k$ vertices. Kohayakawa, Prömel, and Rödl [166] proved that the induced Ramsey number $r^*(G, H)$ satisfies the following bound:
$$r^*(G, H) \le t^{ck \log q}$$

where q denotes the chromatic number of H and c is some absolute constant. This implies

$$r^*(G) < n^{cn \log n}.$$

Extremal graph theory

Turán numbers

For a graph H, let $t(n, H)$ denote the Turán number of H, which is the largest integer m such that there is a graph G on n vertices and m edges which does not contain H as a graph.

(30) *A conjecture on the Turán number for complete bipartite graphs*
Prove that

$$t(n, K_{r,r}) > cn^{2-1/r}$$

where c is a constant depending on r (but independent of n). An upper bound for $t(n, K_{r,r})$ of the same order was proved by Kővári, Sós and Turán [170] and Erdős independently. This long standing problem is well known as the *problem of Zarankiewicz*, first considered by Zarankiewicz [216] in 1951. However, it is included in the favorite problems of Erdős [79], who proposed many variations of this problem. The above conjecture is true for $r = 2$ and 3 (see [124]) and unsolved for $r \geq 4$. The lower bound of $t(n, K_{r,r}) > cn^{2-2/(r+1)}$ can be proved by probabilistic methods [131]. Recently, Kollár, Rónyai and Szabó [167] showed that $t(n, K_{r,s}) > cn^{2-1/r}$ if $r \geq 4$ and $s \geq r! + 1$.

(31) *A conjecture on the Turán number for bipartite graphs* (\$100)
(proposed by Erdős and Simonovits [129], 1984)

If H is a bipartite graph such that every induced subgraph has a vertex of degree $\leq r$, then the Turán number for H satisfies:

$$t(n, H) = O(n^{2-1/r}).$$

This conjecture is open even for $r = 3$.

This problem is essentially the special case which has eluded the power of the celebrated Erdős-Stone Theorem [132] and Erdős-Simonovits-Stone Theorem [126] which can be used to determine $t(n, H)$ asymptotically for all H with chromatic number $\chi(H)$ at least 3. Namely, $t(n, H) = (1 - 1/(\chi(H) - 1))\binom{n}{2} + o(n^2)$.

A variation of the above problem is the following:

(32) Conjecture (proposed by Erdős and Simonovits [129], 1984)
If a bipartite graph H contains a subgraph H' with minimum degree greater than r, then

$$t(n, H) \geq cn^{2-1/r+\epsilon}$$

for some $\epsilon > 0$.

(33) *A conjecture on the exponent of a bipartite graph*
(proposed by Erdős and Simonovits [129], 1984)

For all rationals $1 < p/q < 2$, there exists a bipartite graph G such

$$t(n, G) = \Theta(n^{p/q}).$$

(34) Conversely, is it true that for every bipartite graph G there is a rational exponent $r = r(G)$ such that

$$t(n, G) = \Theta(n^r)?$$

(35) *A Turán problem for even cycles*
(Proposed by Erdős [70])
Prove that
$$t(n, C_{2k}) \geq cn^{1+1/k}.$$

A lower bound of order $n^{1+1/(2k-1)}$ can be proved by probabilistic methods [131]. The bipartite Ramanujan graph [180, 186] gives $t(n, C_{2k}) \geq n^{1+2/3k}$. Recently, Lazebnik, Ustimenko and Woldar [178] constructed graphs which yield $t(n, C_{2k}) \geq n^{1+2/(3k-3)}$. Füredi [146, 148] determined the exact values of $t(n, C_4)$ for infinitely many n. This conjecture is open except for the case of C_4, C_6 and C_{10} (see Benson [21] and also Wenger [215] for a different construction).

(36) *A problem on Turán numbers for an n-cube*
(proposed by Erdős and Simonovits [130], 1970)
Let Q_k denote an k-cube on 2^k vertices.
Determine $t(n, Q_k)$. In particular, determine $t(n, Q_3)$.

Erdős and Simonovits [130] proved that

$$t(n, Q_3) \leq cn^{8/5}.$$

No better lower bound than $cn^{3/2}$ is known.

(37) *A problem on Turán numbers for graphs with degree constraints*
(proposed by Erdős and Simonovits [85], $250 for a proof and $100 for a counterexample)
Prove or disprove
$$t(n, H) < cn^{3/2}$$

if and only if H does not contain a subgraph each vertex of which has minimum degree > 2.

(38) *Turán numbers in an n-cube* ($100, from the 70's, see [84])
Let $f_{2k}(n)$ denote the maximum number of edges in a subgraph of Q_n containing no C_{2k}. Prove or disprove

$$f_4(n) = (\frac{1}{2} + o(1))n2^{n-1}.$$

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It is known that $\sigma_{2k} = \lim_{n\to\infty} f_{2k}(n)/e(Q_n)$ exists [46]. The best bounds for σ_4 are $.623 \geq \sigma_4 \geq 1/2$ (see [46]). For larger k, it is known that $\sqrt{2} - 1 \geq \sigma_6 \geq 1/3$ where the upper bound can be found in [46] and the construction for the lower bound is due to Conder [58]. Also, it was proved that $\sigma_{4k} = 0$ for $k \geq 2$. Is it true that $\sigma_6 = 1/3$? Is it true that $\sigma_{10} = 0$?

(39) *A problem on the octahedron graph*
(proposed by Erdős, Hajnal, Sós and Szemerédi [113])
Let G be a graph on n vertices which contains no $K_{2,2,2}$ and whose largest independent set has $o(n)$ vertices. Is it true that the number of edges of G is $o(n^2)$?

Erdős and Simonovits [127] determined the Turán number for the octahedron graph $K_{2,2,2}$ as well as other Platonic graphs [206, 203].

(40) *A problem on the Turán number of C_3 and C_4*
(proposed by Erdős and Simonovits [128])
Let $t(n, C_3, C_4)$ denote the smallest integer m that every graph on n vertices and m edges must contain C_3 or C_4 as a subgraph. Is it true that

$$t(n, C_3, C_4) = \frac{1}{2\sqrt{2}} n^{3/2} + O(n) ?$$

Erdős and Simonovits [128] proved that $t(n, C_4, C_5) = \frac{1}{2\sqrt{2}} n^{3/2} + O(n)$.

Subgraph enumeration

(41) *Problem* (proposed by Erdős)
For a graph G, let $\#(H, G)$ denote the number of induced subgraphs of G isomorphic to a given graph H.

Determine
$$f(k, n) = \min_G (\#(K_k, G) + \#(K_k, \bar{G}))$$

where G ranges over all graphs on n vertices and $\bar{G}$ denotes the complement of G.

An old conjecture of Erdős stated that a random graph should achieve the minimum which however was disproved by Thomason. In [210], he showed that $f(4, n) < \frac{1}{33}\binom{n}{4}$, $f(5, n) < 0.906 \times 2^{1-\binom{5}{2}}\binom{n}{5}$, and in general, $f(k, n) < 0.936 \times 2^{1-\binom{k}{2}}\binom{n}{k}$. Franek and Rödl [140] gave a different construction which is simpler but gives a slightly larger constant.

(42) *Conjecture* (proposed by Erdős and Simonovits [129])

Every graph G on n vertices and $t(n, C_4) + 1$ edges contains at least two copies of C_4 when n is large.

Radamacher first observed (see [72]) that every graph on n vertices and $t(n, K_3) + 1$ edges contains at least $\lfloor n/2 \rfloor$ triangles. Similar question can be asked for a general graph H, but relatively few results are known for such problems (except for some trivial cases such as stars or disjoint edges).

(43) *A conjecture on enumerating graphs with a forbidden subgraph*
(proposed by Erdős, Kleitman and Rothschild [115])
Denote by $f_n(H)$ the number of (labelled) graphs on n vertices which do not contain H as a subgraph. Then

$$f_n(H) < 2^{(1+o(1))t(n,H)}.$$

If H is not bipartite, this was proved by Erdős, Frankl and Rödl [98]. For the bipartite case, it is open even for $H = C_4$. It is well known that $t(n, C_4) = (1/2 + o(1))n^{3/2}$. On the other hand, Kleitman and Winston [165] proved

$$f_n(C_4) < 2^{cn^{3/2}}.$$

Recently, Kleitman and Wilson [164] proved that $f_n(C_{2k}) < 2^{cn^{1+1/k}}$ for $k = 3, 4, 5$ and Kreuter [171] showed that the number of graphs on n vertices which do not contain C_{2j} for $j = 2, \ldots, k$ is at most $2^{(c_k+o(1))n^{1+1/k}}$ where $c_k = .54k + 3/2$.

(44) *A problem on regular induced subgraphs*
(Proposed by Erdős, Fajtlowicz and Staton [82])
Let $f(n)$ be the largest integer for which every graph of n vertices contains a regular induced subgraph of $\geq f(n)$ vertices. Ramsey's theorem implies that a graph of n vertices contains a trivial subgraph, *i.e.*, a complete or empty subgraph of $c \log n$ vertices.

Conjecture:
$$f(n)/\log n \to \infty.$$

Note that $f(5) = 3$ (since if a graph on 5 vertices contains no trivial subgraph of 3 vertices then it must be a pentagon). $f(7) = 4$ was proved by Fajtlowicz, McColgan, Reid, and Staton [137] and also by Erdős and Kohayakawa (unpublished). McKay (personal communication) found that $f(16) = 5$ and $f(17) = 6$. Bollobás observed that $f(n) < n^{1/2+\epsilon}$ for n sufficiently large (unpublished).

(45) *A problem of Erdős and McKay [82], 1994* ($100)
Let $f(n, c)$ denote the largest integer m such that a graph G on n vertices containing no clique or independent set of size $c \log n$ must contain an induced subgraph with exactly i edges for each i, $0 < i \leq m$.

Prove or disprove that $f(n, c) \geq \epsilon n^2$.

McKay wrote, "It is easy to get bounds of the form $f(n, c) \geq c' \log n$, and Paul had a more complicated way to prove a bound $f(n) \geq c'(\log n)^2$, but I cannot remember it."

Calkin, Frieze and McKay [42] proved that a random graph with pn^2 edges, for a constant p, contains an induced subgraph with exactly i edges for each i, for i ranging from 0 up to $(1 - \epsilon)pn^2$.

(46) *A conjecture of Erdős and Tuza* [85]

Let G denote a graph on n vertices $\lfloor n^2/4 \rfloor + 1$ edges containing no K_4. Denote by $f(n)$ the largest integer m for which there are m edges e in the complement of G so that $G + e$ contains a K_4.

Conjecture:

$$f(n) = (1 + o(1))\frac{n^2}{16}.$$

On triangle-free graphs

(47) *A problem on making a triangle-free graph bipartite*
(proposed by Erdős, Faudree, Pach and Spencer [93])
Is it true that every triangle-free graph on $5n$ vertices can be made bipartite by deleting at most n^2 edges?

This conjecture is proved for graphs with at least $5n^2$ edges [103]. Thus, the general conjecture is open for graphs with e edges for $2n^2 < e \leq 5n^2$.

(48) *A problem on the number of triangle-free graphs* (very recent, [205])
Determine or estimate the number of maximal triangle-free graphs on n vertices.

(49) *A problem on the number of pentagons in a triangle-free graph* [72]
Is it true that a triangle-free graph on $5n$ vertices can contain at most n^5 pentagons?

Győri [152] proved that such graph can have at most $3^3 5^4 / 2^{14} \, n^5 \approx 1.03 n^5$ triangles.

(50) *Conjecture* (proposed by Erdős, Faudree, Rousseau and Schelp [96])
If each set of $\lfloor n/2 \rfloor$ vertices in a graph of n vertices spans more than $n^2/50$ edges, then G contains a triangle.

Krivelevich [172] proved that if each $n/2$ vertices span more than $n^2/36$ edges, then there is a triangle.

(51) *A problem on graphs covered by triangles*
(proposed by Erdős and Rothschild [86])
Suppose G is a graph of n vertices and $e = cn^2$ edges. Assume that every edge of G is contained in at least one triangle. Determine the largest integer $m = f(n, c)$ such that in every such graph there is an edge contained in at least m triangles.

Alon and Trotter showed that $f(n, c) < \alpha_c \sqrt{n}$ (personal communication). In the other direction, Szemerédi observed that the regularity lemma implies that $f(n, c)$ approaches infinity for every fixed c. Is it true that $f(n, c) > n^\epsilon$?

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Graph coloring problems

A graph is said to have chromatic number $\chi(G) = k$ if its vertices can be colored in k colors such that two adjacent vertices have different colors and such a coloring is not possible using $k-1$ colors.

(52) *A problem on graphs with fixed chromatic number and large girth*
(proposed by Erdős, 1962 [68])
Let $g_k(n)$ denote the largest integer m such that there is a graph on n vertices with chromatic number k and girth m.

Is it true that for $k \geq 4$,
$$\lim_{n \to \infty} \frac{g_k(n)}{\log n}$$
exists?

Erdős [64, 68] proved that

$$\frac{\log n}{4 \log k} \leq g_k(n) \leq \frac{2 \log n}{\log(k-2)} + 1.$$

(53) *A problem on the chromatic number and clique number*
(proposed by Erdős, 1967 [89])
Let $\omega(G)$ denote the number of vertices in a largest complete subgraph of G. Let $f(n)$ denote the maximum value of $\chi(G)/\omega(G)$ where G ranges over all graphs on n vertices.

Does the following limit exist?

$$\lim_{n \to \infty} \frac{f(n)}{n/\log^2 n}$$

Erdős [89] proved that

$$\frac{cn}{\log^2 n} \leq f(n) \leq \frac{c'n}{\log^2 n}.$$

(54) *A conjecture on subgraphs of given chromatic number and girth*
(proposed by Erdős and Hajnal [75])
For integers k and r, there is a function $f(k,r)$ such that every graph with chromatic number at least $f(k,r)$ contains a subgraph with chromatic number k and girth r.

Rödl [192] proved the above conjecture for the case of $r = 4$ and for every k. However, his upper bound is quite large. Erdős [75] further conjectured:

$$\lim_{k \to \infty} \frac{f(k,r+1)}{f(k,r)} = \infty.$$

(55) *The problem of Erdős and Lovász [69]*
Suppose a graph G is k-chromatic and contains no K_k. Let a and b denote

two integers satisfying $a, b \geq 2$ and $a+b = k+1$. Do there exist two disjoint subgraphs of G of chromatic numbers a and b, respectively?

The original question of Erdős is for the case $k = 5, a = b = 3$, which was proved affirmatively by Brown and Jung [34]. Several small cases have been solved (for more discussion, see [158]). Of special interest is the following case for $a = 2$:

Suppose the chromatic number of G decreases by 2 by removing any two vertices joining by an edge. Must G be the complete graph?

(56) *A problem on choosability of graphs* (proposed by Erdős, Rubin and Taylor [125])
A graph G is said to be (a, b)-choosable if for any assignment of a list of a colors to each of its vertices there is a subset of b colors of each list so that subsets corresponding to adjacent vertices are disjoint.

Conjecture: If G is (a, b)-choosable, then G is (am, bm)-choosable for every positive integer m.

The conjecture is known [10] to hold for graphs with n vertices, provided m is divisible by all integers smaller than some $f(n)$.

A special case of the above conjecture is the following problem (see [158]):

Let G and H denote two graphs with the same set of vertices. If G is r-choosable (i.e., $(r, 1)$-choosable) and H is s-choosable, then their union is rs-choosable.

(57) *A problem on critical graphs*
(proposed by Erdős in 1949 [76])
A graph with chromatic number k is said to be edge critical or k-critical if the deletion of any edge decreases the chromatic number by 1.
What is the largest number m, denoted by $f(n, k)$, such that there is a k-critical graph on n vertices and m edges? In other words, determine

$$\lim_{n \to \infty} \frac{f(n, k)}{n^2} = c_k.$$

Edge critical graphs were first introduced by Dirac [62] who answered a problem of Erdős from 1949 by showing $f(n, k) > c_k n^2$ for $k \geq 6$ and, in particular, $f(n, 6) > n^2/4 + cn$. Erdős and Simonovits proved that $f(n, 4) < n^2/4 + cn$. Toft [212] showed that $f(n, 4) > n^2/16 + cn$ by using a graph with many vertices of bounded degree. Erdős further raised the following questions:

(58) Is it true that $f(n, 6) = n^2/4 + n$ for $n \equiv 2$ modulo 4?

(59) Is there a 4-chromatic critical graph on n vertices and cn^2 edges which does not contain $K_{t,t}$ for some large t?

Rödl (unpublished) constructed such an example with $t < c \log n$.

(60) *A problem on critical graphs with large degree*
(proposed by Erdős [76, 73])
Let $g(n, k)$ denote the maximum value m such that there exists a k-critical graph on n vertices with minimum degree at least m.

What is the magnitude of $g(n, k)$?

Is it true that $g(n, 4) \geq cn$ for some constant c?

Simonovits [204] and Toft [213] proved that $g(n, 4) \geq cn^{1/3}$.

(61) *A problem on vertex critical graphs* (proposed by Erdős [73])
A graph with chromatic number k is said to be vertex critical or k-vertex-critical if the deletion of any vertex decreases the chromatic number by 1.

Is there some positive function $f(n)$ so that for every $k \geq 4$ there exists a graph G on n vertices which is k-vertex-critical but $\chi(G - A) = k$ for any set A of at most $f(n)$ edges?

Brown [31] gave an example of a 5-vertex-critical graph with no critical edges. Recently, Jensen [157] showed that there exists a k-vertex-critical graph, for $k \geq 5$, such that the chromatic number is not decreased after deleting any m edges all incident to a common vertex.

(62) *A conjecture on strong chromatic index* (proposed by Erdős and Nešetřil [76] 1985)
The strong chromatic index $\chi^*(G)$ of a graph G is the least number r so that the edges of G can be colored in r colors in such a way that any two adjacent vertices in G are not incident to edges of the same color.

Suppose G has maximum degree k. Is it true that $\chi^*(G) \leq 5k^2/4$ if k is even and $\chi^*(G) \leq 5k^2/4 - k/2 + 1/4$ if k is odd?

This conjecture is open for $k \geq 4$ while the cases of $k \leq 3$ are solved by Anderson [13] and Horák, Qing and Trotter [156]. Chung, Gyárfás, Trotter and Tuza [56] proved that if G contains no induced $2K_2$, then G has at most $5k^2/4$ edges.

(63) *A problem on three-coloring* (proposed by Erdős, Faudree, Rousseau and Schelp [139])
Is it true that in every three-coloring of the edges of K_n there is a set of three vertices which are adjacent to at least two-third of all the vertices by edges of the same color?

If true, it is the best possible as shown by an example given by Kierstead [139].

(64) *A problem on anti-Ramsey graphs*
(proposed by Burr, Erdős, Graham and Sós [39])
For a graph G, determine the least integer $r = f(n, e, G)$ so that there is some graph H on n vertices and e edges which can be r-edge-colored such that all edges of every copy of G in H have different colors.

It seems to be a difficult problem to get good bounds for $f(n, e, G)$ for a general graph G (see [39]). Even for special cases, there are large gaps between known bounds. For example, it was shown in [39, 38] that $f(n, e, C_5) \geq c_n n$ for $e = (1/4 + \epsilon)n^2$ and $f(n, e, C_5) = O(n^2/\log n)$ for $e = (1/2 - \epsilon)n^2$. Also, $f(n, e, P_4) > c'_n n$ for $e = \epsilon n^2$ and $f(n, e, P_4) \leq n$ for $n = n^2/\exp(c\sqrt{\log n})$.

Covering and packing

(65) *A conjecture on covering by C_4's* (proposed by Erdős and Faudree [88])
Suppose a graph G has $4n$ vertices with minimum degree at least $2n$. Then G has n vertex-disjoint C_4's.

Alon and Yuster [11] proved that for a fixed bipartite graph H on h vertices, a graph G with n vertices, where h divides n, can be covered by vertex-disjoint copies of H if the minimum degree of G is at least $(1/2+\epsilon)n$ for n sufficiently large.

(66) *A problem on clique covering and clique partition*
(proposed by Erdős, Faudree and Ordman [92])
The clique covering number $cc(G)$ of G is the least number of cliques that covers the graph. The clique partition number $cp(G)$ is the least number of cliques that partition the edge set of G. Here G_n denotes a graph on n vertices.
Determine the largest value c such that

$$\frac{cp(G_n)}{cc(G_n)} > cn^2$$

for an infinite family of graphs G_n.

An example was given in [92] with $c = 1/64$.

Is there a sequence of graphs G_n such that

$$cp(G_n) - cc(G_n) = n^2/4 + O(n)?$$

In [40] it was shown

$$cp(G_n) - cc(G_n) = n^2/4 - n^{3/2}/2 + n/4 + O(1).$$

(67) *The ascending subgraph decomposition problem*
(proposed by Alavi, Boals, Erdős, Chartrand and Oellermann [5])
Suppose G is a graph with $n(n + 1)/2$ edges. Prove that G can be edge-partitioned into subgraphs G_i with i edges such that G_i is isomorphic to a subgraph of G_{i+1} for $i = 1, \ldots, n - 1$.

A special case is the decomposition of star forests into stars (which is the so-called suitcase problem of partitioning integers $1, \ldots, n$ into k parts with given sums $a_1, \ldots, a_k$ for any $a_i \leq n$ and $\sum a_i = n(n + 1)/2$). The suitcase problem was solved by Ma, Zhou and Zhou [185, 184].

General extremal problems

(68) *A conjecture on trees* 1962,
(proposed by Erdős and Sós)

Every graph on n vertices having at least $n(k-1)/2+1$ edges must contain as a subgraph every tree of $k+1$ vertices, for $n \geq k+1$.

This conjecture, if true, is best possible. Some asymptotic approximations of this conjecture were given by Komlós and Szemerédi (unpublished). Also, this conjecture is proved for some special families of trees such as caterpillars.

Brandt and Dobson [30] have proved the conjecture for graphs with girth at least 5.

(69) *The (n/2-n/2-n/2) conjecture*
(proposed by Erdős, Füredi, Loebl and Sós [99])
Let G be a graph with n vertices and suppose at least $n/2$ vertices have degree at least $n/2$. Then G contains any tree on at most $n/2$ vertices.

Ajtai, Komlós and Szemerédi [4] proved the following asymptotic version: If G has n vertices and at least $(1+\epsilon)n/2$ vertices have degree at least $(1+\epsilon)n/2$, then G contains any tree on at most $n/2$ vertices if n is large enough (depending on ϵ). Komlós and Sós conjectured [99]:
Let G be a graph with n vertices and suppose at least $n/2$ vertices have degree at least k. Then G contains any tree with k vertices.

(70) *A conjecture of Erdős and Gallai,* 1959 [100]
Every connected graph on n vertices can be edge-partitioned into at most $\lfloor (n+1)/2 \rfloor$ paths.

Lovász [179] showed that every graph on n vertices can be edge-partitioned into at most $\lfloor n/2 \rfloor$ cycles and paths. Pyber [190] showed that every connected graph on n vertices can be covered by at most $n/2+O(n^{3/4})$ paths.

(71) *A problem on clique transversals*
(proposed by Erdős, Gallai and Tuza [101])
Estimate the cardinality, denoted by $\tau(G)$, of a smallest set that shares a vertex with every clique of G.

Denote by $R(n)$ the largest integer such that every triangle-free graph on n vertices contains an independent set of $R(n)$ vertices. Is it true that $\tau(G) \leq n - R(n)$?

From the results on Ramsey numbers $r(3,k)$, we know that $c\sqrt{n \log n} < R(n) < c'\sqrt{n \log n}$. So far, the best known bound [101] is $\tau(G) \leq n - \sqrt{2n} + c$ for a small constant c.

(72) *A problem on the diameter of a K_r-free graph*
(proposed by Erdős, Pach, Pollack and Tuza [119])
Let G denote a connected graph on n vertices with minimum degree δ.

Show that if G is K_{2r}-free and δ is a multiple of $(r-1)(3r+2)$, then the diameter of G, denoted by $D(G)$, satisfies

$$D(G) \leq \frac{2(r-1)(3r+2)}{(2r^2-1)\delta}n + O(1)$$

when n approaches infinity. If G is K_{2r+1}-free and δ is a multiple of $3r-1$, then
$$D(G) \leq \frac{3r-1}{r\delta}n + O(1)$$

when n approaches infinity.

In [119], bounds for the diameters of triangle-free or C_k-free graphs with given minimum degree were derived.

(73) *Problem* ($100, many years ago)
Is there a sequence A of density 0 for which there is a constant $c(A)$ so that for $n > n_0(A)$, every graph on n vertices and $c(A)n$ edges contains a cycle whose length is in A.

Erdős said [81], "I am almost certain that if A is the sequence of powers of 2 then no such constant exists. What if A is the sequence of squares? I have no guess. Let $f(n)$ be the smallest integer for which every graph on n vertices and $f(n)$ edges contains a cycle of length 2^k for some k. I think that $f(n)/n \to \infty$ but that $f(n) < n(\log n)^c$ for some $c > 0$."

Alon pointed out that $f(n) \leq cn \log n$ using the fact [29] that graphs with n vertices and $ck^{1+1/k}$ edges contain cycles of all even lengths between $2k$ and $2kn^{1/k}$ (by taking k to be about $\log n/2$).

(74) *Conjecture* (proposed by Erdős [74])
For $n \geq 3$, any graph with $\binom{2n+1}{2} - \binom{n}{2} - 1$ edges is the union of a bipartite graph and a graph with maximum degree less than n.

(75) *A decomposition problem about odd cycles*
(proposed by Erdős and Graham [102])
It is known that a complete graph on 2^n vertices can be edge-partitioned into n bipartite graphs (and this is not true for $2^n + 1$). Suppose a complete graph on $2^n + 1$ is decomposed into n subgraphs. Let $f(n)$ denote the smallest integer m such that one of the subgraphs must contain an odd cycle of length less than or equal to m. Determine $f(n)$. Is $f(n)$ unbounded?

(76) *A problem on almost bipartite graphs* (proposed by Erdős [82])

Suppose G has the property that for every m, every subgraph on m vertices contains an independent set of size $m/2 - k$. Let $f(k)$ denote the smallest number such that G can be made bipartite by deleting $f(k)$ vertices.

Recently, Reed (unpublished) proved the existence of $f(k)$ by using graph minors. It would be of interest to improve the estimates for $f(k)$.

Erdős, Hajnal and Szemerédi [114] proved that for every $\epsilon > 0$ there is a graph of infinite chromatic number for which every subgraph of m vertices contains an independent set of size $(1 - \epsilon)m/2$. Erdős remarked that perhaps $(1-\epsilon)m/2$ can be replaced by $m/2 - f(m)$ where $f(m)$ tends to infinity arbitrarily slowly.

(77) *A problem of Erdős and Rado* [122]

What is the least number $k = k(n, m)$ so that for every directed graph on k vertices, either there is an independent set of size n or the graph includes a directed path of size m (not necessarily induced)?

Erdős and Rado [122] give an upper bound of $[2^{m-1}(n-1)^m + n - 2]/(2n - 3)$.

Mitchell and Larson [176] give a recurrence relation and obtain a bound of n^2 for $m = 3$ and, more generally, of n^{m-1} for $m > 3$.

Random graphs

(78) *Problem* (proposed by Erdős, see [9])
Let G denote a random graph on n vertices and cn edges. What is the largest r, denoted by $r(c)$, for which the probability that the chromatic number is r is at least some constant strictly greater than 0 (and independent of n)?

This is open except for $r = 3$ (see [9]). Łuczak [182] gave an asymptotic estimate:
$$r(c) = (1 + o(1))\frac{c}{2 \log c}.$$

However, the exact values are not known.

(79) *Problem* (proposed by Erdős, see [9])
How accurately can one estimate the chromatic number of a random graph (with edge probability $1/2$)? Prove or disprove that the error is more (much more) than $O(1)$.

Shamir and Spencer have an $O(n^{1/2})$ upper bound [201].

An old problem raised by Erdős and Rényi [123] is to determine the chromatic number of a random graph with given edge density. This problem has almost been completely resolved due to the work of Matula [186], Shamir and Spencer [201], and Bollobás [23]. The order of the chromatic number for both sparse and dense random graphs have been determined asymptotically. For random graphs with edge density p satisfying $p \leq n^{-5/6-\epsilon}$, where $\epsilon > 0$, Shamir and Spencer showed that the chromatic number almost surely takes on at most five different values. Łuczak later on [181] proved that for such sparse random graphs, the chromatic number is concentrated at two values. Recently, Alon and Krivelevich [8]

showed that for $p \leq n^{-1/2-\epsilon}$, the chromatic number is concentrated at at most two values (and for some values of p, in a single value). However, the concentration of the limit function of χ for dense graphs is not as well understood yet.

(80) *A conjecture on a spanning cube in a random graph*
(proposed by Erdős and Bollobás, see [88])
A random graph on $n = 2^d$ vertices with edge density $1/2$ contains an d-cube.

Alon and Füredi [7] showed that this conjecture is true if the random graph has edge density $> 1/2$ for n large enough.

(81) *Problem* (proposed by Erdős and Bollobás, see [9])
In a random graph (with edge probability $1/2$), find the best possible c such that every subgraph on n^α vertices will almost surely contain an independent set of size $c \log n$ (where c depends on α).

(82) *Problem* (proposed by Erdős and Spencer [9])
Start with n vertices and add edges at random one by one. If we stop when every vertex is contained in a triangle, is there a set of vertex disjoint triangles covering every vertex (except for at most two vertices)?

The above question can be posed for other configurations as well [9]. In particular, Bollobás and Frieze [26] showed that stopping as soon as there is no isolated vertex, then there already almost surely is a perfect matching if the number of vertices is even. If we stop when every vertex has degree at least 2, Ajtai, Komlós and Szemerédi [3] and Bollobás [24] proved that there already almost surely is a Hamiltonian cycle. Alon and Yuster [12] and Ruciński [196] examined the threshold function of a random graph on n vertices for the existence of $\lfloor n/|V(H)| \rfloor$ vertex-disjoint copies of a given graph H.

(83) *A problem on monotone graph properties*
(proposed by Erdős, Suen and Winkler [133])
A graph property P is said to be monotone if every subgraph of a graph with property P also has property P.
Start with n vertices and add edges one by one at random, subject to the condition that property P continues to hold. Stop when no more edges can be added. How many edges can such a graph have?

The cases when P denotes "triangle-free", "bipartite", or "disconnected" were considered by Erdős, Suen and Winkler [133]. The property "maximum degree bounded by k" was examined by Ruciński and Wormald [197].

Many interesting cases are still open, including "C_4-free", "K_r-free" (for $r \geq 4$), "k-colorable" (for $k \geq 3$), "planar" and "girth $> k$".

Hypergraphs

Ramsey theory for hypergraphs

A t-graph has a vertex set V and an edge set E consisting of some prescribed set of t-subsets of V. For t-graphs G_i, $i = 1, \ldots, k$, let $r_t(G_1, \ldots, G_k)$ denote the smallest integer m satisfying the property that if the edges of the complete t-graph on m vertices are colored in k-colors, then for some i, $1 \le i \le k$, there is a t-subgraph isomorphic to G_i with all t-edges in the i-th color. We denote $r_t(n_1, \ldots, n_k) = r_t(K_{n_1}, \ldots, K_{n_k})$. Clearly $r_2(n_1, \ldots, n_k) = r(n_1, \ldots, n_k)$.

(84) *Conjecture* ($500)

Is there an absolute constant $c > 0$ such that

$$\log \log r_3(n, n) \ge cn?$$

This is true if four colors are allowed [110]. If just three colors are allowed, there is some improvement due to Erdős and Hajnal (unpublished).

$$r_3(n, n, n) > e^{cn^2 \log^2 n}.$$

(85) *Generalized Ramsey problems* ($500)

(proposed by Erdős and Hajnal [78])

Denote by $f_t(n, u, v)$ the largest value of k such that any coloring of the t-tuples of a set of n elements in blue and red, then there are either k elements all of whose t-tuples are in blue or there are v red t-tuples of a set of u elements. Clearly, for $v = \binom{k}{t}$, the Ramsey function $r_t(k, k) = n$ satisfies $f_t(n, k, \binom{k}{t}) = k$.

Conjecture

$$h_t(n, k, g_t(k) + 1) \le (\log n)^c$$

where n is sufficiently large and g_t is defined by

$$g_t(k) = \sum_1^t g_t(u_i) + \Pi_{i=1}^t u_i$$

and the u's are as nearly equal as possible; $g_t(k) = 0$ for $k < t$; and $g_t(t) = 1$.

Turán problems

(86) *Turán's conjecture for 3-graphs*

For an r-uniform hypergraph (or r-graph, for short) H, we denote by $t_r(n, H)$ the smallest integer m such that every r-graph on n vertices with $m + 1$ edges must contain H as a subgraph. When H is a complete graph on k vertices, we write $t_r(n, k) = t_r(n, H)$.

In memory of Turán, Erdős offered \$1000 for the following conjecture:

Conjecture [214]

$$\lim_{n\to\infty} \frac{t_3(n,4)}{\binom{n}{3}} = \frac{5}{9}.$$

There are many different extremal constructions [32, 169] which show $t_3(n,4) \geq \frac{5}{9}\binom{n}{3}(1 + o(n^{-1}))$. The best upper bound for $t_3(n,4)/\binom{n}{3}$ is $(-1+\sqrt{21})/6 = .5971\ldots$ due to Giraud (unpublished, mentioned in [41]). An excellent survey on this problem can be found in Füredi [149].

(87) *Conjecture* [214]

$$\lim_{n\to\infty} \frac{t_3(n,5)}{\binom{n}{3}} = \frac{1}{4}.$$

(88) *A conjecture for triple systems*
(proposed by Brown, Erdös and Sós [33, 78])
Let $f(n,k,r)$ denote the least integer m such that every 3-graph on n vertices with more than m triples contains an induced subgraph on k vertices with at least r edges. Prove that

$$f(n,k,k-3) = o(n^2).$$

Ruzsa and Szemerédi [198] settled an earlier conjecture of Erdős by showing $f(n,6,3) = o(n^2)$.

Δ-systems

A family of sets A_i, $i = 1, 2, \ldots$, is called a *strong Δ-system* if the intersections $A_i \cap A_j$ for $i \neq j$ are all identical. In other words,

$$A_i \cap A_j = \bigcap_t A_t \text{ if } i \neq j.$$

A strong Δ-system of k sets is also called a *k-star*. The family is called a *weak Δ-system* if we only require that the sizes $|A_i \cap A_j|$ are all the same for $i \neq j$.

(89) *A problem on unavoidable stars*
(proposed by Erdős and Rado [120] in 1960)
For given integers n and k, determine the smallest integer m, denoted by $f(n,k)$, for which every family of sets A_i, $i = 1, \ldots, m$, with $|A_i| = n$ for all i, contains a *k-star*.

Erdős and Rado [120, 121] proved that

$$2^n < f(n,3) \leq 2^n n!.$$

Abbott and Hanson [1] proved $f(n,3) > 10^{n/2}$. Spencer [208] showed $f(n) < (1+\epsilon)^n n!$.

(90) *Conjecture* ($1000):
$$f(n,3) \leq c^n$$
for some absolute constant c.

The current best bound is due to Kostochka [168]:
$$f(n,3) < n! \left(\frac{c \log \log n}{\log n} \right)^n.$$

(91) *Conjecture* :
$$f(n,k) \leq c_k^n$$

(92) *A problem on unavoidable stars of an n-set*
(proposed by Erdős and Szemerédi [136])
Determine the least integer m, denoted by $f^*(n,k)$ such that for any family
$\mathcal{A}$ of subsets of an n-set with $|\mathcal{A}| > f^*(n,k)$, $\mathcal{A}$ must contain a k-star.

Erdős and Szemerédi [136] showed that $f^*(n,3) < 2^{(1-1/(10\sqrt{n}))n}$. Recently, Deuber, Erdő, Gunderson, Kostochka and Meyer [61] proved that
$f^*(n,r) > 2^{n(1-\log\log r/2r - O(1/r))}$ for every $r \geq 3$ and infinitely many n.
In particular, $f^*(n,3) > 1.551^{n-2}$ for infinitely many n.

(93) *A problem on weak Δ-systems*
(proposed by Erdős, Milner and Rado [118])
Let $g(n,k)$ denote the least size for a family of n-sets forcing a weak Δ-system of k sets.

Conjecture[118]:
$$g(n,3) < c_k^n.$$

Recently, Axenovich, Fon-der-Flaass and Kostochka [14] proved
$$g(n,3) < (n!)^{1/2+\epsilon}.$$

(94) *A problem on weak Δ-systems of an n-set* (proposed by Erdős and Szemerédi [136])
Determine the least integer m, denoted by $g^*(n,k)$, such that for any family $\mathcal{A}$ of subsets of an n-set with $|\mathcal{A}| > g^*(n,k)$, $\mathcal{A}$ must contain a weak Δ-system of k sets.

Erdős and Szemerédi [136] proved that $g^*(n,3) > n^{\log n/4 \log \log n}$. Recently, Rödl and Thoma [194] proved that $g^*(n,r) \geq 2^{\frac{1}{3}n^{1/5} \log^{4/5}(r-1)}$ for $r \geq 3$. For the upper bound for $g^*(n,r)$, Frankl and Rödl [142] proved that $g^*(n,k) < (2-\epsilon)^n$, where ϵ depends only on k.

(95) *A problem of Erdős, Faber and Lovász* ($500, 1972 [75])
Let $G_1, \ldots, G_n$ be n edge-disjoint complete graphs on n vertices. Then the chromatic number of $\cup_{i=1}^n G_i$ is n.

Recently, Kahn [159] proved that the chromatic number of $\cup_{i=1}^n G_i$ is at most $(1 + o(1))n$. Erdős also asked the question of determining $\cup_{i=1}^n G_i$ if we require that $G_i \cap G_j$, $i \neq j$, is triangle-free, or should have at most one edge.

(96) *A problem on jumps in hypergraph*
($500, proposed by Erdős [75])
Prove or disprove that any 3-uniform hypergraph with $n > n_0$ vertices
and at least $(1/27 + \epsilon)n^3$ edges contains a subgraph on m vertices and at
least $(1/27 + c)m^3$ edges where $c > 0$ does not depend on ϵ and m.

Originally, Erdős asked the question of determining such a jump for the
maximum density of subgraphs in hypergraphs with any given edge den-
sity. However, Frankl and Rödl [143] gave an example showing for hyper-
graphs with a certain edge density, there is no such jump for the density
of subgraphs. Still, the original question for 3-uniform hypergraphs as
described above remains open.

(97) *A problem on property B* (proposed by Erdős 1963, [66])
A family $\mathcal{F}$ of subsets is said to have property B if there is a subset S such
that every subset in $\mathcal{F}$ contains an element in S and an element not in S.
What is the minimum number $f(n)$ of subsets in a family $\mathcal{F}$ of n-sets not
having property B?

Property B is named after Felix Bernstein who first introduced this prop-
erty in 1908 [22]. The best known upper bound is due to Erdős [66, 67]
and the following lower bound was given by Beck [20].

$$n^{1/3-\epsilon}2^n \leq f(n) < (1+\epsilon)\frac{e \log 2}{4}n^2 2^n$$

for $n \geq n_0$ and n_0 depends only on ϵ. This problem was extensively
considered in [9, 158].

(98) *A conjecture on covering a hypergraph*
(proposed by Erdős and Lovász [117])
Let $f(n)$ denote the smallest integer m such that for any n-element sets
$A_1, \ldots, A_m$ with $A_i \cap A_j \neq \emptyset$ for $i \neq j$ and for every set S with at most
$n - 1$ elements, there is an A_i disjoint from S. Erdős and Lovász [117]
proved that
$$\frac{8}{3}n - 3 \leq f(n) \leq cn^{3/2} \log n.$$

Kahn [160] showed that $f(n) = O(n)$ and he wrote an excellent survey
paper [161] on several related hypergraph problems (including this prob-
lem).

Erdős [84] further conjectured a strengthened version of this conjecture:
For every $c > 0$ there is an $\epsilon > 0$ such that if n is sufficiently large and
$\{A_i : 1 \leq i \leq cn\}$ is a collection of intersecting n-sets, then there is a set
S satisfying that $|S| < n(1 - \epsilon)$ and $A_i \cap S \neq \emptyset$ for all $1 \leq i \leq cn$.

(99) *A problem on unavoidable hypergraphs*
(proposed by Chung and Erdős [49])

A r-graph H is said to be (n, e)-unavoidable if H is contained in every r-
graph on n vertices and e edges. Let $f_r(n, e)$ denote the largest integer m

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with the property that there exists an (n, e)-unavoidable r-graph having m edges.

Determine $f_r(n, e)$.

For the case of $r = 2$ and 3, the solutions can be found in [48, 49].

(100) *A problem on unavoidable stars* (proposed by Duke and Erdős [91])

Let $f(n, r, k, t)$ denote the smallest integer m with the property that any r-graph on n vertices and m edges must contain a k-star with common intersection of size t.

Determine $f(n, r, k, t)$.

Duke and Erdős proved that $f(n, r, k, 1) \leq cn^{r-2}$ where c depends only on r and k. For the case of $r = 3$, tight bounds are obtained by Chung and Frankl [54], also see [141, 47].

(101) *A problem on decompositions of hypergraphs*
(proposed by Chung, Erdős and Graham [51])

For r-graphs $H_1, \ldots, H_k$ with the same number of edges, a U-decomposition (first suggested by Ulam) is a family of partitions of each of the edge sets $E(H_i)$ into t mutually isomorphic sets, i.e., say $E(H_i) = \cup_{j=1}^t E_{ij}$, where for each j, all the E_{ij} are isomorphic. Let $U_k(n, r)$ denote the least possible value m such that all families of k r-graphs must have a U-decompositions into t isomorphic sets.

For graphs, it was shown [53, 50] that

$$\frac{2}{3}n - \frac{1}{3} < U_2(n, 2) < \frac{2}{3}n + c$$

and for $k \geq 3$,
$$\frac{3}{4}n - \sqrt{n-1} < U_k(n, 2) < \frac{3}{4}n + c_k.$$

There is still room for improvement.

For hypergraphs, it is of interest to determine $U_2(n, 3)$, for example. It is known (see [51]) that

$$c_1 n^{4/3} \log \log n / \log n < U_2(n, 3) < c_2 n^{4/3}.$$

Also, for $\epsilon > 0$,

$$c_3 n^{2-2/k-\epsilon} < U_k(n, 3) < c_4 n^{2-1/k}.$$

(102) *A problem on the product of the point and line covering numbers*
(proposed by Chung, Erdős and Graham [52])

In a hypergraph G with vertex set V and edge set E, the point covering number $\alpha_0(G)$ denotes the minimal cardinality of a subset of V which has non-empty intersection with every edge e in E. The line covering number $\alpha_1(G)$ denotes the minimal cardinality of a subset S of E such that every vertex is contained in some edge in S. The problem of interest

is to characterize hypergraphs which achieve the maximum and minimum value of $\alpha_0(G)\alpha_1(G)$.

The case for graphs was solved in [52] and it was shown that

$$n - 1 \leq \alpha_0(G)\alpha_1(G) \leq (n-1)\lfloor \frac{n+1}{2} \rfloor$$

which is asymptotically best possible.

Infinite graphs

Erdős wrote over a hundred papers on infinite graphs. In particular, the problem papers by Erdős and Hajnal [105, 106] contain 82 problems which have been the major driving force in this field. Many of the problems have been solved positively, negatively or proved to be undecidable. The problems here are mainly based on the survey papers [86, 155, 153]. Many comments to these problems were graciously provided by András Hajnal and Jean Larson.

Here we use the following arrow notation, first introduced by Rado:

$$\kappa \rightarrow (\lambda_\nu)^r_\gamma$$

which means that for any r-partition $f : [\kappa]^r \rightarrow \gamma$ there are $\nu < \gamma$ and $H \subset \kappa$ such that H has order type λ_ν and $f(Y) = \nu$ for all $Y \in [H]^r$. If $\lambda_\nu = \lambda$ for all $\nu < \gamma$, then we write $\kappa \rightarrow (\lambda)^r_\gamma$. In this language, Ramsey's theorem can be written as

$$\omega \rightarrow (\omega)^r_k$$

for $1 \leq r, k < \omega$.

(103) *A conjecture on ordinary partition relations for ordinals* ($1000)
   (proposed by Erdős and Hajnal [105])
   Determine the α's for which $\omega^\alpha \rightarrow (\omega^\alpha, 3)^2$.

   Galvin and Larson [150] showed that such α must be of the form ω^β. Chang [43] proved $\omega^\omega \rightarrow (\omega^\omega, 3)^2$. Milner [187] generalized the proof of Chang to show $\omega^\omega \rightarrow (\omega^\omega, n)^2$ for $n < \omega$, and Larson [173] gave a simpler proof.

   There have been many recent developments on ordinary partition relations for countable ordinals. Schipperus [199] proved that

$$\omega^{\omega^\beta} \rightarrow (\omega^{\omega^\beta}, 3)^2 \tag{2}$$

   if β is the sum of at most two indecomposables. In the other direction, Schipperus [199] and Larson [174] showed that

$$\omega^{\omega^\beta} \nrightarrow (\omega^{\omega^\beta}, 5)^2 \tag{3}$$

if β is the sum of two indecomposables. Darby [59] proved that

$$\omega^{\omega^\beta} \not\to (\omega^{\omega^\beta}, 4)^2 \tag{4}$$

if β is the sum of three indecomposables. Schipperus [199] also proved that

$$\omega^{\omega^\beta} \not\to (\omega^{\omega^\beta}, 3)^2 \tag{5}$$

if β is the sum of four indecomposables.

(104) *A problem on ordinary partition relations for ordinals*
(proposed by Erdős and Hajnal [105])
Is it true that if $\alpha \to (\alpha, 3)^2$, then $\alpha \to (\alpha, 4)^2$?

The original problem proposed in [105] was "Is it true that if $\alpha \to (\alpha, 3)^2$, then $\alpha \to (\alpha, n)^2$?". However, Schipperus' results (2) and (3) give a negative answer for the case of $n \geq 5$. For the case of $n = 4$, Darby and Larson (unpublished) proved

$$\omega^{\omega^2} \to (\omega^{\omega^2}, 4)^2$$

extending the previous work of Darby on $\omega^{\omega^2} \to (\omega^{\omega^2}, 3)^2$.

(105) [105]
Is it true that $\omega_1 \to (\alpha, 4)^3$ for $\alpha < \omega_1$?

Milner and Prikry [188] gave an affirmative answer for $\alpha \leq \omega 2 + 1$.

(106) [105]
Is it true that $\omega_1{}^2 \to (\omega_1{}^2, 3)^2$?

A. Hajnal [154] proved $\omega_1{}^2 \not\to (\omega_1{}^2, 3)^2$ under CH. Erdős and Hajnal [106] ask if $MA_{\aleph_1} + 2^{\aleph_0} = \aleph_2$ implies $\omega_1{}^2 \to (\omega_1{}^2, 3)^2$? Erdő, Hajnal and Larson [108] asked for the cardinals λ that $\lambda^2 \to (\lambda^2, 3)^2$ holds. Hajnal [154] showed the relation failed at successors of regular cardinals under GCH. Baumgartner [15] showed that the relation failed at successors of singular cardinals under GCH.

(107)
Is it true that $\omega_3 \to (\omega_2 + 2)_\omega^3$?

Baumgartner, Hajnal and Todorčević [16] showed that GCH implies $\omega_3 \to (\omega_2 + \chi)_k^3$ for $\chi < \omega_1$ and $k < \omega$.

(108) *A problem on graphs of infinite chromatic number* ($250)
(proposed by Erdős, Hajnal and Szemerédi, 1982, [114, 82])
Let $f(n) \to \infty$ arbitrarily slowly. Is it true that there is a graph G of infinite chromatic number such that for every n, every subgraph of G of n vertices can be made bipartite by deleting at most $f(n)$ edges?

Prove or disprove the existence of a graph G of infinite chromatic number for which $f(n) = o(n^\epsilon)$ or $f(n) = o((\log n)^c)$.

Rödl [191] solved this problem for 3-uniform hypergraphs.

(109) *A problem on 4-chromatic subgraphs*
(proposed by Erdős, Hajnal [107])
Is it true that if G_1, G_2 are $\aleph_1$-chromatic graphs then they have a common 4-chromatic subgraph?

Erdős, Hajnal and Shelah [112] proved that any ω_1 chromatic graph contains all cycles C_k for $k > k_0$. Consequently, the above problem has an affirmative answer for 3-chromatic graphs.

(110) *A problem on the union of triangle-free graphs* ($250)
(proposed by Erdős and Hajnal [104])
Is there a graph G which contains no K_4 and which is not the union of $\aleph_0$ graphs which are triangle-free?

Shelah [202] proved that the existence of such a graph is consistent but it is not known if this is provable in ZFC.

(111) *A problem on $\aleph_1$-chromatic graphs*
(proposed by Erdős, Hajnal and Szemerédi [114])
Is it true that if $f(n)$ increases arbitrarily fast, then there is an $\aleph_1$-chromatic graph G so that if $g(n)$ is the smallest integer for which G has an n-chromatic subgraph of $g(n)$ vertices, then $f(n)/g(n) \to 0$?

(112) *A problem on odd cycles* (proposed by Erdős and Hajnal [87])
Let G be a graph of infinite chromatic number and let $n_1 < n_2 < \ldots$ be the sequence consisting of lengths of odd cycles in G. Is it true that

$$\sum \frac{1}{n_i} = \infty?$$

Gyárfás, Komlós and Szemerédi [151] proved that the set of all cycle lengths has positive upper density.

(113) *A problem on ordinal graphs and infinite paths*
(proposed by Erdős, Hajnal and Milner [109])

For which limit ordinals α is it true that if G is a graph whose vertices form a set of type α then either G has an infinite path or contains an independent set of type α. In other words, determine the limit α for which

$$\alpha \to (\alpha, \text{infinite path})^2.$$

Erdős, Hajnal and Milner [109] proved that the positive relation is true for all limit $\alpha < \omega^{\omega+2}$. Baumgartner and Larson [17] showed that if Jensen's Diamond Principle holds, then $\alpha \not\to (\alpha, \text{infinite path})^2$ for all α with $\omega_1^{\omega+2} \leq \alpha < \omega_2$. Larson [175] obtained further results under the assumption of GCH.

(114) *A problem on ordinal graphs and down-up matchings*
(proposed by Erdős and Larson [116])

A down-up matching in an ordinal graph is a matching of a set A with a set B where every element of A is less than all elements of B, denoted by $A < B$. Suppose for every graph on an ordinal α there is either an independent set of type β or a down-up matching from a A to a set B. If A has order type γ, then we write $\alpha \rightarrow (\beta, \gamma - \text{matching})^2$.

Suppose that j and k are positive integers with $k \geq 2$ and η is a limit ordinal. Is it true that $\omega^{\eta+jk} \rightarrow (\omega^{\eta+j}, \omega^k - \text{matching})^2$?

If j and $k \geq 2$ are positive integers and η a countable limit ordinal, then Erdős and Larson have shown that $\omega^{\eta+jk+1} \rightarrow (\omega^{\eta+j}, \gamma - \text{matching})^2$ but $\omega^{\eta+jk-1} \nrightarrow (\omega^{\eta+j}, \gamma - \text{matching})^2$.

Acknowledgement

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