

The following facts *might* be useful to solve the questions.

Def: A problem is NP-hard if every NP problem can be reduced to it in polynomial time.

Def: A problem is NP-complete if it is both NP and NP-hard

Theorem: If problem A is NP-hard and A can be reduced to B in polynomial time, then B is also NP-hard.

Independent Set:

Input: A graph $G=(V,E)$ and an integer k

Question: Does G have a set of nodes such that none of the nodes in the set are connected to each other and the size of this set is at least k ?

Clique:

Input: A graph $G=(V,E)$ and an integer k

Question: Does G have a set of nodes such that any two nodes in the set have an edge connecting them and the size of this set is at least k ?

Q1 [5 pts.]> In this question, you are going to show that NP is closed under union. To do this, you have to prove the following statement:

Let A and B be any problems in NP. $A \cup B$ is also in NP.

Sol'n: A and B are in NP, therefore we have two algorithms that check their solutions, i.e. whether a solution is indeed the solution for the problem or not, in polynomial time. Let's call these algorithms V_a and V_b respectively. To show $A \cup B$ is also in NP we have to come up with another algorithm that checks the solution for this new problem in polynomial time. The algorithm is as follows:

Run V_a on the solution

Run V_b on the solution

If any of them outputs yes then output yes

Otherwise output no.

So given a solution to be tested, the algorithm above checks it in polynomial time. Notice that if V_a/V_b says yes then the solution is valid for A/B therefore it is valid for $A \cup B$.

Q2 [5 pts.] > Using the fact that Independent Set problem is NP-complete, show that Clique problem is NP-complete.

Sol'n: If IS is NP-complete, it is NP-hard by definition. So, if we can reduce IS to Clique, then we can say that Clique is NP-hard as well. (by theorem) The reduction is as follows:

Given an instance A of the problem IS, we have to convert this instance to an instance B of Clique problem. Let A be consisting of graph $G=(V,E)$ and an integer k . So B will have another graph $G'=(V',E')$ and another integer k' . What is the relation between G,k and G',k' ? In other words, how is the reduction constructed? The restriction on this reduction is that, if G,k satisfies the requirement of IS, G',k' should satisfy the requirement of Clique. So if G has a set of nodes of size at least k that are not connected to each other, then G' should have a set of nodes of size at least k' , which are fully connected to each other. So let $G'=(G \text{ complemented})$ and $k'=k$. Complementing a graph means connecting all the nodes if they are not connected in the original graph and deleting the edges present in the original graph. So if two nodes have an edge between them in G , now they won't have in G' , otherwise, now they will. If G had a set of nodes that are not connected to each other, in G' those nodes will be fully connected so these nodes will form a clique. And since the size of independent set in G is at least k , we know that the size of the clique will be $k=k'$. Since the reduction is just complementing the graph, it can be done polynomial time. Therefore, **Clique is NP-hard**.

The second thing we have to do is show Clique is in NP. This is trivial because given a solution, in other words, a clique, we can check that all the nodes in this clique are connected to each other and the number of nodes are greater than or equal to the integer k . This can be done in polynomial time so **Clique is NP**.

Since Clique is NP and NP-hard, by definition, it is **NP-complete**.