1 Solution

Claim 1.1. Every 2-coloring of the edges of $K_n$ contains a monochromatic spanning tree.

Proof. Proof is done by induction on $n$. Base case $n = 1$ is trivial. Assume that the statement is true for $K_{n-1}$.

Consider an arbitrary 2-coloring of the edges of $K_n = G(V, E)$. Take any vertex $v \in V$. Denote $V' = V \setminus \{v\}$, $E_v = \{v, u\} \mid u \in V'$, and $E' = E \setminus E_v$. Consider subgraph $K_{n-1} = G(V', E')$. There are two cases:

- if all edges in $E_v$ have the same color then they form a monochromatic spanning tree;
- if there exist edges in $E_v$ of both colors then one of them matches the color of a monochromatic spanning tree in the subgraph $K_{n-1}$, and together they form a monochromatic spanning tree in the graph $K_n$.

\[\square\]

2 Solution

Suppose $G$ is a graph with vertex set $V$ that doesn’t contain $K_{r+1}$.

Claim 2.1. There is an $r$-partite graph $H$ with vertex set $V$ such that for every vertex $z \in V$

\[d_G(z) \leq d_H(z) \tag{\star}\]

where $d_G(z)$ denotes the degree of $z$ in $G$. 
Proof. Proof is done by induction on $r$. Base case $r = 1$ is trivial. Assume that for any graph free from $K_r$ there exists an $(r-1)$-partite graph satisfying the condition (*)

Take vertex $v \in V$ of the maximal degree and partite $V = N \sqcup U$, where $N$ is neighborhood of $v$. Subgraph $G'$ with $V(G') = N$ is $K_r$-free; otherwise subgraph $K_r$ in $G'$ together with $v$ would produce subgraph $K_{r+1}$ in $G$. By the assumption there exists an $(r-1)$-partite graph $H'$ that has vertex set $N$ and satisfies condition $d_{G'}(z) \leq d_{H'}(z)$ for all $z \in N$.

To build graph $H$, we add set $U$ to the graph $H'$ as a new part, and connect every vertex in $U$ with every vertex in $N$. Since $v$ has the maximal degree in $G$, the condition (*) is true for all vertices in $U$. And it is true for vertices in $N$ by the induction assumption. $\Box$

Claim 2.2. If $G$ is not a complete $r$-partite graph, then there is at least one vertex $z$ for which the inequality (*) is strict.

Proof. The claim is directly follows from the previous proof. If $G$ is not a complete $r$-partite graph, then either $N$ is not a complete $(r-1)$-partite graph, or some vertex $u \in N$ is not connected to all vertices in $U$. The first case implies (by induction) strict inequality for some vertex in $N$. In the second case the strict inequality holds for $z = u$. $\Box$

3 Solution

Claim 3.1. The chromatic number $\chi(G)$ of a graph $G$ satisfies

$$\chi(G) \geq \frac{n^2}{n^2 - 2e(G)}$$

where $n = |V(G)|$.

Proof. Let $r = \chi(G)$. By the definition of chromatic number the vertices of $G$ can be colored in $r$ colors such that no two adjacent vertices receive the same color. So $G$ is $r$-partite graph with monochromatic parts.

Turán’s theorem claims that Turán graph $T_{n,r}$ has the maximal number of edges among all $r$-partite graphs on $n$ vertices. Hence,

$$e(G) \leq e(T_{n,r}) \leq \binom{r}{2} \left( \frac{n}{r} \right)^2 = \frac{(r-1)n^2}{2r}.$$
Therefore,
\[ 2re(G) \leq rn^2 - n^2 \]
and
\[ r \geq \frac{n^2}{n^2 - 2e(G)}. \]

\[ \square \]

4 Solution

Claim 4.1. Let \( R_k(3) \) denote the least integer \( m \) such that every \( k \)-coloring of the edges of \( K_m \) contains a monochromatic triangle. Then
\[ R_k(3) \leq k(R_k(3) - 1) + 2. \]

Proof. Consider \( k \)-coloring of a complete graph on \( R_k(3) - 1 \) vertices that does not contain a monochromatic triangle. According to Pigeonhole Principle, for an arbitrary vertex \( v \) there exist at least \( \left\lceil \frac{R_k(3) - 2}{k} \right\rceil \) incident to \( v \) edges that have the same color. Denote by \( U \) a set of ends of these edges except \( v \). Note that edges between vertices in \( U \) are colored in \( k - 1 \) other colors; otherwise some pair of vertices together with \( v \) would form a monochromatic triangle. On the other hand, subgraph on vertices from \( U \) doesn’t contain a monochromatic triangle. Hence,
\[ \left\lceil \frac{R_k(3) - 2}{k} \right\rceil \leq |U| \leq R_{k-1}(3) - 1, \]
that implies
\[ R_k(3) \leq k(R_k(3) - 1) + 2. \]
\[ \square \]

Claim 4.2. \( R_k(3) \leq \lfloor ek! \rfloor + 1. \)

Proof. Note that \( R_1(3) = 3. \) Define a sequence
\[ a_0 = 2, \quad a_k = k(a_{k-1} - 1) + 2, \quad \text{for } k \geq 1. \]
Inequality from the previous claim implies by induction that $R_k(3) \leq a_k$ for all $k \geq 1$. Consider an exponential generating function

$$A(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k.$$ 

It satisfies an equation

$$A(x) = x(A(x) - e^x) + 2e^x.$$ 

Hence,

$$A(x) = e^x \left( 1 + \frac{1}{1-x} \right) = e^x \left( 1 + \sum_{k=0}^{\infty} x^k \right)$$

and

$$a_k = 1 + \sum_{i=0}^{k} \frac{k!}{i!} \leq 1 + k! \sum_{i=0}^{\infty} \frac{1}{i!} = 1 + k!e.$$ 

Since $a_k$ is integer,

$$R_k(3) \leq a_k \leq 1 + \lfloor k!e \rfloor.$$ 

\[ \square \]

5 Solution

Claim 5.1. Every 2-coloring of the edges of $K_{3n-1}$ contains $n$ independent edges of the same color.

Proof. Proof is done by induction on $n$. Base case $n = 1$ is trivial. Assume that every 2-coloring of the edges of $K_{3n-4}$ contains $n-1$ independent edges of the same color.

Consider an arbitrary 2-coloring of the edges of $K_{3n-1} = G(V,E)$. If all edges have the same color we are done. So suppose that there are edges of both colors. Since graph is complete there exist two adjacent edges, say, $\{u, v\}$ and $\{v, w\}$ of different colors. Consider a complete subgraph $K_{3n-4}$ on vertices $V \setminus \{u, v, w\}$. By the assumption it contains $n-1$ independent edges of the same color. Then either $\{u, v\}$, or $\{v, w\}$ has the same color and is independent from any edge in $K_{3n-4}$. So there are $n$ independent edges of the same color. \[ \square \]
Claim 5.2. There is a 2-coloring of the edges of $K_{3n-2}$ in which no set of $n$ independent edges is monochromatic.

Proof. Consider graph $K_{3n-2} = G(V, E)$. Partite set of vertices $V = U \sqcup W$ such that $|U| = 2n - 1$ and $|W| = n - 1$. Color blue all edges inside $U$, and color red all edges inside $W$ as well as edges between $U$ and $W$. This coloring has two properties:

- There are no $n$ independent blue edges. Each blue edge ends belong to $U$, so a set of $n$ independent blue edges requires $|U| \geq 2n$.

- There are no $n$ independent red edges. At least one end of each red edge belongs to $W$, so a set of $n$ independent red edges requires $|W| \geq n$.