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Eigenvalue bounds

1 Introduction and brief review

We continue studying the spectrum of the Laplacian, finding bounds for λ_1.

Recall that a random walk on an undirected graph G converges to its unique stationary distribution if and only if it is connected (so $\lambda_1 \neq 0$) and nonbipartite (so $\lambda_{n-1} \neq 2$).

Also recall that

$$\lambda_1 = \inf_{f\colon \sum_x f(x)d_x = 0} R(f) = \inf_{f\colon \sum_x f(x)d_x = 0} \frac{\sum_{x\sim y}(f(x)-f(y))^2}{\sum_x f^2(x)d_x} \tag{1}$$

2 Deriving a lower bound

Given a connected undirected graph G, define $d(u,v)$ to be the length of a shortest path between two vertices u and v, with $d(u,u)=0$. The diameter of a graph is then naturally defined as

$$D(G) = \max_{u,v} d(u,v).$$

One interpretation for diameter is thatHence each eigenvalue $\frac{2k}{n}$, for $0 \leq k \leq n$, has multiplicity $\binom{n}{k}$. it is the cost in the worst-case situation to send a message in the graph.

Now, suppose we have an optimal f for equation (1). Then, after possibly reversing the sign of f (which doesn't change the value of (1)), there is a maximal vertex x_0 such that $f(x_0) = \max_x |f(x)|$. Since f can't be uniformly nonnegative, there must also exist another vertex y_0 such that $f(y_0) < 0$, and a path P joining x_0 and y_0 containing $t \leq D$ edges.

Let $x_0 = v_0, v_1, \ldots, v_t = y_0$ (where $v_i \sim v_{i+1}$) denote the vertices of P. Then, since all components of the numerator sum in (1) are nonnegative, we can simply throw out all edges not contained in P and get

$$\lambda_1 \geq \frac{\sum_{(x,y)\in P}(f(x)-f(y))^2}{\sum_x f^2(x)d_x}.$$

Since we made $f(x_0)$ maximal, replacing all instances of $f^2(x)$ with $f^2(x_0)$ in the denominator will not decrease it, so

$$\lambda_1 \geq \frac{\sum_{i=0}^{t-1}(f(v_i)-f(v_{i+1}))^2}{f^2(x_0)\operatorname{vol}(G)}.$$

The Cauchy-Schwarz inequality states that $\sum_{i=1}^n a_i^2 \geq \frac{1}{n}(\sum_{i=1}^n a_i)^2$. Applying it to our equation, we get

$$\lambda_1 \geq \frac{\frac{1}{t}(\sum_{i=0}^{t-1} f(v_i)-f(v_{i+1}))^2}{f^2(x_0)\operatorname{vol}(G)}.$$

The sum in the numerator nicely telescopes, leaving

$$\lambda_1 \geq \frac{(f(x_0)-f(y_0))^2}{tf^2(x_0)\operatorname{vol}(G)} \geq \frac{f^2(x_0)}{tf^2(x_0)\operatorname{vol}(G)} \geq \frac{1}{D\times\operatorname{vol}(G)} \tag{2}$$

If G is disconnected then we define $D=\infty$. In this case $\lambda_1 = 0$ so the result still holds.

3 Upper bounds

Any function f satisfying $\sum_x f(x)d_x = 0$ will give an upper bound. In particular, if G is connected and is k-regular, then we can show

$$\lambda_1 \leq 1 - \frac{2\sqrt{k-1}}{k}(1-\frac{2}{D}) + \frac{2}{D} \tag{3}$$

Proof (from [1]): A weighted graph is a graph in which each edge uv has a numerical weight w_{uv}. If there isn't an edge between u and v, then $w_{uv} = 0$. An ordinary graph can be treated as a weighted graph where all edges that exist have weight 1; this is what we will do in our proof.

A *contraction* of G is formed by identifying two distinct vertices u and v into a single vertex v^*, and updating edge weights in the obvious manner:

$$\begin{aligned} w_{xv^*} &= w_{xu} + w_{xv} \\ w_{v^*v^*} &= w_{uu} + w_{vv} + 2w_{uv} \end{aligned}$$

Note that a contraction never decreases λ_1, since any eigenfunction f defined on the contracted graph yields the same Rayleigh quotient as the following function f' defined on G:

$$f'(x) = \begin{cases} f(v^*) & \text{if } x = u \text{ or } x = v \\ f(x) & \text{otherwise} \end{cases}$$

Now, let u and v denote two vertices at maximal distance D from each other, and for convenience define t to be the largest integer such that $D \geq 2t+2$. We contract G into a path H with $2t+2$ edges, with vertices $x_0, x_1, \ldots, x_t, z, y_t, \ldots, y_1, y_0$ such that vertices at distance $i \leq t$ from u are contracted to x_i, vertices at distance $j \leq t$ from v are contracted to y_j, and all remaining vertices are contracted to z. We then define f by

$$f(x_i) = a(k-1)^{-i/2} f(y_j) = b(k-1)^{-j/2} f(z) = 0$$

where a and b are chosen to satisfy the initial condition $\sum_x f(x)d_x = 0$.

Then after some simplifying the Rayleigh quotient satisfies

$$\frac{\sum_{u \sim v}(f(u) - f(v))^2 w_{uv}}{\sum_v f^2(v) d_v} \leq 1 - \frac{2\sqrt{k-1}}{k}(1 - \frac{1}{t+1}) + \frac{1}{t+1}$$

since it is maximized when $w_{x_i x_{i+1}} = w_{y_i y_{i+1}} = k(k-1)^{i-1}$.

This bound is good for *Ramanujan graphs* (which have applications in coding theory). These graphs are connected k-regular graphs with $\lambda_1 \leq 1 - \frac{2\sqrt{k-1}}{k}$; much research has been done on such graphs and numerical evidence indicates that most k-regular graphs are "almost" Ramanujan.

Note that (3) actually isn't a very good bound in general, since it assumes a regular graph. An open research problem is to discover a nontrivial upper bound for general graphs.

4 Convergence and lazy walks

We now look at a type of walk where λ_1 is uniquely important.

Recall from last Monday's lecture that convergence is dependent on the quantity

$$\rho = \max_{i \neq 0} |\rho_i| = \max\{|\rho_1|, |\rho_{n-1}|\}$$

which corresponds to $\min\{|\lambda_1|, |2 - \lambda_{n-1}|\}$; the larger this last value is, the faster the convergence. It turns out that, when λ_{n-1} is limiting convergence, we can sidestep that by changing our random walk into a *lazy walk* – thus, λ_1 is in a sense the only eigenvalue that matters.

Given an ordinary random walk defined by a set of edge weights w_{uv}, with $P(u,v) = \frac{w_{uv}}{d_u}$ and $d_u = \sum_v w_{uv}$, define its associated lazy walk by adding a $\frac{1}{1+c}$ chance of standing in place, so

$$\begin{aligned} w'_{uv} &= \begin{cases} w_{uv} & \text{if } u \neq v; \\ w_{vv} + cd_v & \text{if } u = v; \end{cases} \\ d'_v &= d_v + cd_v = (1+c)d_v, \\ P'(u,v) &= \frac{w'_{uv}}{d'_u}. \end{aligned}$$

Then $\lambda'_k = \frac{\lambda_k}{1+c}$. If we choose $c = \frac{\lambda_1 + \lambda_{n-1}}{2} - 1$, then $\lambda'_1 = 2 - \lambda'_{n-1}$, so we've neutralized λ_{n-1}. (Note, any $c > 0$ suffices to guarantee convergence when G is connected. This c was chosen to get λ_1 and λ_{n-1} close to 1 to speed up the rate of convergence.)

Since $\lambda_{n-1} < 2$ if $\lambda_1 > 0$, a lazy walk converges if and only if the graph is connected.

References

[1] F. Chung, "Lectures on Spectral Graph Theory", AMS, 1992.