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Cheeger Constant

Introduction

1 Introduction

In many areas of mathematics the questions of "best" comes into play. What is the best bound for a given constant? What is the best way of row reducing a certain matrix? In this section, we will describe a way to make the "best possible cut" of a graph $G = (V, E)$, where a cut may be either an edge-cut or a vertex-cut, and this cut will split G into two disconnected pieces.

2 Cheeger Constant

We would like a way to measure the quality of a cut that is made to G. That is, would it be better to cut 4 edges which cause us to lose 20 vertices, or is it better to cut 10 edges which would result in the removal of 120 vetices?

2.1 The Cheeger Ratio and The Cheeger Constant

Suppose we are given a graph $G = (V, E)$ and a subset $S \subseteq V$. We wish to define the folling two sets:

$$\partial S = \{\{u, v\} | u \in S, v \notin S\} \tag{1}$$

and

$$\delta S = \{v \notin S | v \sim u, u \in S\}. \tag{2}$$

Definition 1 For any vertex set W, the volume of W is given by

$$vol(W) = \sum_{x \; W} d_x, \tag{3}$$

where d_x is the degree of x in W.

Definition 2 The *Cheerger Ratio* for a vertex set S is

$$h(S) = \frac{|\partial S|}{min\{vol(S), vol(\overline{S})\}}, \tag{4}$$

where $\overline{S} = V - S$.

It is first worth noting that in terms of this defintion of the Cheeger ratio, we are gauging the quality of our cut by taking a measure of what's been cut off of G. There are other forms of the Cheeger ratio as well. For example, we can use $|\delta S|$ instead of $|\partial S|$, $|S|(or \overline{S})$ instead of $vol(S)$ $(or\ vol(\overline{S}))$, or $|S||\overline{S}|$ instead of $min\{vol(S), vol(\overline{S})\}$.

Definition 3 For any graph $G = (V, E)$, the *Cheeger Constant of G* is given by

$$h_G =^{min}_{S} h(S). \tag{5}$$

Now, if we consider the case where $vol(S) \leq \frac{1}{2}vol(G)$, then we can see that

$$|\partial S| \geq h_G(vol(S)).$$

3 The Cheeger Inequality

Given a graph G, we can define λ_1 to be the first nontrivial eignevalue of the Laplacian, $\mathcal{L}$, of G.

Theorem 1. *For any graph G,*

$$2h_G \geq \lambda_1 \geq \frac{h_G^2}{2} \tag{6}$$

Proof of Theorem 1. Suppose h_G is acheived by $h(A)$ such that $vol(A) \leq vol(\overline{A})$. Now

$$h(A) = \frac{\partial A}{min\{vol(A), vol(\overline{A})\}}.$$

From before, we know that

$$\lambda_1 =^{inf}_{\sum_x f(x) d_x = 0} \frac{\sum_{x \sim y}(f(x) - f(y))^2}{\sum_x f^2(x) d_x},$$

$$\chi(A) = \begin{cases} 1 & \text{if } A \text{ is true,} \\ 0 & \text{if } A \text{ is false,} \end{cases}$$

so define

$$g(x) = \begin{cases} \frac{1}{vol(A)} & \text{if } x \in A, \\ \frac{-1}{vol(\overline{A})} & \text{if } x \notin A. \end{cases}$$

Then,

$$\begin{aligned}
\lambda_1 &\leq R(g) \\
&= \frac{|\partial A|(\frac{1}{vol(A)} + \frac{1}{vol(\overline{A})})^2}{\frac{1}{vol(A)} + \frac{1}{vol(\overline{A})}} \\
&= |\partial A|(\frac{1}{vol(A)} + \frac{1}{vol(\overline{A})}) \\
&= |\partial A| \frac{vol(G)}{vol(A)vol(\overline{A})} \\
&\leq \frac{2|\partial A|}{min\{vol(A), vol(\overline{A})\}} = 2h_G.
\end{aligned}$$

□

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We will now prove the lower bound from the inequality stated in the previous Theorem, namely that $\lambda_1 \geq \frac{h_G^2}{2}$.

Proof. First we define,

$$V_+ = \{v : f(v) \geq 0\}$$
$$g(x) = \left\{ \begin{array}{ll} f(x) & x \in V_+ \\ 0 & \text{otherwise} \end{array} \right\}.$$

We also know that by the definition of eigenvaules we have $\forall v$,

$$\lambda f(v) d_v = \sum_{\substack{u \\ u \sim v}} f(v) - f(u)$$

Multiplying both sides of the above equation by $f(v)$ yields

$$\lambda f^2(v) d_v = f(v) \sum_{\substack{u \\ u \sim v}} f(v) - f(u)$$

Now divide both sides by $f^2(v)d_v$ and sum over all $v \in V_+$. Then,

$$\lambda = \frac{\sum_{v\in V_+} f(v)\sum_{u\sim v}(f(v)-f(u))}{\sum_{v\in V_+} f^2(v)d_v} \geq \frac{\sum_{v\in V} g(v)\sum_{u\sim v}(g(v)-g(u))}{\sum_{v\in V} g^2(v)d_v}$$

We will now multiply the top and bottom by $\sum_{u\sim}(g(u)+g(v))^2$ to get

$$\frac{\sum_{u\sim v}(g(u)-g(v))^2\sum_{u\sim v}(g(u)+g(v))^2}{\sum_{v\in V} g^2(v)d_v\sum_{u\sim v}(g(u)+g(v))^2} \tag{7}$$

$$\begin{aligned}
&> \frac{(\sum_{u\sim v} g^2(u)-g^2(v))^2}{\sum_{v\in V} g^2(v)d_v 2\sum_{u\sim v}(g^2(u)+g^2(v))^2} \\
&\geq \frac{(\sum_i (g^2(x_i)-g^2(x_{i+1}))|C_i|)^2}{2(\sum_{v\in V} g^2(v)d_v)^2} \\
&\geq \frac{(\sum_i (g^2(x_i)-g^2(x_{i+1}))\alpha\sum_{j\leq i} d_j)^2}{2(\sum_{v\in V} g^2(v)d_v)^2} \\
&\geq \frac{\alpha^2}{2}\left(\frac{\sum_i (g^2(x_i)-g^2(x_{i+1}))\sum_{j\leq i} d_j)}{\sum_{v\in V} g^2(v)d_v}\right)^2 \\
&= \frac{\alpha^2}{2} \geq \frac{h_G^2}{2}
\end{aligned}$$

□

Recall that $\alpha = \frac{\min |C_i|}{\min(\sum_{j\leq i} d_j, \sum_{j>i} d_j)}$ and so $|C_i| \geq \alpha\sum_{j\leq i} d_j$.

It turns out that you can get a better lower bound by applying a better estimate at a step (7). If you call the expression before you multiply the top and bottom by $\sum_{u\sim v}(g(u)+g(v))^2$, W, all the 2's in the above equations become $(2-W)$'s and arrive at the following.

$$\begin{aligned}
W &\geq \frac{\alpha^2}{2-W} \\
2W - W^2 &\geq \alpha^2 \\
0 &\geq W^2 - 2W + \alpha^2 \\
&= (W - \frac{2+\sqrt{4-4\alpha^2}}{2})(W - \frac{2-\sqrt{4-4\alpha^2}}{2}) \\
&= (W-(1+\sqrt{1-\alpha^2}))(W-(1-\sqrt{1-\alpha^2}))
\end{aligned} \tag{8}$$

So we arrive at the following bound $W \geq 1 - \sqrt{1-\alpha^2}$. Notice that the bound from the theorem is just the first term in the taylor expansion of $1 - \sqrt{1-\alpha^2}$.