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Problems on the Hypercube

There are many interesting problems on the hypercube. We will briefly discuss some of them here.

1 A Turán problem on the hypercube

Conjecture 1. *If G is a subgraph of Q_n (not necessarily induced), and G does not contain C_4 as a subgraph, then*

$$e(G) = \left(\frac{1}{2} + o(1)\right) e(Q_n),$$

where $e(G)$ is the number of edges in G.

This is known as a Turán-type problem. Using the language of such problems, we are interested in

$$t(C_4, Q_n) = \max_G \{e(G) : C_4 \nsubseteq G \subseteq Q_n\}.$$

In the reverse direction, it is not hard to see that it is possible for $e(g) \geq \frac{1}{2} e(Q_n)$. This follows from noting that each copy of C_4 in Q_n contains four vertices, two of which must have the same Hamming weight w, while the other two must have Hamming weights $w + 1$ and $w - 1$. If we take only those edges uv where $w(u) < w(v)$, $w(u)$ is odd, and $w(v)$ is even, then we have taken at least half the edges in the graph, and the resulting subgraph does not contain a C_4.

One can do better by taking every other level, as before, and adding a matching within the levels not taken.

2 Induced subgraphs of the hypercube

Let $S \subseteq V(Q_n)$ be a subset of vertices of the hypercube, and let $G = Q_n[S]$ be the induced subgraph on those vertices. Let $\Delta(G)$ be the maximum degree in G.

Theorem 1. *If $\Delta(n)$ is the minimum of $\Delta(G)$ for any graph $G = Q_n(S)$ with $|S| > 2^{n-1}$. Then,*

$$\sqrt{n} + 1 \geq \Delta(n) \geq \frac{1}{2} \log n - \frac{1}{2} \log \log n + \frac{1}{2}.$$

Here we will focus on the lower bound for $\Delta(n)$. The following Lemma, which we will use in proving the theorem, was reproved in [1].

Lemma 1. *If $G \subseteq Q_n$, and G has average degree $\bar{d}$, then $|V(G)| \geq 2^{\bar{d}}$.*

Theorem 2. *If $G = Q_n[S]$ with $|S| = 2^{n-1}$, and if G contains edges from all the n directions, then*

$$\Delta(G) \geq \frac{1}{2}\log n - \frac{1}{2}\log\log n + \frac{1}{2}.$$

We remark that Theorem 2 implies Theorem 1, and now proceed to prove Theorem 2.

Proof. For a vertex x in Q_n, let $x^{(i)}$ be the vertex that differs from x in the ith digit. Also, let

$$X_i = \{x \in S : x^{(i)} \in S\},$$

and

$$Y_i = \{x \notin S : x^{(i)} \notin S\}.$$

Notice that $|X_i| = |Y_i|$. (This follows by our choice of $|S| = Z^{n-1}$.)

Define $A_i = V(Q_n) - X_i - Y_i$.

Claim: For a vertex x with $x^{(i)} \in X_i$, x has at most $2\Delta(G) - 2$ neighbors in A_i.

Proof of Claim: Let x have s neighbors in A_i, which we denote $x^{(j_1)}, \ldots, x^{(j_s)}$. For each k, either $x^{(j_k)}$ or $(x^{(j_k)})^{(i)}$ belongs to S. Thus, $s \leq 2(\Delta(G) - 1)$.

This implies that every x in X_i has at least $(n - 2\Delta(G) + 1)$ neighbors in Y_i. Thus,

$$e(G(X_i \cup Y_j)) \geq \frac{1}{2}|X_i| + \frac{1}{2}|Y_i| + (n - 2\Delta(G) + 1)|X_i|,$$

which implies

$$\bar{d}(G(X_i \cup Y_i)) \geq n - 2\Delta(G) + 2.$$

Applying Lemma 1, we obtain

$$|X_i| \geq 2^{n-2\Delta(G)+1}.$$

Counting the degrees in $V(G)$ yields

$$\Delta(G)2^{n-1} \geq \sum_{x \in V(G)} \deg_G(x) \geq \sum_{i \in [1,n]} |X_i| = n2^{n-2\Delta(G)+1}.$$

Since $\Delta(G)2^{n-1} \geq n2^{n-2\Delta(G)+1}$, we have $\Delta(G)2^{2\Delta(G)} \geq 4n$, and an easy calculation shows that

$$\Delta(G) \geq \frac{1}{2}\log n - \frac{1}{2}\log\log n + \frac{1}{2}.$$

□

References

[1] F.R.K. Chung, Z. Füredi, R.L. Graham, and P. Seymour, “On induced subgraphs of the cube”, *Journal of Combinatorial Theory*, Series A. 49 (1988). 180-187