Math 261A Fall 2005
Notes for Wednesday, November 23rd

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Today we discussed convergence of lazy walks on strongly connected directed graphs. Recall that if G is such a graph on n vertices, then we denote by $0 = \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{n-1}$ the eigenvalues of the Laplacian of G.

We began by recalling a theorem from Chung, "Laplacians and the Cheeger inequality for directed graphs."

Theorem 1 *If G is a directed graph with eigenvalues $\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{n-1}$, then λ_1 satisfies*

$$2h_G \geq \lambda_1 \geq \frac{h_G^2}{2}$$

where h_G denotes the Cheeger constant of G.

Example 2 *If G is the n-cycle, then $\lambda_1 = 1 - \cos\frac{2\pi}{n} \sim \frac{2\pi^2}{n^2}$. Indeed, in this case we have $h_G = \frac{2}{n}$ and the theorem is verified.*

Example 3 *The transition probability matrix for the lazy walk on a random graph on n vertices is*

$$\mathcal{P} = \frac{I+P}{2}$$

where P is the transition probability matrix for the random walk. The Laplacian of $\mathcal{P}$ is

$$I - \frac{\Phi^{\frac{1}{2}}P\Phi^{\frac{-1}{2}}P^*\Phi^{\frac{1}{2}}}{2}$$

where Φ is as in the definition of the Laplacian of a directed graph.

We need the following theorem to prove our main result on convergence of random walks.

Theorem 4 *Let G be a strongly connected directed graph on n vertices with eigenvalues $\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{n-1}$. Let P denote the transition probability matrix of G and let $\mathcal{P} = \frac{I+P}{2}$ be a lazy random walk. Then the matrix $M = \Phi^{\frac{1}{2}}\mathcal{P}\Phi^{\frac{-1}{2}}$ satisfies*

$$\frac{||fM||^2}{||f||^2} \leq 1 - \frac{\lambda_1}{2}$$

for all vectors f satisfying $f\Phi^{\frac{1}{2}}\mathbf{1} = 0$, i.e., for all vectors f which are orthogonal to the eigenvector associated with $\lambda_0 = 0$.

Proof We have

$$\frac{||fM||^2}{||f||^2} = \frac{fMM^*f^*}{ff^*}$$

$$= \frac{f\Phi^{\frac{1}{2}}\mathcal{P}\Phi^{-1}\mathcal{P}^*\Phi^{\frac{1}{2}}f^*}{ff^*}$$

Replacing f by $g\Phi^{\frac{1}{2}}$ and $\mathcal{P}$ by $\frac{I+P}{2}$ gives

$$= \frac{g\Phi(I+P)\Phi^{-1}(I+P^*)\Phi g^*}{4g\Phi g^*}$$

$$= \frac{g(\Phi + \Phi P + P^*\Phi + \Phi P\Phi^{-1}P^*\Phi)g^*}{4g\Phi g^*}$$

$$= \frac{g(-2\Phi + \Phi P + P^*\Phi)g^* + g(3\Phi + \Phi P\Phi^{-1}P^*\Phi g^*)}{4g\Phi g^*}$$

$$\leq -\frac{R(g)}{2} + 1$$

$$\leq 1 - \frac{\lambda_1}{2}$$

by the definition of the Laplacian, where $R(g)$ denotes the Rayleigh quotient. ■

We can now prove the following theorem.

Theorem 5 *Let G be a strongly connected directed graph on n vertices with eigenvalues $\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{n-1}$. Then G has a lazy walk with rate of convergence of order $2\lambda_1^{-1}(-log\ min_x\phi(x))$. In other words, after at most $t \geq 2\lambda_1^{-1}((-log\ min_x\phi(x)) + 2c)$ steps, we have*

$$\Delta'(t) \leq e^{-c}$$

Proof Recall that the chi-squared distance $\Delta'(t)$ is given by

$$\Delta'(t) = max_y \sqrt{\sum_x \frac{|\mathcal{P}^t(y,x) - \phi(x)|^2}{\phi(x)}}$$

We write

$$\Delta'(t) = max_x||(\chi_y - \phi)\mathcal{P}^t\Phi^{\frac{1}{2}}||$$

$$= max_y||(\chi_y - \phi)\Phi^{\frac{-1}{2}}(\Phi^{\frac{1}{2}\mathcal{P}^t\Phi^{\frac{-1}{2}}})||$$

$$= max_y||g\Phi^{\frac{1}{2}}\mathcal{P}^t\Phi^{\frac{-1}{2}}||$$

$$= max_y||gM^t||$$

where $g = (\chi_y - \phi)\Phi^{\frac{-1}{2}}$ is orthogonal to $\phi^{\frac{1}{2}}$ and $M = \Phi^{\frac{1}{2}}\mathcal{P}\Phi^{\frac{-1}{2}}$. Now Theorem 4 gives

$$\Delta'(t)^2 = max_y||(\chi_y - \phi)\Phi^{\frac{-1}{2}}M^t||^2$$

$$\leq (1 - \frac{\lambda_1}{2})^t max_y||(\chi_y - \phi)\Phi^{\frac{-1}{2}}||^2$$

$$\leq (1 - \frac{\lambda_1}{2})^t max_y\phi(y)^{-1}$$

So if $t \geq 2\lambda_1^{-1}((-log(min_x\phi(x))) + 2c)$ then we have

$$\Delta'(t) \leq \frac{(1 - \frac{\lambda_1}{2})^{t/2}}{\sqrt{min_x\phi(x)}}$$

$$\leq e^{t\lambda_1/4 - \frac{1}{2}log(min_x\phi(x))}$$

$$\leq e^{-c}$$ ■

Remark. For undirected graphs, there is a lot of work involved in making the $-log(min_x\phi(x))$ term small (the term is roughly *log n*). With the log Sobolev method, we can get *log log n*. Not much work has been done in the directed case; even for regular graphs, this term could be exponential.