The isoperimetric problem on the hypercube

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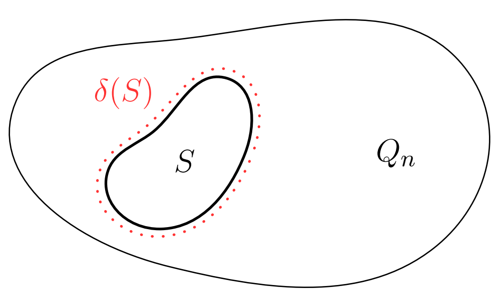
1 The isoperimetric problem

We will consider the n-dimensional hypercube Q_n. Recall that the hypercube Q_n is a graph whose vertices can be represented by binary words of length 1, i.e., $V = \{0, 1\}^n$. The edges of Q_n join vertices whose words differ in exactly one coordinate. Another interpretation is that the vertices are the possible subsets of an n-set, i.e., $V = \{s : s \subseteq [n]\}$, and edges join two subsets s, t if the symmetric difference of the two subsets, $(s \cap \bar{t}) \cup (\bar{s} \cap t)$, is a one element set.

The original isoperimetric problem deals with finding a figure in the Euclidean plane that will enclose a given area with minimal perimeter (in other words, of all figures with the same area, find the minimal perimeter). To generalize this to graphs we need to decide what to consider as the area and perimeter.

The area for the hypercube will intuitively be the number of vertices. (Note the hypercube is regular and we are leaning on that for our intuition, for nonregular graphs we might try to capture more vertices of higher degree and so then volume as defined in previous lectures for graphs would become a better choice.) For the perimeter (or boundary) we have two choices, we can use either edges or vertices.

The vertex boundary of a set S is $\delta(S) = \{v \notin S : v \sim u, u \in S\}$, the vertices which are adjacent to an element in S but are not in S. This is the minimum number of vertices that must be removed to separate S from the other vertices. The edge boundary is similarly defined as all edges which connect a vertex in S to a vertex not in S and is the minimum number of edges whose removal disconnects S from the other vertices of the graph.



It is important to note that the vertex boundary and edge boundary have very different behaviors and so we must analyze them separately. As an example if we want to cut Q_n in half then the optimal way to do it using an edge boundary is to split the vertices into two subcubes and then remove all edges between the cubes. On the other hand if we wanted to split the cube in half using vertex boundaries then the optimal thing is to use Hamming balls (which we define below).

1.1 Hamming balls

Since we have a notion of distance on a graph (i.e., the length of the shortest path joining the two vertices), then the notion of a ball generalizes easily. Since the ball solved the isoperimetric problem in Euclidean space it is reasonable to expect that it might again solve the problem in graphs. So let $B_r(v) = \{u : d(u,v) \leq r\}$ denote a ball of radius r centered at v in our graph, in Q_n these are called Hamming balls. For the hypercube it is easy to calculate the number of vertices contained in such a ball. Namely we have

$$b_r = |B_r(v)| = \binom{n}{r} + \binom{n}{r-1} + \cdots + \binom{n}{0}.$$

[Here we can exploit the high symmetry of the hypercube and note that the behavior at one point is the same as any other point by an easy automorphism, in this case the volume of the ball is independent of v.]

The volume equation follows by recalling that every vertex can be represented by a binary word of length n and it is easy to show that the distance between two vertices in Q_n is the number of entries in their index which disagree. So in the above calculation $\binom{n}{r}$ is the number of vertices distance r away from v (i.e., we choose r out of n entries at which their words disagree), and in general $\binom{n}{k}$ is the number of vertices at distance k away from v.

1.2 The isoperimetric problem defined

We will answer the following question: Given $m > 0$ find

$$f(m) = \min_{S \subseteq V} \{|\delta(S)| : |S| = m, S \subseteq V(Q_n)\}.$$

In other words, what is the smallest number of vertices needed to separate m vertives from the rest of the graph. We want to determine both $f(m)$ and the sets S which achieve this minimum value. While in general this is difficult to calculate for a graph, for the hypercube the results have long been known.

When $m = b_r$, a good guess would be that the set S is a Hamming ball and that $f(m) = \binom{n}{r+1}$ (the next layer of vertices from the center vertex). We will show that is the case and that in general that the sets S are near Hamming balls.

2 Main results

We have two ways to state the main results, the first will be a rough estimate but is useful. The second will be a precise estimate but is somewhat messy.

Theorem 1. *Suppose* $m = b_r + \binom{x}{n-r-1} < b_{r+1}$ *then* $f(m) \geq b_{r+1} - m + \binom{x}{n-r-2}$.

In the statement of the theorem the x in $\binom{x}{k}$ need not be an integer. In general we have for x real and $k \geq 0$ that

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}.$$

We must be careful in using these "binomial" coefficients as some properties no longer hold in this more general setting, for instance $\binom{x}{k} \neq \binom{x}{x-k}$ in general.

Theorem 2. *Suppose*

$$m = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{k+1} + s, \qquad 0 \leq s < \binom{n}{k}.$$

Let $a_k > a_{k-1} > \cdots > a_t \geq t \geq 1$ *be as large as possible and satisfying*

$$s = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_t}{t}.$$

Then $f(m) = b_{n-k} - m + \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_t}{t-1}$.

[Note: the expression for $f(m)$ is very suggestive about what to include in the boundary.]

Our first step will be to show that the right object to study are the near Hamming balls (these are Hamming balls which are possibly missing some vertices in the last level). In particular, the above results will then follow.

Theorem 3. *Given* m*, then* $f(m) = \min\{|\delta(S)| : |S| = m,\ S \text{ a near Hamming ball}\}$.

2.1 Pushing Hamming balls

To show Theorem 3 the idea will be to push any set into a Hamming ball without increasing the size of the vertex boundary. In particular we need the following result.

Theorem 4. *Given* $A, B \subseteq V(Q_n)$*, with* $d(A,B) = \min\{d(a,b) : a \in A, b \in B\}$ *being the minimal distance between the sets. Then there are two antipodal near Hamming balls* A', B' *with* $|A'| = |A|$*,* $|B'| = |B|$ *and* $d(A', B') \geq d(A, B)$.

Antipodal means that the center of the two near Hamming balls are at vertices whose indices are complements of each other. The vague idea behind this is that if we have two sets we push one to the top and one to the bottom and our distance can only increase.

We now show that Theorem 3 follows from Theorem 4. Suppose that we have a set A with $|A| = m$, $|\delta(A)| = f(m)$. Then let $B = V - A - \delta(A)$, note that $d(A, B) = 2$ (i.e., the shortest path is go from a vertex in A adjacent to the boundary to the boundary to a vertex in B). By Theorem 4 there exists sets A', B' which are near-Hamming balls where $|A'| = |A| = m$ and $|B'| = |B|$ with $d(A'B') \geq d(A, B) = 2$. It suffices to show that the set A' has $|\delta(A')| \leq f(m)$. Note that points in $\delta(A')$ cannot be in A' and cannot be in B' (since A' and B' are at least distance 2 apart) and so

$$|\delta(A')| \leq 2^n - |A'| - |B'| = 2^n - |A| - |B| = |\delta(A)| = f(m),$$

and the result follows.

Proof of Theorem 4. We first consider the following sets

$$\begin{aligned} S_1 &= \{(a, a') : a \in A,\ a' \notin A,\ \text{and } |a| < |a'|\}, \\ S_2 &= \{(b, b') : b \in B,\ b' \notin B,\ \text{and } |b| > |b'|\}. \end{aligned}$$

By $|a|$ we mean either the number of 1's in the binary word indexing the vertex, or the size of the subset indexing the vertex (depending on which interpretation we use). Intuitively we are trying to push A up to be centered around the all 1s vertex (respectively $[n]$) and pushing B down to be centered around the antipodal point of the all 0s vertex (respectively $\emptyset$). If $S_1 = \emptyset$ then A is a near Hamming ball centered at the top and if $S_2 = \emptyset$ then B is a near Hamming ball centered at the bottom.

If both S_1 and S_2 are empty then we are done. If one of them is nonempty then we can push, we now assume that we are in the case where one of the sets is nonempty.

Without loss of generality let us suppose that $(a, a') \in S$ and (a, a') has minimum symmetric difference, i.e., $|(a - a') \cup (a' - a)|$ is minimized over all elements in S_1 and S_2. Let $x = a - a'$ and $y = a' - a$ where $|x| < |y|$. We define two operators $\mathcal{U}$ for $a \in A$ and $\mathcal{D}$ for $b \in B$ (these are the "up" and "down" operators respectively). For the following we will think of the subset convention of denoting vertices.

$$\mathcal{U}(a) = \begin{cases} (a \cup y) - x & \text{if } x \subseteq a,\ y \cap a = \emptyset,\ (a \cup y) - x \notin A; \\ a & \text{otherwise.} \end{cases}$$

$$\mathcal{D}(b) = \begin{cases} (b \cup x) - y & \text{if } y \subseteq b,\ b \cap x = \emptyset,\ (b \cup x) - y \notin B; \\ b & \text{otherwise.} \end{cases}$$

What is happening is that for $a \in A$ we (if possible) cut out the small set x and replace it by the larger set y moving that element up (we know that this is possible for one such element and so we will reduce the size of S_1). Similarly for $b \in B$ we (if possible) cut out the large set y and replace it by the smaller set x moving that element down.

Because we do not move an element if the new element already exists in the set it is easy to see that $|\mathcal{U}(A)| = |A|$ and $|\mathcal{D}(B)| = |B|$. Since we also have reduced the size of S_1 we can repeat this procedure only finitely many times until $S_1 = \emptyset$ and $S_2 = \emptyset$ at which point we will have antipodal Hamming balls.

The only remaining step is to check that for $a \in A$, $b \in B$ with $a' = \mathcal{U}(a)$ and $b' = \mathcal{D}(b)$ that

$$d(a', b') \geq d(a'', b'') \qquad \text{for some } a'' \in A, b'' \in B.$$

[Note: in general we need not have $a = a''$ and $b = b''$.] $\square$

Exercise 1. Complete the proof of Theorem 4. (Hint: there are a few cases to consider, but all of the key ideas are already given above.)