Some boundary conditions

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1 Analogs of methods from spectral geometry

In spectral geometry we study a space, usually a plane or more generally a Riemannian manifold. (A Riemannian manifold is a topological object where locally around every point it behaves like d-dimensional space for some d.) Instead of studying the whole structure we instead focus on pieces of it and how to fit pieces together.

For us we are looking at big graphs (some of them very big, the Internet graph for instance). In studying such an object it is much easier and more practical to study a small part. So we imagine that we have some subset S of the vertices of our graph G and we want to know what is happening on S. An important piece of information is what is happening on the boundary of S.

As we have seen in previous lectures there are two types of boundaries, namely edge boundaries $\partial(S)$ and vertex boundaries $\delta(S)$. We will be focusing on the vertex boundary and give natural conditions for functions $f : V \to \mathbb{R}$ that are of interest for random walks on S.

1.1 Dirichlet boundary condition

One approach to the random walks on S is that when it hits the boundary the walk is absorbed (in practical terms that means we restart the walk). Another way of thinking of this is that the boundary is acting as a sink for the random walk. This translates into the function $f : V \to \mathbb{R}$ vanishing on the boundary S (i.e., for all $v \in \delta(S)$ we have $f(v) = 0$).

This condition is motivated by our desire to use the Laplacian operator on S. Let L denote the combinatorial Laplacian of G and L_S denote the L restricted to the vertices of S. (Note that L_S is not the same as the combinatorial Laplacian of the induced subgraph on S, the reason for this is that we are accounting for edges which lie in the edge boundary when computing vertex degrees.) Using our definition of L_S

we have

$$\langle f, L_S f\rangle \;=\; \sum_{x\in S} f(x)\sum_{y\sim x}\big(f(x)-f(y)\big). \tag{$*$}$$

We still want this to become $\sum\big(f(x)-f(y)\big)^2$. For edges $x{\sim}y$ with x and y both in S then the right hand sum in $(*)$ will have terms

$$f(x)\big(f(x)-f(y)\big)+f(y)\big(f(y)-f(x)\big) \;=\; \big(f(x)-f(y)\big)^2,$$

as needed. But for edges $x{\sim}y$ in the edge-boundary (say $x\in S$ and $y\notin S$) then the right hand sum in $(*)$ will have $f(x)\big(f(x)-f(y)\big)$ but will miss the other term. The easiest way to guarantee that this will be $\big(f(x)-f(y)\big)^2$ is if we set $f(y)=0$, and hence our condition.

1.2 Dirichlet eigenvalues

The eigenvalues of L_S are naturally known as the Dirichlet eigenvalues. We should note that while the smallest Laplacian eigenvalue is always 0 this may no longer hold for the Dirichlet eigenvalues (though they will always be nonnegative). We can use the Dirichlet eigenvalues to get good control of the local cuts. More information about this can be found in [1].

1.3 Neumann boundary condition

The other natural approach for the random walks on S is that when it hits the boundary the walk reflects, in other words if we exit S the next step is to return to S. In this case the condition translates into having for all vertices $x\in\delta(S)$

$$\sum_{\substack{y\in S\\ y\sim x}}\big(f(x)-f(y)\big) \;=\; 0,$$

another way to think of it is that $f(x)$ is the average of f on the neighbors in S of x. The derivation of this boundary condition is similar to that of the Dirichlet boundary condition, except now instead of focusing on individual terms we sum up over all terms.

References

[1] F. Chung, "Random walks and cuts in directed graphs", preprint.