On the cover polynomial of a digraph

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1. Introduction

There are many polynomials which can be associated with a graph G, the most well known perhaps being the Tutte polynomial $T(G; x, y)$ (*cf.* [B74] or [T54]). In particular, for specific values of x and y, $T(G; x, y)$ enumerates various features of G. For example, $T(G; 1, 1)$ is just the number of spanning trees of G, $T(G; 2, 0)$ is the number of acyclic orientations of G, $T(G; 1, 2)$ is the number of connected subgraphs of G, etc. (see [JVW90] for details).

However, for directed graphs, no analogue of the Tutte polynomial is known. In this paper we introduce the *cover polynomial* $C(D; x, y)$ for a directed graph D and examine its relationships to other graph polynomials. While the cover polynomial is not exactly the directed analogue of the Tutte polynomial, it does have a number of properties which are comparable to those of $T(G; x, y)$.

2. The cover polynomial

Let $D = (V, E)$ be a directed graph, or *digraph*, for short. That is, V is some (finite) set of *vertices* of D, and $E \subseteq V \times V$ is the set of *edges* of D. For an edge $e = uv \in E$, $u \neq v$, denote by $D \setminus e$ the digraph with vertex set V, and edge set $E \setminus \{e\}$; this we call a *deletion* of D. Similarly, we define the *contraction* D/e as the digraph with vertex set obtained by replacing two vertices u and v in V by a single new vertex w, and with edge set formed from E by removing exactly those edges of the form ux or yv. We illustrate these operations in Figure 1.

In the case that $e = uu$ (usually called a *loop*), the corresponding operations of deletion and contraction are shown in Figure 2. In $D \setminus e$, the vertex u has been removed. Note that in Figure 1, if $vu \in E$ then D/e will have the loop ww in its edge set.

We now introduce the basic object of study in this paper, the *cover polynomial* $C(D) = C(D; x, y)$ of D. The cover polynomial of D is a polynomial in two indeterminates x and y with integer coefficients, and is defined recursively as follows:

(i) For I_n, the digraph with n (independent) vertices and no edges,

$$C(I_n) = x^{\underline{n}} := x(x-1)\ldots(x-n+1)\ ; \tag{1}$$

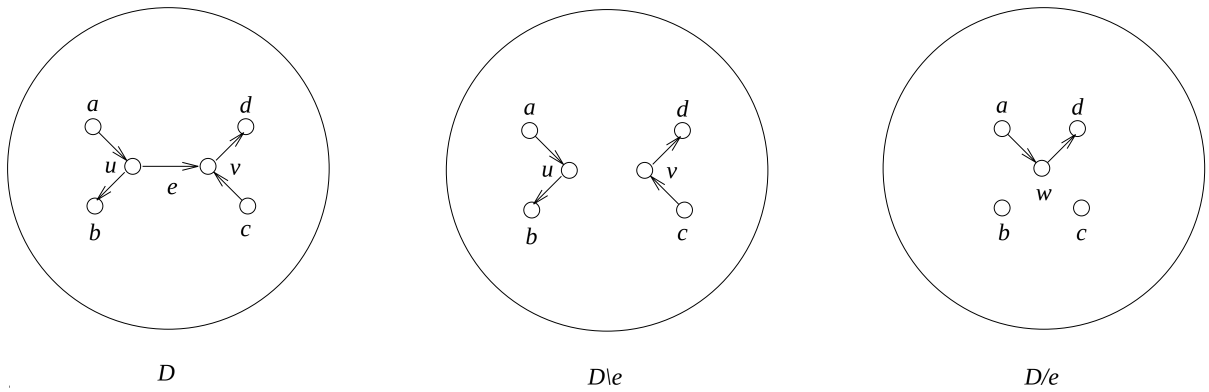


Figure 1: Deleting and contracting a non-loop edge.

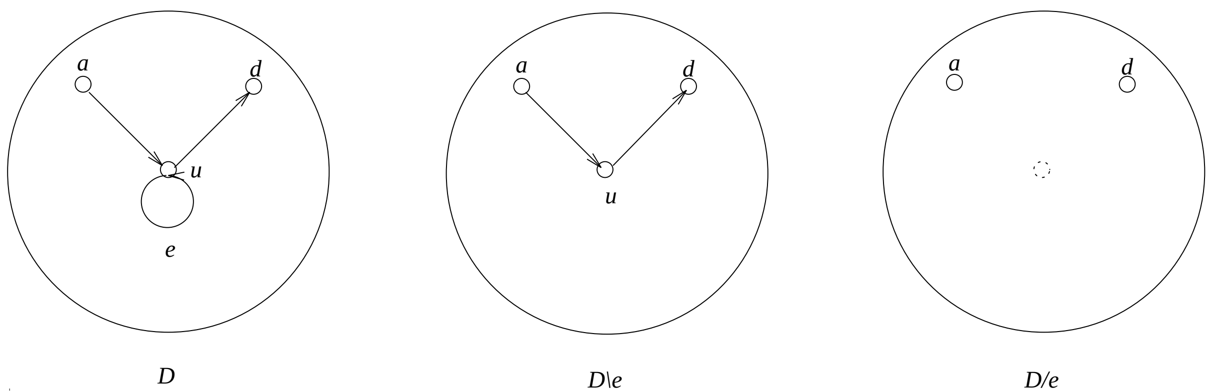


Figure 2: Deleting and contracting a loop.

For the special case of $n = 0$, the corresponding digraph D_ϕ having no vertices or edges has cover polynomial

$$C(D_\phi) = 1 \; ; \tag{2}$$

(ii) If e is an edge of D which is not a loop then

$$C(D) = C(D \setminus e) + C(D/e) \; ; \tag{3}$$

(iii) If e is a loop of D then

$$C(D) := C(D \setminus e) + y\, C(D/e) \; .$$

Of course, it is not clear at this point that $C(D)$ is even well-defined, i.e., independent of the order that various edges are chosen in the recursive procedure. We will show that $C(D)$ is an invariant of D in the next section. In Figure 3, we tabulate $C(D)$ for some small digraphs D.

3. Basic properties of the cover polynomial

Let us write $C(D; x, y)$ in the following "falling factorial" form:

$$C(D; x, y) = \sum_{i,j} c_D(i,j) x^{\underline{i}} y^j$$

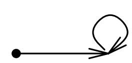
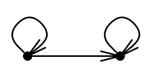
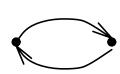
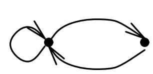
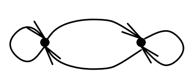
D	$C(D;x,y)$
$D_\emptyset$	1
	x
	$x+y$
	x^2-x
	x^2+xy-x
	x^2
	x^2+xy
	x^2+xy
	$x^2+2xy+y^2-x$
	$x^2+2xy+y^2$
	x^2+x+y
	$x^2+xy+x+y$
	$x^2+2xy+y^2+x+y$

Figure 3: Cover polynomials for small digraphs.

where, as usual, $x^{\underline{i}} := x(x-1)\dots(x-i+1)$ and $x^{\underline{0}} = 1$. In general, $c_D(i,j)$ will be taken to be 0 when it is not defined, e.g., when $i < 0$ or $j < 0$.

Theorem 1. *$c_D(i,j)$ is the number of ways of (disjointly) covering all the vertices of D with i directed paths and j directed cycles.*

Note that a (directed) path can consist of a single vertex, and a cycle can consist of a single loop. For example, for the next to the last digraph shown in Figure 3, we have

$$c_D(1,0) = 2, \quad c_D(2,0) = 1, \quad c_D(0,1) = 1, \quad c_D(1,1) = 1$$

and

$$\sum_{i,j} c_D(i,j) x^{\underline{i}} y^j = 2x^{\underline{1}} + x^{\underline{2}} + y + x^{\underline{1}} y = x^2 + xy + x + y \ .$$

In general, unless otherwise specified, summation indices range over all integers.

Proof: The proof will proceed by double induction on the number n of vertices of D, and then on the number m of edges of D. The theorem clearly holds for any digraph having no edges (by (i)). Assume it holds for all D with fewer than n vertices, and for all D with n vertices and fewer than m edges, for some fixed $m > 0$, $n > 0$. We will use the recurrence formulas (ii) and (iii). Let e be an edge of D. The set of path/cycle covers of D can be partitioned into those which actually use the edge e in a path or cycle, and those which do not use e. It is clear that $c_{D\backslash e}(i,j)$ counts the number of covers of D by i paths and j cycles which do not use e.

Now, if e is not a loop then $c_{D/e}(i,j)$ counts the number of covers of D by i paths and j cycles which use e (just insert e into the appropriate path or cycle covering the contracted vertex w). In this case the induction step follows by (ii).

On the other hand, if e is a loop then the number of covers of D by i paths and j cycles is just $c_{D/e}(i,j-1)$. Namely, each cover of D/e by i paths and $j-1$ cycles augmented by the loop (= cycle) e is such a cover of D. In this case the induction step follows by (iii). This proves Theorem 1. ■

In particular, this shows that $C(D)$ is an invariant of D, and so, is well-defined.

For fixed positive integers λ and μ, we can assign to each cover of D by paths and cycles certain colorings of the vertices of D by $\lambda + \mu$ colors as follows: (i) any two vertices in the same path or cycle have the same color; (ii) vertices in different paths have different colors; (iii) vertices in paths have colors from a set of λ colors; (iv) vertices in cycles have colors from a disjoint set of μ colors.

Let us call such an assignment a (λ,μ)-coloring of D.

Since each of the $c_D(i,j)$ covers of D by i paths and j cycles generates $\lambda^{\underline{i}}, \mu^j$ such (λ,μ)-colorings then (as noted by Richard Stanley [S92]) we have:

Corollary 1. *$C(D;\lambda,\mu)$ is the number of (λ,μ)-colorings of D.*

Corollary 2. *Suppose $D = (V, E)$ is formed by joining the disjoint digraphs $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ with all the edges v_1v_2, $v_1 \in V_1$, $v_2 \in V_2$. Then*

$$C(D) = C(D_1)C(D_2) \ . \tag{4}$$

Proof: For each $\lambda, \mu > 0$, each pair of valid (λ, μ)-colorings of D_1 and D_2 can be extended to a unique (λ, μ)-coloring of D (since any two paths in D_1 and D_2 are joined on D to form a single path). Conversely, each (λ, μ)-coloring of D generates unique (λ, μ)-colorings of D_1 and D_2. Since this is true for all choices of λ and μ then this implies the polynomial identity (4). ■

Note that this product formula is analogous to the corresponding result for the Tutte polynomial

$$T(G) = T(G_1)T(G_2)$$

which holds whenever G is the disjoint union of two graphs G_1 and G_2.

Corollary 3. *$C(D)$ is a polynomial in x only (i.e., y is absent) if and only if D is acyclic.*

Corollary 4. *Let $D^{(r)}$ be formed by adding r independent vertices to D. Then*

$$C(D^{(r)}; x, y) = x^{\underline{r}} C(D; x - r, y) \ .$$

Proof: Each of the added vertices must be covered by a unique (trivial) path.

Corollary 5. *Let $D^{[s]}$ be formed by adding s disjoint independent loops to D. Then*

$$C(D^{[s]}; x, y) = \sum_{k=0}^{s} x^{\underline{k}} y^{s-k} C(D; x - k, y) \ .$$

Proof: Follows by induction, after computing what happens when a single loop is added.

Corollary 6. *Let $\widehat{D}$ be formed from D by* reversing *all the edges of D (i.e., uv is an edge of $\widehat{D}$ if and only if vu is an edge of D). Then*

$$C(\widehat{D}) = C(D) \ .$$

Proof: Just observe that $c_D(i, j)$ and $c_{\widehat{D}}(i, j)$ are equal for all i and j.

4. Non-attacking rooks, bipartite matchings and path covers

In this section we point out connections between the numbers $c_D(i, j)$ and several other well-studied quantities in combinatorics.

By a *board* B we mean an n by n array of cells on which certain cells have been designated as forbidden. An *arrangement* of i *non-attacking rooks* on B corresponds to a placement of i (chess) rooks on non-forbidden cells so that no rook attacks any other rook, i.e., so that no two rooks lie on the same row or column, (e.g., see [R58]). Let $r_B(i)$ denote the number of possible such arrangements.

Of course, B can be specified by a matrix $M = M(B)$ where $M(u,v) = 0$ if the (u,v)-cell of B is forbidden, and $M(u,v) = 1$ otherwise (e.g., see [LP86]). With M we can also associate a bipartite graph G on the set of vertices $\{1,2,\dots,n\}$ and $\{1',2',\dots,n'\}$ by setting $\{u,v'\}$ to be an *edge* of G if and only if $M(u,v) = 1$. In this way, an arrangement of i non-attacking rooks on B is equivalent to a *matching* in G, a selection of i disjoint edges. Thus, if we let $m_G(i)$ denote the number of such matchings, then $r_B(i) = m_G(i)$.

Finally, with M we can also associate a digraph D on $\{1,2,\dots,n\}$ by defining uv to be an edge of D if and only if $M(u,v) = 1$. Let $c_D(i)$ denote the number of ways of covering the vertices of D with cycles and exactly i (directed) paths. Thus,

$$c_D(i) = \sum_j c_D(i,j) \ .$$

Fact 1. For all i,

$$c_D(n-i) = r_B(i) \ . \tag{5}$$

Proof: Any arrangement of i non-attacking rooks on B corresponds to selections of i edges on D which form some number of (directed) paths and cycles, say r paths and s cycles. Hence, if t is the total number of vertices covered by these i edges then $t = i + r$. Let us cover the remaining $n - t$ vertices with trivial paths, each consisting of a single vertex. Since none of the r paths above are of this type, then this is a bijection between arrangements of i rooks on B, and coverings of the vertices of D by cycles and $r + n - t = n - i$ paths. Thus,

$$c_D(n-i) = r_B(i)$$

as claimed. ■

This relationship allows us to connect various properties of $C(D;x,1) = \sum_i c_D(i)x^{\underline{i}}$ with corresponding properties of rook polynomials and matching polynomials, both of which have a substantial literature (e.g., see [R58], [LP86], [GJW78], [F88]). In fact, $C(D;x,1)$ is just the "n-factorial rook polynomial" introduced by Goldman, Joichi and White in [GJW78], who also prove Corollary 2 for the case $y = 1$.

5. Drop polynomials and chromatic polynomials

Given a digraph $D = (V,E)$ with $|V| = n$, denote by $Sym(D)$ the set of all $n!$ bijections $\pi : V \to V$. If $u\pi(u) \in E$, we say that π has a *drop* at u. Denote by $\delta_D(k)$ the number of $\pi \in Sym(D)$ having k drops. A basic result from the theory of rook polynomials asserts the following (*cf.* [S86]).

For all j

$$\sum_k \delta_D(k)\binom{k}{j} = r_B(j)(n-j)! \tag{6}$$

where of course B is the board corresponding to D. This follows at once by interpreting the RHS of (6) as placing j special non-attacking rooks on B, and then $n - j$ other non-attacking rooks in the other rows and columns of B, possibly in forbidden cells. The LHS then counts the same thing, by choosing for each $\pi \in Sym(D)$ having k drops, j of the drops to be special (corresponding to the special rooks). Consequently,

$$\sum_k \delta_D(k) \binom{k}{j} \frac{x^{\underline{n-j}}}{(n-j)!} = r_B(j) x^{\underline{n-j}}$$

which implies

$$\Delta(D;x) := \sum_j \sum_k \delta_D(k) \binom{k}{j} \binom{x}{n-j} = \sum_k \delta_D(k) \binom{x+k}{n} = \sum_j c_D(n-j) x^{\underline{n-j}} = C(D;x,1) \quad (7)$$

where we have used (5) and Vandermonde convolution (see [GKP89]). The polynomial $\Delta(D;x)$, which we call the *binomial drop polynomial* for D, has a number of interesting properties. One of these is the following. Suppose $D = (V,E)$ is acyclic. Then by Corollary 1, for any positive integer λ, $C(D;\lambda,1)$ is just the number of ways of λ-coloring the vertices of D so that all the points of each color form a path (since any set of vertices can be arranged into a path in at most one way). Further, if D is transitive, i.e., $uv \in E$ and $vw \in E$ imply $uw \in E$,then a set S of vertices can be assigned the same color (i.e., arranged into a path) if and only if for all $x, y \in S$, either $xy \in E$ or $yx \in E$. To each such D can be associated a *poset* $P = (V, \prec)$ in the natural way, namely, $x \prec y$ on $P \Leftrightarrow xy \in E$. Finally, let $inc(P)$ denote the *incomparability graph* associated to P, so that G has vertex set V, and $\{x,y\}$ is an edge of $G \Leftrightarrow x$ and y are incomparable in P (i.e., neither $x \prec y$ nor $y \prec x$). It then follows that $C(D;\lambda,1)$ just counts the number of valid λ-colorings of $inc(P)$, that is, maps of V into $\{1,2,\ldots,\lambda\}$ so that adjacent vertices in $inc(P)$ are assigned distinct values. Of course, the number of such colorings of $inc(P)$ is exactly given by $\chi(inc(P);\lambda)$, the value of the *chromatic polynomial* of $inc(P)$ evaluated at λ. The chromatic polynomial $\chi(G;x)$ of a graph is another graph polynomial which has been treated extensively in the literature (e.g., see [B74], [T84], [S86]).

By (7) we have:

Corollary 7. *For any poset P with n elements,*

$$\Delta(P;x) = \sum_k \delta_p(k) \binom{x+k}{n} = \chi(inc(P);x) \qquad (8)$$

where $\delta_p(k)$ is interpreted in the obvious way.

This result first appeared (to the best of the authors' knowledge) in Goldman, Joichi and White [GJW78] with later proofs given in [BG93] and [St91].

In view of (8), it is natural to ask whether for the corresponding expansion

$$\chi(G;x) = \sum_k b_G(k) \binom{x+k}{n}$$

for an arbitrary graph G with n vertices, the coefficients $b_G(k)$ have any natural interpretation. It is not even obvious that the $b_G(k)$ are integers or non-negative, for example. The integrality of the $b_G(k)$

was first observed by Vo in 1981 (see [V87]). The fact that they are non-negative follows from a result of Linial [L86] who showed that if we write

$$F(G;x) := \sum_{k=1}^{\infty} \chi(G;k)x^k$$

then

$$F(G;x) = \frac{Q(x)}{(1-x)^{n+1}}$$

for some polynomial $Q(x)$ having non-negative integer coefficients and degree n. In fact, the $b_G(k)$ do have an interpretation as enumerating certain invariants of G which also implies they are always non-negative integers. This was first shown by Gansner and Vo ([GV87]; see also [V87]) and independently by Brenti [Br92]. We describe a different interpretation here.

Let G be a graph on the vertex set $V = \{1, 2, \ldots, n\}$. For a permutation $\pi : V \to V$, and $i \in V$, define the *rank* $\rho(\pi(i))$ of $\pi(i)$ to be the largest integer r so that there are values $i_1 < i_2 < \cdots < i_r = i$ with all pairs $\{\pi(i_j), \pi(i_{j+1})\}$ being edges of G. We say that π has a *G-descent* at $\pi(i)$ if either

(i) $\rho(\pi(i)) > \rho(\pi(i+1))$, or

(ii) $\rho(\pi(i)) = \rho(\pi(i+1))$ and $\pi(i) > \pi(i+1)$.

Finally, let $\partial_G(k)$ denote the number of π having exactly k G-descents.

Theorem 2.

$$\chi(G;x) = \sum_k \partial_G(k)\binom{x+k}{n}\ . \tag{9}$$

This can be proved by first applying Möbius inversion to (9), and then using sieving arguments and a classical result of Stanley [S73] interpreting $\chi(G;k)$ for negative integer values of k (we omit the proof). In principal, (9) must follow the expansions of Gansner and Vo [GV87] and Brenti [Br92], although this implication is not so direct.

6. An expansion of the cover polynomial

We next turn to the problem of finding reasonable expansions of the general cover polynomial $C(D;x,y)$. As seen in (7), the cover polynomial with $y = 1$ is equal to the drop polynomial $\Delta(D;x)$. It is natural to ask the question if such relations can be generalized to the general cover polynomial $C(D;x,y)$. To start with, we seek appropriate bases for $\mathbb{Z}[x,y]$ so that the corresponding coefficients have some natural (or at least understandable) interpretations. In view of (8) and (9), it is natural to guess something like $\sum_{k,\ell} \delta_D(k,\ell)\binom{x+k}{n}y^{\ell}$, $\sum_{k,\ell} \delta'_D(k,\ell)\binom{x+k}{n}y^{\underline{\ell}}$ or $\sum_{k,\ell} \delta''_D(k,\ell)\binom{x+k}{n}\binom{y+\ell}{n}$, for example. However, all of these (and numerous others) end up having *negative* coefficients for various digraphs D, which is something we would like to avoid, if possible. In fact, it is possible to avoid this, as the following result shows.

For our given digraph $D = (V, E)$ with $|V| = n$, let $A(c, s)$ denote the collection of all sets $C \subseteq E$ of c edges which form s disjoint cycles. For $\pi \in Sym(D)$, define *drop* (π) to be the set of all pairs $u\pi(u) \in E$, and define $d(\pi)$ to be $|drop(\pi)|$. Finally, set

$$\delta_D(k, c, s) := \sum_{y \in A(c,s)} |\{\pi \in Sym(D) : y \subseteq drop(\pi),\ d(\pi) = k + c\}| \ . \tag{10}$$

Theorem 3.

$$C(D; x, y) = \sum_{k,c,s} \delta_D(k, c, s) \binom{x+k}{n-c} (y-1)^s \ . \tag{11}$$

Proof: By (5) and (7) we can write

$$C(D; x, y) = \sum_{k,\ell} c_D(k, \ell) x^{\underline{k}} y^\ell = \sum_{k,\ell} r_B(k, \ell) x^{\underline{n-k}} y^\ell \tag{12}$$

where $r_B(k, \ell)$ is defined to be the number of ways of placing k non-attacking rooks on the board B (associated with D) so that the corresponding edges in D form exactly ℓ cycles and any number of paths. Expanding to the basis $\binom{x+k}{n}$ we have

$$C(D; x, y) = \sum_k f_k(y) \binom{x+k}{n} \tag{13}$$

where

$$f_k(y) := \sum_{j=0}^{n} \sum_{\ell} r_B(j, \ell)(n-j)!(-1)^{j-k} \binom{j}{k} y^\ell \ . \tag{14}$$

We can rewrite (14) as

$$f_k(y) = \sum_{j,\ell} \sum_{b \in B(j,\ell)} y^\ell \sum_{drop(\pi) \supseteq b} (-1)^{j-k} \binom{j}{k} \tag{15}$$

where $B(j, \ell)$ consists of the set of all ways of placing j non-attacking rooks so as to form ℓ cycles (and any number of paths) in D. Interchanging the order of summation, we have

$$f_k(y) = \sum_{\pi} \sum_{j,\ell} \sum_{\substack{b \in B(j,\ell) \\ b \subseteq drop(\pi)}} (-1)^{j-k} \binom{j}{k} y^\ell = \sum_{\pi} f_{k,\pi}(y) \tag{16}$$

where

$$f_{k,\pi}(y) := \sum_{j,\ell} \sum_{\substack{b \in B(j,\ell) \\ b \subseteq drop(\pi)}} (-1)^{j-k} \binom{j}{k} y^\ell \tag{17}$$

and the sum is over all j and ℓ. Thus,

$$C(D; x, y) = \sum_k f_k(y) \binom{x+k}{n} = \sum_{\pi} f_\pi(x, y) \tag{18}$$

where

$$f_\pi(x, y) := \sum_k f_{k,\pi}(y) \binom{x+k}{n} \ . \tag{19}$$

In particular,

$$f_\pi(x,1)=\sum_{j,k}(-1)^{j-k}\binom{j}{k}\binom{d(\pi)}{j}\binom{x+k}{n}=\sum_k(-1)^{k-d(\pi)}\binom{0}{k-d(\pi)}\binom{x+k}{n} \tag{20}$$

where we have used the identity

$$\sum_j(-1)^j\binom{a}{j}\binom{b+j}{c}=(-1)^a\binom{b}{c-a}\,.$$

Let us now examine several special cases.

(i) Suppose π contains no cycle. Then

$$f_{k,\pi}(1)=\sum_j\sum_{\substack{b\in B(j,0)\\ b\in drop(\pi)}}(-1)^{j-k}\binom{j}{k}=\sum_j(-1)^{j-k}\binom{d(\pi)}{j}\binom{j}{k}=(-1)^{k-d(\pi)}\binom{0}{k-d(\pi)}$$

and

$$f_\pi(x,y)=f_\pi(x,1)=\sum_k(-1)^{k-d(\pi)}\binom{0}{k-d(\pi)}\binom{x+k}{n}=\binom{x+d(\pi)}{n}\,.$$

(ii) Suppose π has a non-cycle drops and one cycle having c edges in D. Then

$$\begin{aligned}
f_{k,\pi}(y) &= \sum_j\binom{a}{j-c}\binom{j}{k}(-1)^{j-k}y\\
&\quad+\sum_j\left(\binom{a+c}{j}-\binom{a}{j-c}\right)\binom{j}{k}(-1)^{j-k}\\
&= (-1)^{a+c-k}\binom{c}{k-a}(y-1)+(-1)^{a+c-k}\binom{0}{a+c-k}
\end{aligned}$$

and

$$\begin{aligned}
f_\pi(x,y) &= \sum_k f_{k,\pi}(y)\binom{x+k}{m}\\
&= (-1)^{a+c}\sum_k\left[(-1)^k\binom{c}{k-a}\binom{x+k}{m}(y-1)+(-1)^k\binom{0}{a+c-k}\binom{x+k}{n}\right]\\
&= \binom{x+a}{n-c}(y-1)+\binom{x+a+c}{n}\,.
\end{aligned}$$

(iii) Suppose π has a non-cycle drops and two cycles of sizes c_1 and c_2, respectively. Then

$$\begin{aligned}
f_{k,\pi}(y) &= \sum_j\binom{a}{j-c_1-c_2}(-1)^{j-k}\binom{j}{k}y^2\\
&\quad+\sum_j\left(\binom{a+c_1}{j-c_2}+\binom{a+c_2}{j-c_1}-2\binom{a}{j-c_1-c_2}\right)(-1)^{j-k}\binom{j}{k}y\\
&\quad+\sum_j\left(\binom{a+c_1+c_2}{j}-\binom{a+c_1}{j-c_2}-\binom{a+c_2}{j-c_1}+\binom{a}{j-c_1-c_2}\right)\\
&= (-1)^{a+c_1+c_2-k}\left(\binom{c_1+c_2}{k-a}(y-1)^2+\left(\binom{c_2}{k-a-c_1}+\binom{c_1}{k-a-c_2}\right)(y-1)\right.\\
&\quad\left.+\binom{0}{k-a-c_1-c_2}\right)
\end{aligned}$$

and so,

$$\begin{aligned}
f_\pi(x,y) &= \sum_k f_{k,\pi}(y)\binom{x+k}{n} \\
&= (-1)^{a+c_1+c_2}(y-1)^2\sum_k \binom{c_1+c_2}{k-a}(-1)^k\binom{x+k}{n} \\
&\quad + \ (-1)^{a+c_1+c_2}(y-1)\sum_k\left(\binom{c_2}{k-a-c_1}+\binom{c_1}{k-a-c_2}\right)(-1)^k\binom{x+k}{n} \\
&\quad + \ \sum_k(-1)^{a+c_1+c_2-k}\binom{0}{k-a-c_1-c_2}\binom{x+k}{n} \\
&= \binom{x+a}{n-c_1-c_2}(y-1)^2+\left(\binom{x+a+c_1}{n-c_2}+\binom{x+a+c_2}{n-c_1}\right)(y-1)+\binom{x+a+c_1+c_2}{n}.
\end{aligned}$$

(iv) Now, suppose in general that π has a non-cycle drops and s cycles of sizes $c_1, c_2, \ldots, c_s$. Let c denote $c_1+c_2+\cdots+c_s$. Then, arguing as before, we finally obtain after some inclusion-exclusion computations:

$$\begin{aligned}
f_\pi(x,y) &= \binom{x+a}{n-c}(y-1)^s+\sum_i\binom{x+a+c_i}{n-c+c_i}(y-1)^{s-1} \\
&\quad + \ \cdots+\binom{x+a+c}{n} \\
&= \sum_I\binom{x+d(\pi)-c_I}{n-c_I}(y-1)^{|I|}
\end{aligned} \tag{21}$$

where I ranges over all subsets of $[s] := \{1,2,\ldots,s\}$ and $c_I := \sum_{i\in I} c_i$. Thus,

$$\begin{aligned}
C(D;x,y) &= \sum_\pi f_\pi(x,y) \\
&= \sum_\pi\sum_{I\subseteq[s]}\binom{x+d(\pi)-c_I}{n-c_I}(y-1)^{|I|} \\
&= \sum_{k,c,s}\delta_D(k,c,s)\binom{x+k}{n-c}(y-1)^s
\end{aligned} \tag{22}$$

where $\delta_D(k,c,s)$ is defined in (10). This completes the proof. ∎

We isolate the other expression in (22) for $C(D;x,y)$ for ease of reference.

Corollary 8.

$$C(D;x,y)=\sum_{\pi\in Sym(D)}\sum_{I\subseteq[s]}\binom{x+d(\pi)-c_I}{n-c_I}(y-1)^{|I|} \tag{23}$$

where s is the number of cycles of π, $c_1,\ldots,c_s$ are the sizes of these cycles, and for $I\subseteq[s]$, $c_I := \sum_{i\in I} c_i$.

Of course, when we substitute $y=1$ into (23), then we obtain the relevant part of (7). On the other hand, by substituting $y=0$ and $y=2$ into (23), we have:

Corollary 9. *Let $c_D(i,0)$ denote the number of ways of covering all the vertices of D with exactly i paths. Then*

$$\begin{aligned} C(D;x,0) &= \sum_i c_D(i,0)x^{\underline{i}} \\ &= \sum_\pi \sum_{I\subseteq[s]} (-1)^{|I|}\binom{x+d(\pi)-c_I}{n-c_I} \ . \end{aligned} \tag{24}$$

Corollary 10. *Let $c_D(i,\geq j)$ denote the number of ways of covering all the vertices of D with exactly i paths and at least j cycles and let*

$$c_D^*(i) = \sum_j c_D(i,\geq j)2^j \ .$$

Then

$$\begin{aligned} C(D;x,2) &= \sum_i c_D^*(i)x^{\underline{i}} \\ &= \sum_W \sum_k \delta(W,k)\binom{x+k-|W|}{|W|} \end{aligned}$$

where W ranges over all subsets of cycles in D, $|W|$ denotes the number of edges in cycles of W, and $\delta(W,k)$ denotes the number of permutations containing W which have k drops.

7. Special cases

To begin with, suppose $D=(V,E)$ is the complete acyclic digraph on $V=\{1,2,\ldots,n\}$ so that $E=\{ij: \ i<j\}$. Then $c_D(k)=\left\{ {n \atop k} \right\}$, a Stirling number of the second kind which enumerates the number of partitions of an n element set in k nonempty subsets. Also, $\delta_D(k)=\left\langle {n \atop k} \right\rangle$, an Eulerian number which enumerates the number of permutations on $\{1,2,\ldots,n\}$ having k drops (e.g., see [GKP89]). Substituting these values into (8) we obtain

$$C(D;x,1)=\sum_k \left\{ {n \atop k} \right\} x^{\underline{k}} = \sum_k \left\langle {n \atop k} \right\rangle \binom{x+k}{n} = x^n \ . \tag{25}$$

This expresses x^n as a linear combination of $x^{\underline{k}}$ or $\binom{x+k}{n}$, $0\leq k\leq n$. The last equality is often called Worpitsky's identity (see [GKP89]).

In a similar spirit, by taking D^+ to be the digraph formed by adding a loop to each vertex in D (and noting that $\delta_{D^+}(k)=\left\langle {n \atop k-1} \right\rangle$), we obtain an identity due to Frobenius (see [Co74], Theorem E, p. 244):

$$\sum_j (-1)^{n-k-j} j! \binom{n-j}{k} \left\{ {n+1 \atop j+1} \right\} = \left\langle {n \atop k-1} \right\rangle \ , \tag{26}$$

Of course, applying this approach to more complex digraphs will lead to the corresponding (more complex) identities.

For specific values of x and y, $C(D;x,y)$ can often be given natural interpretations. (Many examples of this for the Tutte polynomial can be found in [JVW90].)

(i) $C(D;1,0) = c_D(1,0)$, the number of Hamiltonian paths on D;

(ii) $C(D;0,1) = \sum_j c_D(0,j)$, the number of ways of covering D with disjoint cycles;

(iii) $\left.\frac{\partial C(D;x,y)}{\partial y}\right|_{\substack{x=0\\y=0}} = c_D(0,1)$, the number of Hamiltonian cycles in D;

(iv) $\frac{1}{2}C(D;2,0) = c_D(1,0) + c_D(2,0)$, the number of ways of covering D with at most two disjoint paths;

(v) $\frac{1}{2}\{C(D;0,1)+C(D;0,-1)\} = \sum_j c_D(0,2j)$, the number of ways of covering D with an even number of cycles.

(vi) If D is acyclic and transitive then

$$(-1)^n C(D;-1,1) = (-1)^n \sum_{k=0}^{n} \delta_D(k) \binom{-1+k}{n} = \delta_D(0) \ , \tag{27}$$

the number of $\pi \in Sym(D)$ having no drops. Note the similarity of (27) to the classic result of Stanley [S73], which asserts that $(-1)^n \chi(G;-1)$ is equal to the number of acyclic orientations of G.

8. Concluding remarks

We close by mentioning a number of open problems and promising directions.

(a) Is there a natural analogue of the Tutte polynomial for directed graphs? The polynomial introduced here, the cover polynomial $C(D;x,y)$ has many properties which are similar to those of the Tutte polynomial $T(G;x,y)$, but yet is different enough to make us believe that we are not yet quite there. At least, it is not obvious how to convert back and forth between results for $T(G;x,y)$ and those for $C(D;x,y)$. Of course, the Tutte polynomial for graphs arises naturally from the Tutte polynomial for matroids. At present there does not seem to be a satisfactory definition of a Tutte polynomial for oriented matroids. We remark that in Gordon [Go93], a Tutte-like polynomial is defined for posets. Also, Gessel [Ge89] has introduced a two variable version of the rook polynomial which takes into account the cycle structure of the rook placements.

(b) Is it possible to characterize those graphs G for which $\chi(G;x) \neq \Delta(D;x)$ for any digraph D? The only example known of such a graph is C_6, the cycle on 6 vertices (see [GJW78]. Presumably, *most* graphs are like this.

(c) For a digraph $D = (V,E)$, define the dual digraph $\overline{D} = (V, V \times V \setminus E)$. Very recently, T. Chow [Ch94] and I. Gessel [Ge94] independently proved the following beautiful reciprocity formula:

$$C(\overline{D};x,y) = (-1)^n C(D;-x-y,y) \ .$$

In particular, this shows that the cover polynomial for $\overline{D}$ is determined by the cover polynomial for D, something already known to happen for the rook polynomials of a board and its complement.

(d) Recently, Stanley [S93] has introduced the following multi-variable generalization $X(G)$ of the chromatic polynomial of a graph $G = (V, E)$, $V = \{v_1, \ldots, v_n\}$, defined by:

$$X(G) = \chi(G; x_1, x_2, \ldots) = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \ldots x_{\kappa(v_n)}$$

where the sum ranges over all proper colorings $\kappa : V \to \{1, 2, 3, \ldots\}$. Then $X(G)$ is a homogeneous symmetric polynomial of degree n, and

$$\chi(G; \overbrace{1, 1, \ldots, 1}^{i}, 0, 0, \ldots) = \chi(G; i) \ .$$

Stanley derives a number of interesting properties of $X(G)$ by expanding it to different bases for symmetric functions, such as the elementary symmetric functions, the power sum symmetric functions, and Schur functions, for example. What are the correspondings results for $C(D; x, 1)$ or $C(D; x, y)$? Very recently, T. Chow [Ch94] has made some very nice progress on this problem.

In this direction, it is possible to generalize the definitions of $C(D; x, y)$ to a polynomial $C(D; y, x_1, x_2, \ldots)$ where in (1) we define $C(I_n) = x_n$. What can be said about this polynomial? Clearly much more remains to be done.

The authors wish to thank F. Brenti, M. Dworkin, I. Gessel, and R. Stanley who were instrumental in the evolution of this paper into its current form.

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