Combinatorics for the East model

Fan Chung * Persi Diaconis †‡ Ronald Graham *

Abstract

We study the number of configurations in the East model of statistical physics. This may be pictured as sites in a line. The site at zero is always occupied. The site at $i > 0$ can only be changed if site $i - 1$ is occupied. If at most n occupied sites are permitted, we establish upper and lower bounds of the form $2^{\binom{n}{2}}\, n!\, c^n$ where $c < 1$ for the number of possible configurations.

1 Introduction

This paper is motivated by a variety of Markov chains used by chemists and physicists to study properties of glasses and super-cooled liquids. The chains are called 'facilitated kinetic Ising spin models'. They are based on a graph or lattice with various sites occupied or empty. At each time, a site is chosen at random and changed or not according to the familiar Metropolis dynamics for a given stationary distribution. The difference is that the change is allowed only if the neighbors of the chosen site are in a prescribed configuration; otherwise, no change is made. These neighborhood restrictions do not change the long-term stationary distribution but can lead to dramatic changes in approach to equilibrium.

The earliest such chains were introduced by Andersen and Fredrickson [2, 3] who allowed a change when k neighbors on a d-dimensional lattice were occupied. Reiter, Jäckle and coworkers [10] studied asymmetric rules, e.g., on a two-dimensional lattice, change is allowed if sites North and East are occupied. The simplest such model is the East model; this takes place on a one-dimensional lattice or ring with a transition permitted only if the neighbor to the immediate left is occupied (This should probably be called the West model but historically East is East).

Reiter and Jäckle [10] studied how the kinematic "East" restriction changes relaxation and cor-

*University of California, San Diego
†Stanford University
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relation times. One of their conjectures was proved by Aldous and Diaconis [1]. Pitts, Young and Andersen [8] (following Pitts [9]) studied the autocorrelation function of a single site in the East model, started in stationarity. They derive various approximations paralleling mode-coupling approximations used in the study of real glasses and super-cooled liquids. They found that spin systems give illuminating toy models for studying the validity of mode-coupling – just as in more complex systems, mode-coupling works well in some regions but not in others.

The present paper studies the combinatorics of the East model if at most n occupied sites are allowed. We give bounds for the entropy (number of possible states). It is convenient to study the subset of occupied positions. Thus we consider a graph $G(n)$ formed as follows. The vertex set $V(n)$ of G is the set of all subsets $X \subseteq \mathbb{P} = \{1, 2, 3, \dots\}$ of cardinality at most n. A pair $\{X, X'\}$ forms an *edge* of G, written $X \sim X'$, provided X' can be obtained from X by adjoining to (or removing from) X the element $x + 1$ for some $x \in X$, or by adjoining (or removing) the element 1.

We will be interested in investigating various properties of G. In particular, we will establish upper and lower bounds on $|V(n)|$ of the form

$$2^{\binom{n}{2}}\ n!\ c^n$$

for various constants $c < 1$ (see Theorems 2, 4 and 5).

In Figure 1, we show the graph $G(3)$. With the help of Susan Holmes and Glenn Tesler, we have computed the first few values of $|V(n)|$.

n	1	2	3	4	5
$\lvert V(n)\rvert$	2	5	26	373	15193

We did not find this sequence in standard lists of integer sequences. Our bounds show that $|V(6)|$ is about 2.4×10^6 which is too large for the brute force algorithm we employed. The exact value $|V(5)|$ gives an estimate of $c = .6583$ if $|V(n)| \sim 2^{\binom{n}{2}} n!\ c^n$.

2 Elementary facts

Fact 1 *(i)* $A(n) := \max\{x \in X \in V(n) \ :\ |X| = 1\} = 2^{n-1}$;

(ii) $B(n) := \max\{x \in X \in V(n)\} = 2^n - 1$.

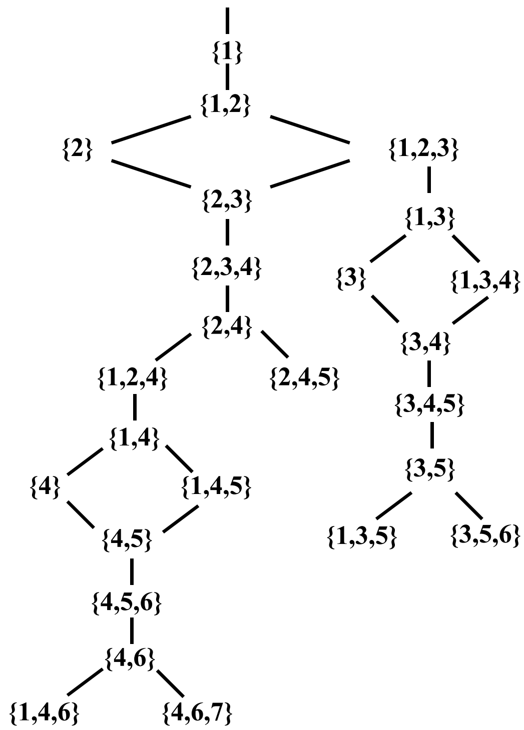

Figure 1: $G(3)$

Proof: (By induction on n.) The assertion certainly holds for $n = 1$ since $A(1) = 1 = B(1)$. Assume for some $n \geq 1$ that $A(k) = 2^{k-1}$ and $B(k) = 2^k - 1$ for all $k \leq n$. Observe that in general if $X \in V(n)$ with $|X| = r$ and $Y \in V(n-r)$ then $X \cup (x + Y) \in V(n)$ for any $x \in X$ (where $x + Y$ denotes $\{x + y \ : \ y \in Y\}$). In this case we can think of building a copy of Y on the "base" $x \in X$. Thus, taking $X = \{2^{n-1}\} \in V(n) \subset V(n+1)$ and $Y = \{2^{n-1}\} \in V(n)$, we get $X' = \{2^{n-1}, 2^n\} \in V(n+1)$. Now we can reverse the process of generating the element 2^{n-1} in $V(n)$ to remove 2^{n-1} from X', forming $X'' = \{2^n\} \in V(n+1)$, which shows that $A(n+1) \geq 2^n$.

Now, with $X = \{2^n\} \in V(n+1)$ (as we just showed) and $Y \in V(n)$ with $\max Y = 2^n - 1$ (by the induction hypothesis), we can construct $X' = X \cup (2^n + Y) \in V(n+1)$ with $\max X' = 2^n + 2^n - 1 = 2^{n+1} - 1$, which shows $B(n+1) \geq 2^{n+1} - 1$.

In the other direction, if $\{x_0\} \in V(n+1)$ with $x_0 \geq 2^n + 1$, then in order to remove it (i.e., reach $\emptyset$ through a sequence of edges), we would have to create a set $Y \in V(n)$ with $x_0 - 1 \in Y$. But since $x_0 - 1 \geq 2^n$ then by (ii), this is impossible. Thus, $A(n+1) = 2^n$. Finally, suppose $X \in V(n+1)$

where, without loss of generality, we can assume $|X| = n+1$. Since by hypothesis there is a path in $G(n+1)$ from X to $\emptyset$ then X must contain a pair of consecutive integers, say x_0 and x_0+1 (since otherwise we couldn't move at all from X). Removing x_0+1 to form X_1, we see (by induction) that X_1 much have a pair of elements x_1, x_1+g_1, with $g_1 \leq 2$. (Again, since otherwise X_1 wouldn't be connected to $\emptyset$). Remove x_1+g_1 to form X_2. The general step in this process forms the (sub)set $X_k \subset X$ of size $n+1-k$, which must then possess a pair of elements x_k, x_k+g_k with $g_k \leq 2^k$. We remove x_k+g_k from X_k to form X_{k+1}, etc. Eventually, we reach $X_n \subset X$ of size 1, which must consist of a single element $x_n \leq 2^n = A(n+1)$. Combining all the preceding inequalities shows that

$$\max X \leq 2^n + 2^{n-1} + \ldots + 2 + 1 = 2^{n+1} - 1$$

Thus, $B(n+1) \leq 2^{n+1} - 1$ and Fact 1 is proved. □

The same argument can be used to prove the more general fact:

Fact 2 *For* $1 \leq k \leq n$,

$$\max\{x \in X \in V(n) \ : \ |X| = k\} = 2^n - 2^{n-k}$$

3 Upper bounds on $|V(n)|$

For a set $X = \{X(1) < X(2) < \ldots < X(r)\} \in V(n)$, define the sequence of *gaps* of X to be the sequence $g = g(X) = (g_1, g_2, \ldots, g_r)$ where $g_i := X(i) - X(i-1)$, and by convention, we always take $X(0) = 0$. The preceding considerations show that the following (polynomial-time) algorithm can always be used to decide whether a particular set $X \subseteq \mathbb{P}$ is in $V(n)$.

(1) If $g(X)$ has no gap of size $\leq 2^{n-|X|}$ then HALT. We can conclude that $X \notin V(n)$. Otherwise, if $g_i = X(i) - X(i-1) \in g(X)$ has $g_i \leq 2^{n-|X|}$ then remove $X(i)$ from X to form X'.
(2) Repeat (1) with X replaced by X'.
(3) If we succeed in reaching $\emptyset$ this way then $X \in V(n)$, and, in fact, by reversing the preceding steps (and using Fact 1), this shows how to construct it. Otherwise, we conclude $X \notin V(n)$. Notice that there may be many choices for the elements to be removed at each step. This reduction algorithm allows for *any* choice to be made at each step.

Let us assume for now that $X \in V(n)$ with $|X| = n$. We are going to specify a *particular choice* to be made at each of the removal steps. Namely, let R denote the preceding reduction algorithm in which we always remove the *largest* possible integer satisfying the required gap size condition. This process results in the elements of X being removed in some particular order, generating a *permutation* $\pi = \pi_X$ on $\{1, 2, \ldots, n\}$. In particular, for $X = \{X(1) < X(2) < \ldots < X(n)\} \in V(n)$, where $X(i)$ is removed at step $\pi(i)$.

It will be convenient to denote a set X by its corresponding gaps sequence $g(X) = (g_1, g_2, \cdots, g_n)$ where $g_i = X(i) - X(i-1)$. What we will do is to derive upper bounds on the number $N(\pi)$ of $X \in V(n)$ which generate the permutation $\pi = \pi_X$ for each permutation π of $\{1, 2, \ldots, n\}$. We first illustrate this idea with several examples.

Example 1: $n = 4, \pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$. $g(X) = (g_1, g_2, g_3, g_4)$
At the first step of the reduction, since $\pi(4) = 1$, then $X(4)$ is removed, leaving $g(X^{(1)}) := (g_1, g_2, g_3)$. This implies in particular that $g_4 \leq 1$.

At the second step, since $\pi(3) = 2$, then $X(3)$ is removed (so $g_3 \leq 2$), leaving $g(X^{(2)}) := (g_1, g_2)$. We continue this process for two more steps, finally reaching $\emptyset$. For the permutation π to be valid, we need the inequalities

$$\begin{aligned} g_1 &\leq 1 \\ g_2 &\leq 2 \\ g_3 &\leq 4 \\ g_4 &\leq 8 \end{aligned}$$

Hence, the total number $N(\pi)$ of possible $X \in V(4)$ is at most $g_1 g_2 g_3 g_4 \leq 1 \cdot 2 \cdot 4 \cdot 8 = 2^6$. The same argument shows that for general n, the *reverse* permutation π with $\pi(k) = n + 1 - k, 1 \leq k \leq n$, has $N(\pi) \leq \prod_{k=1}^{n} 2^{k-1} = 2^{\binom{n}{2}}$. In general, since each X is determined by its gap sequence $g(X)$, then in fact $N(\sigma) \leq 2^{\binom{n}{2}}$ for *any* permutation $\sigma = \sigma_X$, which gives the (trivial) estimate

$$|V(n)| \leq \sum_{\pi} N(\pi) \leq n!\ 2^{\binom{n}{2}} \tag{1}$$

Theorem 1 will improve upon this estimate by an exponential factor.

Example 2: $n = 4, \pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$. $g(X) = (g_1, g_2, g_3, g_4)$
Proceeding as before we find $X(3)$ is the first number removed, so that $g_3 \leq 1$. However, since $X(4)$ was *not* removed (and is to the right of $X(3)$) then we must have $g_4 > 1$. Removing $X(3)$ leaves us with the set X' with gap sequence $g(X') = (g_1, g_2, g_3 + g_4)$. In general, whenever an *internal*

number $X(i)$ is removed, the new gap formed is the *sum* of the two gaps $X(i)$ is currently adjacent to. Now at the second step, $X(4)$ is removed, so we must have its (new) gap $g_3 + g_4 \le 2$. However, this is not possible since $g_3 = 1$ and $g_4 > 1$. Hence, no X can have this permutation, i.e., $N(\pi) = 0$.

We now consider the general case. We begin with a permutation π on $\{1, 2, \ldots, n\}$ where $X(i)$ is removed at step $\pi(i)$ by the (greedy) algorithm R. Let $g_i(k)$ denote the gap associated with $X(i)$ at the beginning of step k (i.e., when only $k-1$ elements have been removed), assuming that $X(i)$ has not yet been removed. Thus, $g_i(k) = \sum_{j=0}^{r} g_{i-j}$ where r is the largest index such that $\pi(i-r) < k$. In particular $g_i(1) = g_i$. Define $h_i = g_i(\pi(i))$. Then h_i is the gap associated with $X(i)$ just prior to its being removed at step $\pi(i)$. By the definition of algorithm R, we always have

$$h_i \le 2^{\pi(i)-1}, \quad 1 \le i \le n \tag{2}$$

Now, suppose that for some i, we find there is a $j < i$ such that $\pi(j) = \pi(i) - 1$

$$\begin{array}{cc} j & i \\ \bullet & \bullet \\ \pi(i)-1 & \pi(i) \end{array}$$

Thus, at step $\pi(i) - 1$, $X(i)$ was passed as a candidate for removal, and $X(j)$ was selected instead. This implies that

$$2^{\pi(i)-2} < g_i(\pi(i) - 1) \le g_i(\pi(i)) = h_i.$$

Combining this with (2), we have

$$2^{\pi(i)-2} + 1 \le h_i \le 2^{\pi(i)-1} \tag{3}$$

(i.e., we lose a factor of 1/2 over the trivial estimate of $2^{\pi(i)-1}$ for the number of choices for h_i). Hence, if there are k such i's for π, then total number of choices for all the h_i is at most

$$2^{-k} \cdot 2^{0+1+\ldots+(n-1)} = 2^{\binom{n}{2}} \cdot 2^{-k}.$$

It is easy to see by considering the inverse permutation π^{-1} that the number of permutations π having exactly k values i with $\pi(j) = \pi(i) - 1$ for some $j < i$ is just the *Eulerian* number $\left\langle {n \atop k} \right\rangle$, which also counts the number of permutations π of $\{1, 2, \ldots, n\}$ with k *rises*, i.e., k occurrences of an s such that $\pi(s) < \pi(s+1)$ (see [4] for an in-depth discussion of Eulerian numbers). Hence, we have the estimate:

Theorem 1

$$|V(n)| \leq 2^{\binom{n}{2}} \sum_k \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle 2^{-k} \tag{4}$$

The sum $S_n := \sum_k \langle \begin{matrix} n \\ k \end{matrix} \rangle 2^{-k}$ has occurred in various forms in the literature. In particular, one finds in page 627 of [7], the sum

$$P_n := \sum_k \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle 2^{k-1} \tag{5}$$

and references where it is shown that

$$\sum_{n\geq 0} P_n \frac{z^n}{n!} = \frac{1}{1-e^z} \tag{6}$$

which implies

$$\frac{P_n}{n!} = \frac{1}{2}(\ln 2)^{-n-1} + \sum_{k\geq 1} Re((\ln 2 + 2\pi i k)^{-n-1}) \tag{7}$$

One also finds the interesting equality of Gross [5]

$$P_n = \sum_{k\geq 1} \frac{k^n}{2^{k+1}}, \quad n \geq 1 \tag{8}$$

Note that by the symmetry property of $\langle \begin{matrix} n \\ k \end{matrix} \rangle = \langle \begin{matrix} n \\ n-k-1 \end{matrix} \rangle$, we have

$$\begin{aligned} \frac{1}{2^{n-2}} P_n &= \sum_k \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle 2^{-n+k+1} \\ &= \sum_k \left\langle \begin{matrix} n \\ n-k-1 \end{matrix} \right\rangle 2^{-k} \\ &= \sum_k \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle 2^{-k} \\ &= S_n \end{aligned} \tag{9}$$

which implies

$$S_n = \frac{1}{2^{n-1}} \sum_{k\geq 1} \frac{k^n}{2^k} \tag{10}$$

Using dominated convergence in (7) along with (8) shows

$$S_n \sim \frac{n!}{(\ln 4)^n} \tag{11}$$

Hence, we have

Theorem 2

$$|V(n)| \leq 2^{\binom{n}{2}} S_n < 2^{\binom{n}{2}} n! \frac{1}{(\ln 4)^n} \tag{12}$$

for n sufficiently large.

A more refined version of this argument can be used to obtain the following stronger upper bound. For a permutation π of $\{1, 2, \ldots, n\}$, define for $1 \leq i \leq n$ the quantity $d_\pi(i)$ to be the least integer d (if it exists) such that $\pi(i) < \pi(i + d)$. If d does not exist then set $d_\pi(i) = \infty$. Finally, define

$$d(\pi) := \prod_{i=1}^{n} (1 - \frac{1}{2^{d_\pi(i)}}).$$

Then it can be shown that the following generalization of Theorem 1 holds.

Theorem 3

$$|V(G)| \leq 2^{\binom{n}{2}} \sum_{\pi} d(\pi) \tag{13}$$

The bound in Theorem 1 comes from (13) by just taking account of those i in π for which $d_\pi(i) = 1$ (counted by Eulerian numbers). An intermediate result arises by just considering those i in π for which $d_\pi(i) \leq 2$ (and taking other factors in the product $d(\pi)$ equal to 1). It is straightforward to show that this results in the following bound.

For a permutation π of $\{1, 2, \ldots, n\}$, if $\pi(i) < \pi(i + 1)$ we say that π has a *rise* at i. Similarly, if $\pi(i + 1) < \pi(i) < \pi(i + 2)$, we say that π has a "213" at i.

Let $\langle {n \atop k,l} \rangle$ denote the number of permutations π of $\{1, 2, \ldots, n\}$ which have k rises and l 213's for $0 \leq l \leq k < n$. Thus, $\sum_l \langle {n \atop k,l} \rangle = \langle {n \atop k} \rangle$

Theorem 4

$$|V(n)| \leq 2^{\binom{n}{2}} \sum_{k,l} \langle {n \atop k,l} \rangle \, 2^{-k} (4/3)^{-l} \tag{14}$$

It is easy to see that these “generalized Eulerian” numbers $\langle \begin{matrix} n \\ k,l \end{matrix} \rangle$ satisfy the recurrence

$$\begin{aligned} \langle \begin{matrix} n \\ k,l \end{matrix} \rangle &= (l+1)\langle \begin{matrix} n-1 \\ k,l \end{matrix} \rangle + (l+1)\langle \begin{matrix} n-1 \\ k-1,l+1 \end{matrix} \rangle \\ &\quad +(n-k-l)\langle \begin{matrix} n-1 \\ k-1,l \end{matrix} \rangle + (k-l+1)\langle \begin{matrix} n-1 \\ k,l-1 \end{matrix} \rangle \qquad (15) \\ \langle \begin{matrix} 0 \\ 0,0 \end{matrix} \rangle &= 1, \qquad \langle \begin{matrix} a \\ b,c \end{matrix} \rangle = 0 \ \text{ if } \ a,b \ \text{ or } \ c<0 \end{aligned}$$

We show some small values of $\langle \begin{matrix} n \\ k,l \end{matrix} \rangle$ in Table 1.

0	1
l / k	0

$n=1$

1	0	0
0	1	1
l / k	0	1

$n=2$

2	0	0	0
1	0	1	0
0	1	3	1
l / k	0	1	2

$n=3$

2	0	0	0	0	0
1	0	5	3	0	0
0	1	6	8	1	0
l / k	0	1	2	3	4

$n=4$

3	0	0	0	0	0	0
2	0	0	3	0	0	0
1	0	16	32	6	0	0
0	1	10	31	20	1	0
l / k	0	1	2	3	4	5

$n=5$

Table 1:

We have not analyzed the asymptotic behavior of the sum in (14). However, preliminary computations indicate that

$$\sum_{k,l} \langle \begin{matrix} n \\ k,l \end{matrix} \rangle \, 2^{-k}(4/3)^{-l} = O(n! \; c_2^n) \qquad (16)$$

where $c_2 < 0.95/\ln 4 = 0.68528\ldots$, which represents a modest (but real) improvement over the bound (12).

4 Lower bounds on $|V(n)|$

To show that $|V(n)|$ is relatively large, we will describe a method for constructing large subsets of $V(n)$. We begin with a simple version of the construction. Suppose $d = (d_1 > d_2 > \ldots > d_n)$ is a sequence of integers satisfying $d_i \in (2^{n-i-1}, 2^{n-i}]$, $1 \le i \le n$. Form a set $X = \{X(1), X(2), \ldots, X(n)\}$ from d as follows (where, as usual, we define $X(0) = 0$):

For the first two steps, choose $X(1) = d_1$, and $X(2) = X(1)+d_2$. Now, in general, at the k-th step, select $X(k)$ to be one of $X(i) + d_k$, $0 \leq i < k$, where $X(i)$ is required to be *different* from the $X(i')$ used in forming $X(k-1) = X(i') + d_{k-1}$. Note that the intermediate set $X_k = \{X(1), \ldots, X(k)\}$ has the property that the unique smallest gap between consecutive elements is just d_k. This follows by induction since when $X(k)$ is added then either it is the largest element of X_k, or it falls between two consecutive elements of X_{k-1}, say, $X(i) < X(k) < X(i')$. Thus, the two new gaps created in this case are $X(k) - X(i) = d_k$ and

$$\begin{aligned} X(i') - X(k) &= d_{i'} - d_k \\ &\geq d_{k-2} - d_k \qquad \text{by hypothesis on the choice of } X_k \\ &> 2^{n-k+1} - 2^{n-k} \\ &= 2^{n-k} \geq d_k \end{aligned}$$

Hence, in either case, d_k is the unique minimum gap size of X_k.

Now observe that we can reduce X to $\emptyset$ by removing its elements sequentially, always choosing the point having the *smallest current gap* to be removed. Doing this will remove the $X(k)$ exactly in the reverse order $X(n), X(n-1), \ldots, X(1)$ by the minimum gap size property of the d_k just mentioned. In fact, given the final set X, this reduction will recover both the sequence d, and the points $X(i)$ on which each $X(k)$ was "based", (i.e., $X(k) = X(i) + d_k$). Hence, the total number of different X's which can be constructed this way is

$$(n-1)!\ 2^{1+2+\ldots+(n-2)} = (n-1)!\ 2^{\binom{n-1}{2}}$$

This implies the estimate

$$|V(n)| \geq 2^{\binom{n}{2}} n!\ \{(\frac{1}{2})^n/n\} \tag{17}$$

For the next approximation, we will allow more choices for each d_k than before, but fewer choices for the number of ways that $X(k)$ can be chosen, still however, so that when $X(k)$ is selected, say $X(k) = X(i)+d_k$, then $d_k = X(k)-X(i)$ is always the unique smallest gap in $X_k = \{X(1), X(2), \ldots, X(k)\}$. Now for $d = \{d_1 > d_2 > \ldots > d_n\}$, we will only require that $d_i \in (2^{n-i-2}, 2^{n-i}], 1 \leq i \leq n$. However, we will now require in choosing $X(k) = X(i) + d_k$ that $X(k)$ is different from any $X(i')$ used in defining $X(k-1)$ *and* $X(k-2)$. Thus, the number of ways of choosing the "base points" $X(i)$ in forming X is now only $(n-2)!$ (instead of $(n-1)!$ as in the preceding construction). However, we will more than make up for this with the increased number of choices of the d_i. Our next job is to estimate this number of choices, which we will denote by $f_0(n)$. Further, define $f_1(n)$ to be the number of choices of $d = \{d_1 > d_2 > \ldots > d_n\}$, with $d_1 \in (2^{n-2}, 2^{n-1}]$ and $d_i \in (2^{n-i-1}, 2^{n-i+1}]$, $2 \le i \le n$, where, for convenience, we will henceforth assume $n \ge 10$. Thus by considering where d_1 is chosen, we have the recurrences

$$\begin{aligned} f_0(n) &= 2^{n-2} f_0(n-1) + f_1(n-1), \\ f_1(n) &= \binom{2^{n-2}}{2} f_0(n-1) + 2^{n-2} f_1(n-1), \quad n \ge 10 \end{aligned} \tag{18}$$

Let $F_0(m) = \frac{f_0(m)}{2^{\binom{m-1}{2}}}$, $F_1(m) = \frac{f_1(m)}{2^{\binom{m}{2}}}$, $1 \le m \le n$. Then (18) implies

$$\begin{aligned} F_0(n) &= F_0(n-1) + F_1(n-1), \\ F_1(n) &= (\frac{1}{4} - \frac{1}{2^n}) F_0(n-1) + \frac{1}{2} F_1(n-1), \quad n \ge 10 \end{aligned} \tag{19}$$

Finally, for $i = 0$ and 1, define

$$F_i'(n) = F_i(n) \prod_{j=5}^{n} (1 - \binom{j}{2} 2^{-j+2})^{-1} \tag{20}$$

Substituting into (19), we obtain

$$\begin{aligned} F_0'(n)(1 - \binom{n}{2} 2^{-n+2}) &= F_0'(n-1) + F_1'(n-1), \\ F_1'(n)(1 - \binom{n}{2} 2^{-n+2}) &= (\frac{1}{4} - \frac{1}{2^n}) F_0'(n-1) + \frac{1}{2} F_1'(n-1), \end{aligned} \tag{21}$$

which implies

$$\begin{aligned} F_0'(n) &\ge F_0'(n-1) + F_1'(n-1), \\ F_1'(n) &\ge \frac{1}{4} F_0'(n-1) + \frac{1}{2} F_1'(n-1), \end{aligned} \tag{22}$$

for $n \ge 10$. Hence, if we define F_0'' and F_1'' recursively by

$$\begin{aligned} F_0''(n) &= F_0''(n-1) + F_1''(n-1), \\ F_1''(n) &= \frac{1}{4} F_0''(n-1) + \frac{1}{2} F_1''(n-1), \end{aligned} \tag{23}$$

then we find

$$F_0''(n) > c(\frac{3+\sqrt{5}}{4})^n$$

for a suitable constant $c > 0$ as $n \to \infty$. This implies

$$F_0(n) > c'(\frac{3+\sqrt{5}}{4})^n$$

for some $c' > 0$, and so,

$$\begin{aligned} f_0(n) &> c'(\frac{3+\sqrt{5}}{4})^n 2^{\binom{n-1}{2}} \\ &= c'(\frac{3+\sqrt{5}}{8})^n 2^{\binom{n}{2}} \end{aligned}$$

Thus, by the previous remark on the number of choices for base points, we have the lower bound :

$$|V(n)| \geq c'(\frac{3+\sqrt{5}}{8})^n 2^{\binom{n}{2}}(n-2)! \tag{24}$$

for a suitable constant $c' > 0$.

Before proceeding to the general construction, we will sketch the next stage in this approach. Now, we will relax the constraints on choosing $d = \{d_1 > d_2 > \ldots > d_n\}$ even further, while at the same time, increasing the constraints on selecting the $X(i)$. Namely, we now only require that $d_i \in (2^{n-i-3}, 2^{n-i}]$. However, in choosing $X(k) = X(i) + d_k$, we require that $X(i)$ is different from any $X(i')$ used in defining $X(k-j)$ for $j = 1, 2, 3$. As usual, this will guarantee that d_k is always the current smallest gap (and consequently, the d_k and (something) where they are attached can be recovered uniquely from X). However, the number of choices for the $X(i)$ is now only $(n-3)!$. To count the number of choices for d, define

$$\begin{aligned} g_0(n) &= \#\text{ of choices for } d \text{ with } && d_i \in (2^{n-i-3}, 2^{n-i}], 1 \leq i \leq n. \\ g_1(n) &= \#\text{ of choices for } d \text{ with } && d_1 \in (2^{n-3}, 2^{n-1}], \\ & && d_i \in (2^{n-i-2}, 2^{n-i+1}], 2 \leq i \leq n. \\ g_2(n) &= \#\text{ of choices for } d \text{ with } && d_1 \in (2^{n-2}, 2^{n-1}], \\ & && d_2 \in (2^{n-3}, 2^{n-1}], \\ & && d_i \in (2^{n-i-1}, 2^{n-i+2}], 3 \leq i \leq n. \end{aligned}$$

Again, by considering where d_1 and d_2 are chosen, we have the recurrences:

$$g_0(n) = 2^{n-2} g_0(n-1) + g_1(n-1), \tag{25}$$

$$g_1(n) = \binom{2^{n-2}}{2} g_0(n-1) + 2^{n-2} g_1(n-1) + g_2(n-1),$$

$$g_2(n) = \binom{2^{n-2}}{3} g_0(n-1) + \binom{2^{n-2}}{2} g_1(n-1) + 2^{n-2} g_2(n-1), \quad n \geq 10$$

As before, setting $G_i(n) = g_i(n) 2^{-\binom{n-1+i}{2}}, 1 \leq i \leq 3$, and defining

$$G_i'(n) = G_i(n)\prod_{j=5}^{n}(1-\binom{j}{2}2^{-j+2})^{-1},$$

we obtain the system of inequalities:

$$G_0'(n) \geq G_0'(n-1) + G_1'(n-1), \tag{26}$$

$$G_1'(n) \geq \frac{1}{4}G_0'(n-1) + \frac{1}{2}G_1'(n-1) + G_2'(n-1),$$

$$G_2'(n) \geq \frac{1}{48}G_0'(n-1) + \frac{1}{16}G_1'(n-1) + \frac{1}{4}G_2'(n-1),$$

This implies that

$$G_0'(n) > c\rho^n$$

for a suitable $c > 0$ where $\rho \approx 1.34259\ldots$ is the largest root of $x^3 - \frac{7}{4}x^2 + \frac{9}{16}x - \frac{1}{48}$, i.e., ρ is the largest eigenvalue of the matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \\ \frac{1}{48} & \frac{1}{16} & \frac{1}{4} \end{pmatrix}$$

This implies

$$|V(n)| > c_2(\frac{\rho}{2})^n 2^{\binom{n}{2}}(n-3)! \tag{27}$$

Now, for the general case of this construction, we choose a fixed integer $r > 0$, and we want to estimate the number of $d = \{d_1 > d_2 > \ldots > d_n\}$, this time with $d_i \in (2^{n-i-r}, 2^{n-i}]$, $1 \leq i \leq n$, where $n \geq 10$. Correspondingly, in choosing $X(k) = X(i) + d_k$, we require that $X(i)$ is different from any $X(i')$ used in defining $X(k-j)$ for $1 \leq j \leq r$. Thus, we will have a factor of $(n-r)!$ when counting the number of choices for X.

Next, for $0 \leq u \leq r-1$, let $h_u(n)$ denote the number of ways of choosing $d = \{d_1 > d_2 > \ldots > d_n\}$ with

$$d_i \in (2^{n-r+u-i}, 2^{n-1}] \text{ for } 1 \leq i \leq u,$$
$$d_i \in (2^{n-r+u-i}, 2^{n+u-i}] \text{ for } u+1 \leq i \leq n.$$

By analyzing where the initial u d_i's are chosen, we obtain the following recurrence equations:

$$h_u(n) = \sum_{i=0}^{u+1} \binom{2^{n-2}}{u-i+1} h_i(n-1), \qquad 0 \le u \le r-1 \tag{28}$$

Substituting

$$H_i(n) = h_i(n) 2^{-\binom{n-1+i}{2}},$$

we obtain

$$H_u(n) = \sum_{i=0}^{u+1} \left(\prod_{j=0}^{u-i} (1 - \frac{j}{2^{n-2}}) \right) \frac{1}{(u-i+1)!} \frac{2^{\binom{i}{2}}}{2^{\binom{u+1}{2}}} H_i(n-1) \tag{29}$$

As before, if we make the substitution

$$H'_i(n) = H_i(n) \prod_{j=5}^{n} (1 - \binom{j}{2} 2^{-j+2})^{-1}$$

then we find

$$H'_u(n) \ge \sum_{i=0}^{u+1} \frac{1}{(u-i+1)!} \frac{2^{\binom{i}{2}}}{2^{\binom{u+1}{2}}} H'_i(n-1) \tag{30}$$

This implies that for a suitable constant $c_r > 0$,

$$H_0(n) > c_r (\frac{\rho_r}{2})^n 2^{\binom{n}{2}} (n-r)!$$

where ρ_r is the largest eigenvalue of the $r \times r$ matrix

$$M_r = \left(\frac{2^{\binom{j}{2}}}{(i+1-j)! 2^{\binom{i+1}{2}}} \right)_{0 \le i,j \le r-1}$$

Note that

$$M_r = U_r A_r U_r^{-1}$$

where U_r is the $r \times r$ diagonal matrix with i-th entry $2^{-\binom{i}{2}}$ and

$$A_r = \left(\frac{1}{2^i (i+1-j)!} \right)_{0 \le i,j \le r-1}$$

Thus, ρ_r is just the largest eigenvalue of A_r. We note that ρ_r, $r \to \infty$, is an increasing sequence. Computation produces the following bounds on the ρ_r:

r	ρ_r
1	$1.309\ldots = \frac{3+\sqrt{5}}{4}$
2	$1.34259\ldots$
3	$1.34399\ldots$
4	$1.344014945\ldots$
5	$1.344015076\ldots$
6	$1.344015076\ldots$

This rapid convergence is to be expected because of the smallness of the entries of A_r as their row indices increase.

Thus, we have the lower bound:

Theorem 5

$$|V(n)| > (0.672)^n 2^{\binom{n}{2}} n! \quad \textit{for} \quad n > n_0 \tag{31}$$

Recall the bound in (14) gives (via (16) which we do not prove)

$$|V(n)| < (0.6852)^n 2^{\binom{n}{2}} n!$$

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