Montecarlo Combinatorics

or better

How to transform Probabilistic Existence Proofs

into

Constructions

Problem 1

Given an $n \times n$ matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad a_{ij} = \pm 1.$$

Construct two ± 1 vectors

$$X = (x_1, x_2, \ldots, x_n)\,, \qquad Y = (y_1, y_2, \ldots, y_n)$$

Such that

$$\sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} y_j a_{i,j} \right) \geq \sqrt{\frac{2}{\pi}} \times n^{3/2} \qquad (1)$$

Algorithm:

Step 1 Produce a random Y, with $P[y_i = \pm 1] = 1/2$

Step 2 Compute for each i the sum $S_i = \sum_{j=1}^{n} y_j a_{i,j}$

Step 3 Set $x_i = sign\, S_i$

Step 4 Test if (1) is satisfied.

Repeat until it is

next

Why does this work?

This gives

$$\sum_{i=1}^{n} x_i \Big(\sum_{j=1}^{n} y_j a_{i,j} \Big) \; = \; \sum_{i=1}^{n} \Big| \sum_{j=1}^{n} y_j a_{i,j} \Big|$$

Since for each i the random variables

$$a_{i,1} y_1 \; , \; a_{i,2} y_1 \; , \; \dots \; , \; a_{i,n} y_n$$

are independent identically distributed and mean zero, the Central Limit Theorem gives

$$E\Big[\Big| \sum_{j=1}^{n} y_j a_{i,j} \Big|\Big] \approx \sqrt{\frac{2}{\pi}} \times n^{1/2} \qquad (2)$$

This suggests that if we repeat steps 1,2,3,4 we are bound to find the desired X and Y.

How good is the approximation in (2)? This is easily verified by computing

$$E\Big[\Big| \sum_{j=1}^{n} y_j a_{i,j} \Big|\Big] \; = \; \sum_{i=0}^{n} \binom{n}{i} |2i - n| \times \frac{1}{2^n}$$

Problem 2

Given a set of n integers

$$\mathbf{S} = \{\mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_n}\}$$

construct a "*sum free*" subset $\mathbf{T}$ of cardinality $\mathbf{n/3}$.

(Erdos proved that such a subset exists)

Algorithm:

Step 1 Produce a Random subset $\mathbf{T} \subseteq \mathbf{S}$

Step 2 Check if T is sum free.

Repeat until you find it.

Note: It is good to test if $\mathbf{S}$ itself is sum free. If so we are done.

Computer experimentation with this algorithm reveals two interesting facts:

a) The overwhelming majority of cardinality $n/3$ subsets are sum free.

b) If the integers are all positive then any subset of the largest $n/2$ elements of S is very likely to be sum free.

Note : Define for a given $\mathbf{S}$

$$\mathbf{i_0} = \max\{\mathbf{i} \; : \; \text{there exists } k > j > i \text{ such that } r_i + r_j = r_k\}$$

By choosing at random n distinct positive integers we could not find an example for which $\mathbf{i_0} > \mathbf{n/2}$

next

On 2-Colorings of the edges of Kn

Theorem

For each $n > 1$ there exists a 2-coloring of the edges of $\mathbf{K_n}$ with no monochromatic $\mathbf{a}$-cliques,

provided

$$a \geq 2Log_2[n]$$

Proof

Let $\mathcal{C}$ be a random 2-coloring of the edges of $\mathbf{K_n}$ and set

$$N(\mathcal{C}) \;=\; \sum_{T \subseteq [1,n] \& |T|=a} \chi(K_T \text{ is monochromatic })$$

Under a random 2-coloring $\mathcal{C}$ the probability that the clique with vertices in T is monochromatic is

$$2 \times \frac{1}{2^{\binom{|T|}{2}}}$$

Thus the expected number of monochromatic a-cliques is

$$E[N(\mathcal{C})] \;=\; \binom{n}{a} \times \frac{2}{2^{\binom{a}{2}}}$$

In particular it follows that

$$P[N(\mathcal{C}) \geq 1] \leq \binom{n}{a} \times \frac{2}{2^{\binom{a}{2}}}$$

Now it can be shown that $\binom{n}{a} \times \frac{2}{2^{\binom{a}{2}}} < 1$ when $a > 2Log_2[n]$. For such choices of a we have

$$1 - P[N(\mathcal{C}) \geq 1] = P[N(\mathcal{C}) = 0] > 0$$

Q.E.D.

Problem 3

For a given n construct the 2-coloring of the edges of $\mathbf{K_n}$ guaranteed by the previous theorem

Algorithm:

Step 1 Produce a Random 2-coloring of the edges of $\mathbf{K_n}$

Step 2 Count the number of Monochromatic a-cliques.

Repeat until you get a zero count

The previous theorem guarantees that you will find the desired coloring when $a \geq 2Log_2[n]$

Note: In the following table, under each $10 \leq n \leq 30$, I placed the smallest a such that

$$\binom{n}{a} 2 \times \frac{1}{2^{\binom{a}{2}}} < 1$$

and under it I placed the ceiling of $2Log_2[n]$:

10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
5	5	6	6	6	6	6	6	7	7	7	7	7	7	7	7	7	7	8	8	8
7	7	7	7	8	8	8	8	8	8	9	9	9	9	9	9	9	10	10	10	10

In the table below, under each $1 \leq a \leq 8$, we have the value of $\binom{20}{a} \frac{2}{2^{\binom{a}{2}}}$

1	2	3	4	5	6	7	8
40.	190.	285.	151.406	30.2813	2.36572	0.0739288	0.00093855

In particular we see that in a random 2-coloring of the edges of $\mathbf{K_{20}}$

we should expect to find about 30 monochromatic 5-cliques.

next