

Seeking for n! Derivatives

↓ \$1,000\$ ↑

Reward

A remarkable determinant

(1,0)	(1,1)
(0,0)	(0,1)



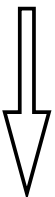
$$\Delta_{2,2}(X, Y) = \det$$

$$\begin{pmatrix} x_1^0 y_1^0 & x_2^0 y_2^0 & x_3^0 y_3^0 & x_4^0 y_4^0 \\ x_1^0 y_1^1 & x_2^0 y_2^1 & x_3^0 y_3^1 & x_4^0 y_4^1 \\ x_1^1 y_1^0 & x_2^1 y_2^0 & x_3^1 y_3^0 & x_4^1 y_4^0 \\ x_1^1 y_1^1 & x_2^1 y_2^1 & x_3^1 y_3^1 & x_4^1 y_4^1 \end{pmatrix}$$

$$= \det$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \\ x_1 y_1 & x_2 y_2 & x_3 y_3 & x_4 y_4 \end{pmatrix}$$

(2,0)		
(1,0)	(1,1)	(1,2)
(0,0)	(0,1)	(0,2)



$$\Delta_{331}(X, Y) = \det$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ y_2 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ x_1 y_1 & x_2 y_2 & x_3 y_3 & x_4 y_4 & x_5 y_5 & x_6 y_6 & x_7 y_7 \\ x_1 y_1^2 & x_2 y_2^2 & x_3 y_3^2 & x_4 y_4^2 & x_5 y_5^2 & x_6 y_6^2 & x_7 y_7^2 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 & x_7^2 \end{pmatrix}$$

General definition

If $(p_1, q_1), (p_2, q_2), \dots, (p_n, q_n)$ are the cells of the Ferrers diagram of $\mu \vdash n$ then

$$\Delta_\mu(X, Y) = \det \| x_j^{p_i} y_j^{q_i} \|_{i,j=1}^n$$

As a starter

Theorem (easy)

For any $\mu \vdash n$ the dimension of the linear span of the derivatives of $\Delta_\mu(X, Y)$ is at most $n!$

In symbols

$$\dim M_\mu[\mathbf{X}, \mathbf{Y}] \leq n!$$

where

$$M_\mu[X, Y] = L \left[\delta_{x_1}^{r_1} \delta_{y_1}^{s_1} \delta_{x_2}^{r_2} \delta_{y_2}^{s_2} \cdots \delta_{x_n}^{r_n} \delta_{y_n}^{s_n} \Delta_\mu(X, Y) : r_i, s_i \geq 0 \right]$$

For example [using MAPLE]

```
D21:=det([");
```

$$\begin{bmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{bmatrix}$$

```
D21:=det([");
```

$$D21 := y_2 x_3 - y_3 x_2 - y_1 x_3 + y_1 x_2 + x_1 y_3 - x_1 y_2$$

```
diff(D21, x1);
```

$$y_3 - y_2$$

```
diff(D21, x3);
```

$$y_2 - y_1$$

```
diff(D21, y1);
```

$$-x_3 + x_2$$

```
diff(D21, y3);
```

$$-x_2 + x_1$$

```
diff(D21, x3, y2);
```

$$1$$

6 independent derivatives!

Could it be ??????

next

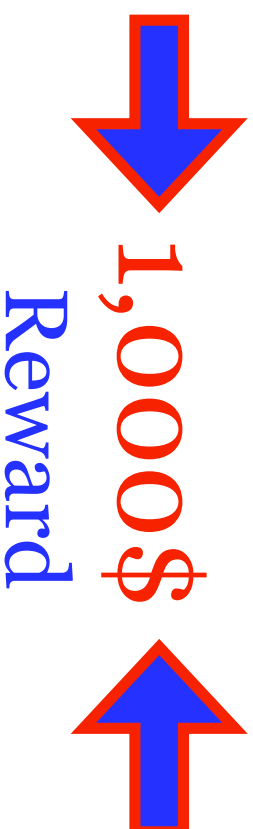
The $n!$ Conjecture

(1990 - 2000)

For $\mu \vdash n$

$$\dim M_\mu[X, Y] = n!$$

Proved by Mark Haiman using algebraic geometry



for an “elementary” proof

“Elementary” means:

By calculus or/and combinatorics
give an algorithm

that produces a “triangular” set of $n!$ derivatives

The basic construction

from a tableau to a point in $2n$ dimensional space

α_3	6		
α_2	7	1	4
α_1	2	5	3



$(\alpha_2 \alpha_1 \alpha_1 \alpha_2 \alpha_1 \alpha_3 \alpha_2 \beta_2 \beta_1 \beta_3 \beta_3 \beta_2 \beta_1 \beta_1)$
 $x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \ x_7 \ y_1 \ y_2 \ y_3 \ y_4 \ y_5 \ y_6 \ y_7$

some polynomials vanishing at this orbit point

$$x_1 - \alpha_2 \quad x_4 - \alpha_2 \quad x_7 - \alpha_2 \quad y_2 - \beta_1 \quad y_5 - \beta_2$$

Polynomials vanishing at all these tableau points:

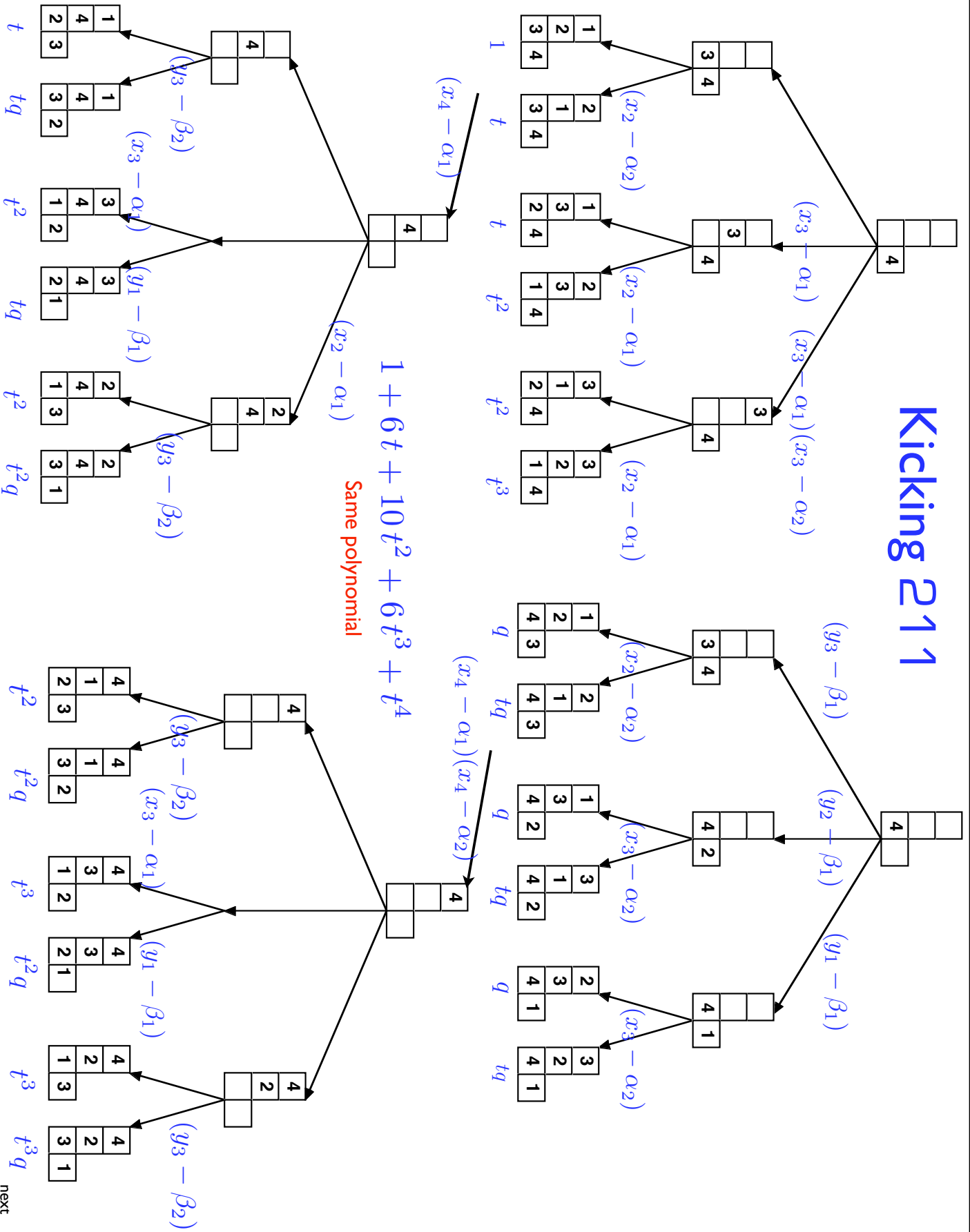
$$(x_2 - \alpha_1)(x_2 - \alpha_2)(x_2 - \alpha_3)$$

$$(y_5 - \beta_1)(y_5 - \beta_2)(y_5 - \beta_3)$$

$$(x_3 - \alpha_1)(x_3 - \alpha_2)(y_3 - \beta_1)$$

next

Kicking 211



next

The Algorithm

Step 1

Fill the Ferrers diagram of μ with $1, 2, \dots, n$ in all possible ways to get the $n!$ tableaux.

Step 2

Construct the corresponding orbit points

5040

In this case

3		
6	1	2
4	7	5

Step 4

Compute the kicking statistics.

1
12
75
306
807
1319
1319
807
306
75
12
1

Theorem

If the kicking statistic are symmetric then the top components of the kicking polynomials

give the $n!$ independent derivatives of $\Delta_\mu(X, Y) = \det \left\| x_j^{p_i} y_j^{q_i} \right\|_{i,j=1}^n$

MAGICS?

No

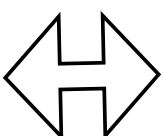
Only Algebraic Combinatorics

Triangularity

A typical term order

The degree-Lexicographic order

$$x^P = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} <_{\text{dlex}} x^Q = x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n}$$



$$\text{degree}(x^P) < \text{degree}(x^Q) \quad \text{or} \quad \text{degree}(x^P) = \text{degree}(x^Q) \quad \text{and} \quad P <_{\text{lex}} Q$$

Where we have $P <_{\text{lex}} Q$ if and only if for some $1 \leq i \leq n-1$

$$P_1 = Q_1, P_2 = Q_2, \dots, P_i = Q_i, \quad \text{and} \quad P_{i+1} < Q_{i+1}$$

The “*Leading monomial*” in a polynomial $P = \sum_p c_p x^p$

is the dlex highest monomial in P

Definition

In a vector space V of polynomials a collection $\mathcal{C} = \{P_1, P_2, \dots, P_m\}$ is said to be “*triangular*” if

$$\text{ldm}(P_1) <_{\text{dlex}} \text{ldm}(P_2) <_{\text{dlex}} \cdots <_{\text{dlex}} \text{ldm}(P_m)$$

Note:

The polynomials in a triangular collection are necessarily independent!

A basic tool

Let V be a vector space of polynomials and suppose that $\dim V \leq d$ then $\dim V = d$ if and only if we can find in V

a collection of d polynomials with distinct leading monomials.

Theorem

The vandermonde determinant $\Delta_n(x_1, x_2, \dots, x_n)$ has $n!$ independent derivatives

Proof

$$\Delta_n(x_1, x_2, \dots, x_n) = \det \|x_i^{n-j}\|_{i,j=1}^n = x_1^{n-1} x_2^{n-2} x_3^{n-3} \dots x_n^{n-n} + \geq_{\text{dlex}} \dots$$

thus if

$$0 \leq a_1 \leq n-1, \quad 0 \leq a_2 \leq n-2, \quad 0 \leq a_3 \leq n-3, \quad \dots, \quad 0 \leq a_{n-1} \leq 1$$

then

$$\partial_{x_1}^{a_1} \partial_{x_2}^{a_2} \partial_{x_3}^{a_3} \dots \partial_{x_{n-1}}^{a_{n-1}} \Delta_n(x_1, x_2, \dots, x_n) = x_1^{n-1-a_1} x_2^{n-2-a_2} x_3^{n-3-a_3} \dots x_{n-1}^{1-a_{n-1}} + \geq_{\text{dlex}} \dots$$

because

$$x^p \leq_{\text{dlex}} x^q \implies x^{p-a} \leq_{\text{dlex}} x^{q-a}$$

Note:

This proves the $n!$ conjecture for $\lambda = 1^n$

(2,0)
(1,0)
(0,0)

 $\implies \Delta_{1^3}[X] = \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix}$

next

Hilbert series

A vector space V is called “*graded*” if and only if

$$V = H_0(V) \oplus H_1(V) \oplus H_2(V) \oplus \dots \oplus H_m(V) \oplus \dots$$

The subspace “ $H_m(V)$ ” is called the “ m^{th} homogeneous component” of V .

its elements are called homogeneous of degree m

If $\dim H_m(V) < \infty$ for all m , we set

$$F_V(t) = \sum_{m \geq 0} t^m \dim H_m(V)$$

the “Hilbert series” of V

For instance for $R = \mathbb{Q}[X_1, X_2, \dots, X_n]$ we have $R = H_0(R) \oplus H_1(R) \oplus H_2(R) \oplus \dots$ with

$$H_m(R) = L[x_1^{p_1} x_2^{p_2} \dots x_n^{p_n} : p_1 + p_2 + \dots + p_n = m]$$

In this case

$$\dim H_m(R) = \binom{m+n-1}{n-1} \quad \text{and} \quad F_R(t) = \frac{1}{(1-t)^n}$$

Our spaces $M_\mu[X, Y]$ are “*bigraded*” that is we have the double decomposition

$$M_\mu[X, Y] = \bigoplus_{r=0}^{n(\mu)} \bigoplus_{s=0}^{n(\mu')} H_{r,s}(M_\mu[X, Y])$$

With $H_{r,s}(M_\mu[X, Y])$ the linear span of derivatives of $\Delta_\mu(x, y)$

that are homogeneous of degree \mathbf{r} in x_1, x_2, \dots, x_n and degree \mathbf{S} in y_1, y_2, \dots, y_n

Here and after we set

$$F_\mu(\mathbf{q}, t) = \sum_{\mathbf{r}=0}^{n(\mu)} \sum_{\mathbf{s}=0}^{n(\mu')} t^{\mathbf{r}} q^{\mathbf{s}} \dim H_{\mathbf{r},\mathbf{s}}(M_\mu[X, Y])$$

next

Some Hilbert series

Theorem

If V is graded and B is a homogeneous basis of V then

$$F_V(t) = \sum_{b \in B} t^{\text{degree}(b)}$$

Using this we get for the linear span of derivatives of the Vandermonde

$$F_{1^n}(t) = (1+t)(1+t+t^2)\cdots(1+t+\cdots+t^{n-1}) = [n]_t!$$

Using Maple

`hilb([3,2]);`

$$q^4 t^4 + 4 q^4 t + 4 q^3 t^2 + 5 q^4 + 15 q^3 t + 9 q^2 t^2 + 11 q^3 + 22 q^2 t + 11 q t^2 + 9 q^2 + 15 q t + 5 t^2 + 4 q + 4 t + 1$$

The dimension of

$$H_{2,1}(M_{3,2}[X, Y])$$

The dimension of $H_{1,3}(M_{3,2}[X, Y])$

The determinant

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 \\ x_1 y_1 & x_2 y_2 & x_3 y_3 & x_4 y_4 & x_5 y_5 \end{bmatrix}$$

has 15 independent bihomogeneous derivatives of degree **1** in x_1, x_2, \dots, x_5 and degree **3** in y_1, y_2, \dots, y_5 .

`hilb([2,2,1]);`

$$q^2 t^4 + 4 q^2 t^3 + 4 q t^4 + 9 q^2 t^2 + 15 q t^3 + 5 t^4 + 11 q^2 t + 22 q t^2 + 11 t^3 + 5 q^2 + 15 q t + 9 t^2 + 4 q + 4 t + 1$$

$$\begin{bmatrix} 5 & 4 & 1 \\ 11 & 15 & 4 \\ 9 & 22 & 9 \\ 4 & 15 & 11 \\ 1 & 4 & 5 \end{bmatrix}$$

next

Flip

Definition

We say that a vector space V of polynomials is a "cone " if it is the linear span of derivatives of a single homogeneous polynomial $\Delta(\mathbf{x})$.

In this case V has an automorphism "Flip " defined by

$$\text{flip } P(\mathbf{x}) = P(\partial_{\mathbf{x}})\Delta(\mathbf{x})$$

Note this is a non-singular linear map of V onto V since if

$$P(\mathbf{x}) = Q(\partial_{\mathbf{x}})\Delta(\mathbf{x}) \quad \text{and} \quad P(\partial_{\mathbf{x}})\Delta(\mathbf{x}) = 0$$

then

$$P(\partial_{\mathbf{x}})P(\mathbf{x}) = P(\partial_{\mathbf{x}})Q(\partial_{\mathbf{x}})\Delta(\mathbf{x}) = Q(\partial_{\mathbf{x}})P(\partial_{\mathbf{x}})\Delta(\mathbf{x}) = 0$$

Thus

$$\text{flip } P(\mathbf{x}) = 0 \implies P(\partial_{\mathbf{x}})P(\mathbf{x}) = 0 \implies P(\mathbf{x}) = 0$$

Note that this implies that if degree $\Delta(\mathbf{x}) = n_0$ then

$$F_V(t) = t^{n_0} F_V(1/t)$$

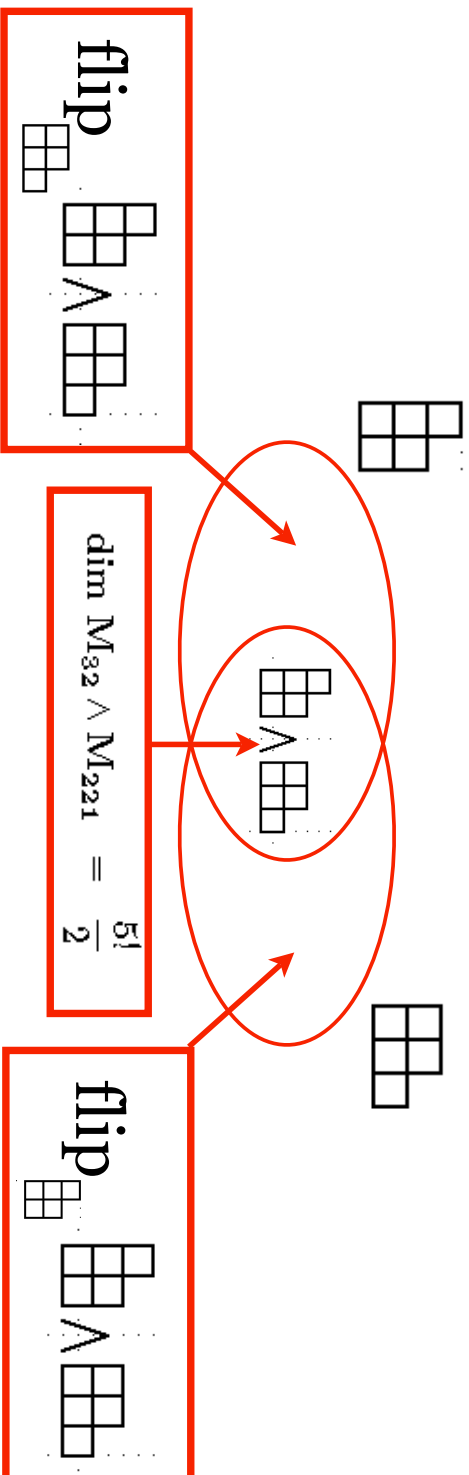
The same is true if V is a cone with summit a bihomogeneous polynomial $\Delta(\mathbf{x}, \mathbf{y})$ of bi-degree n_x, n_y In this case

$$F_V(\mathbf{q}, t) = t^{n_x} \mathbf{q}^{n_y} F_V(1/\mathbf{q}, 1/t)$$

This explains the symmetry

$$F_{32}(\mathbf{q}, t) = \begin{bmatrix} 5 & 11 & 9 & 4 & 1 \\ 4 & 15 & 22 & 15 & 4 \\ 1 & 4 & 9 & 11 & 5 \end{bmatrix}$$

Now the miracles



$$M_{221} = M_{32} \wedge M_{221} \oplus \text{flip}_{221} M_{32} \wedge M_{221}$$

$$M_{32} = M_{221} \wedge M_{32} \oplus \text{flip}_{32} M_{221} \wedge M_{32}$$

$$\dim M_{32} \wedge M_{221} = \frac{5!}{2}$$

$\frac{n!}{2}$ Conjecture (still open!)

If $\alpha, \beta \vdash n$ differ only by the position of one corner cell then $\dim M_\alpha \wedge M_\beta = \frac{n!}{2}$

SF Conjecture (still open!)

If $\alpha, \beta \vdash n$ differ only by the position of one corner cell then

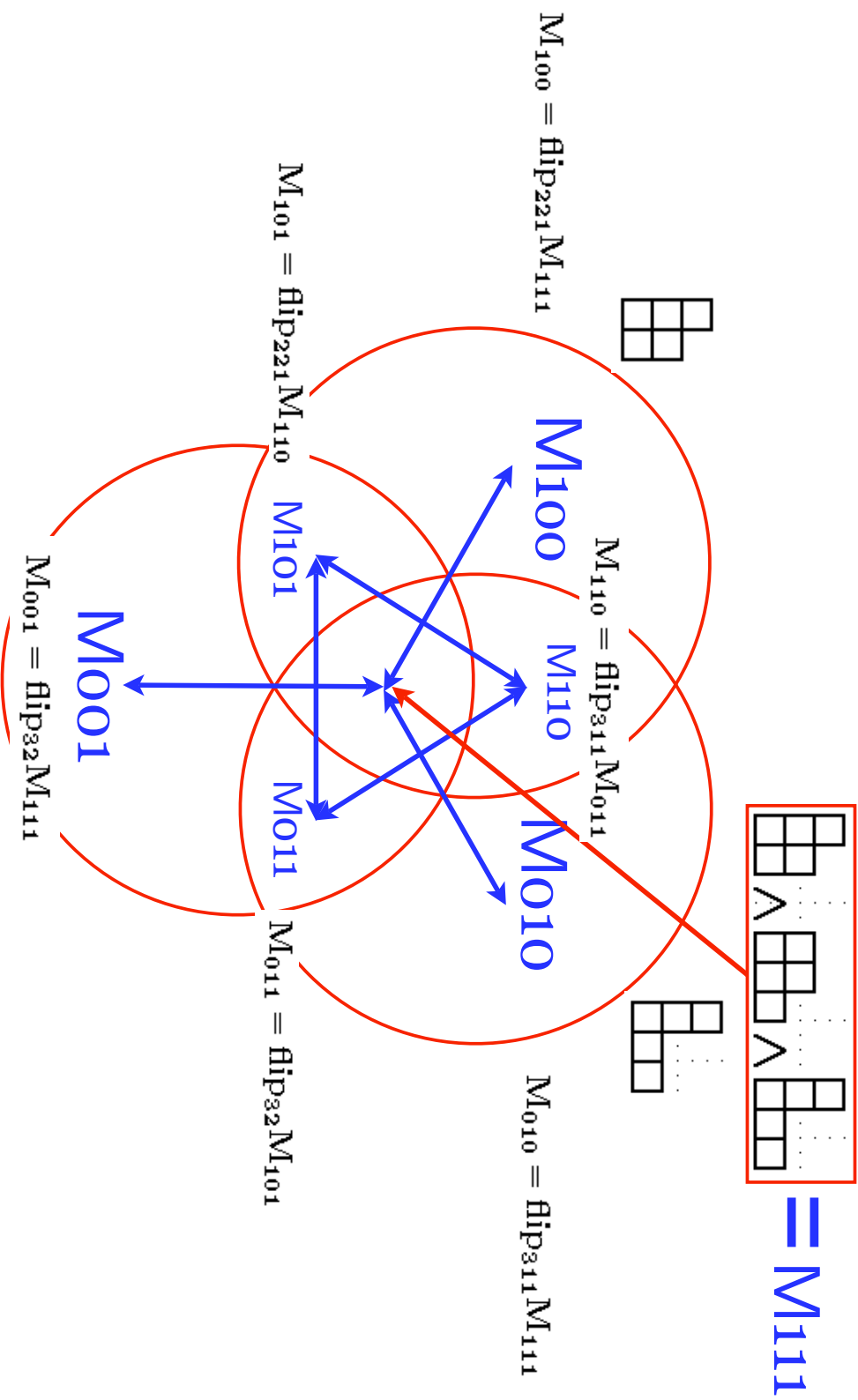
$$M_\alpha = M_\alpha \wedge M_\beta \oplus \text{flip}_\alpha M_\alpha \wedge M_\beta, \quad M_\beta = M_\alpha \wedge M_\beta \oplus \text{flip}_\beta M_\alpha \wedge M_\beta$$

This implies

$$F_{M_\alpha \wedge M_\beta}(\mathbf{q}, t) = \frac{T_\beta F_{M_\alpha}(\mathbf{q}, t) - T_\alpha F_{M_\beta}(\mathbf{q}, t)}{T_\alpha - T_\beta}$$

with $T_\mu = t^{n(\mu)} q^{n(\mu')}$ and $n(\mu) = \sum_i (i-1)\mu_i$

More miracles



$\frac{n!}{k!}$ Conjecture (still open!)

If $\alpha_1, \alpha_2, \dots, \alpha_k \vdash n$ differ only by one cell from each other then

$$\dim M_{\alpha_1} \wedge M_{\alpha_2} \wedge \dots \wedge M_{\alpha_k} = \frac{n!}{k!}$$

Etc Etc Etc

next

Some facts about ideals of polynomials

Definition

We say that x^p contains x^q if and only if $p_1 \geq q_1, p_2 \geq q_2, \dots, p_n \geq q_n$

Gordan:

Under the containment partial order every collection of monomials in x_1, x_2, \dots, x_n has a finite number of minimal elements

Let $J \subseteq \mathbb{Q}[x_1, x_2, \dots, x_n]$ be an ideal and let $M(J)$ the collection of all the $f_1, f_2, \dots, f_N \in J$ be such that s of J

let m_1, m_2, \dots, m_N be the minimal elements of $M(J)$ and

$\text{leadmon}(f_i) = m_i$

then

$$J = (f_1, f_2, \dots, f_N)$$

Let B_J be the collection of monomials that do not contain any of m_1, m_2, \dots, m_N then B_J is a basis for the quotient

$$R_J = \mathbb{Q}[x_1, x_2, \dots, x_n]/J$$

For a polynomial $P(x)$ we will denote by $h(P)$ the homogeneous component of P of highest degree.

Define

$$\text{gr } J = (h(P) : P \in J)$$

and

$$\text{gr } R_J = \mathbb{Q}[x_1, x_2, \dots, x_n]/\text{gr } J$$

we have:

B_J is also basis for $\text{gr } R_J$.

Indeed $M(J) = M(\text{gr } J)$ and

$$\text{gr } J = (h(f_1), h(f_2), \dots, h(f_N))$$

In particular

$$\dim R_J = \dim \text{gr } R_J$$

Orthogonal complements of Ideals

Definitions:

- (1) An ideal is called "homogeneous" if it is generated by homogeneous polynomials
- (2) For $P, Q \in \mathbb{Q}[x_1, x_2, \dots, x_n]$ set

$$\langle P, Q \rangle = P(\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n})Q(x_1, x_2, \dots, x_n) \Big|_{x_1, x_2, \dots, x_n=0}$$
- (3) For any homogeneous ideal J set

$$J^\perp = \{Q : \langle P, Q \rangle = 0 \ \forall \ P \in J\}$$

Theorem

For J homogeneous

- 1) $J^\perp = \{Q : P(\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n})Q(x_1, x_2, \dots, x_n) = 0 \ \forall \ P \in J\}$
- 2) $(J^\perp)^\perp = J$
- 3) $\mathbb{Q}[x_1, x_2, \dots, x_n]/J$ and J^\perp have the same Hilbert series

Note:

For all $P \in J$ and all $p_1, p_2, \dots, p_n \geq 0$ we have

$$\partial_{x_1}^{p_1} \partial_{x_2}^{p_2} \dots \partial_{x_n}^{p_n} P(\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n})Q(x_1, x_2, \dots, x_n) \Big|_{x_1, x_2, \dots, x_n=0} = 0$$

and the assertion in 1) follows from Taylor's Theorem.

The assertion in 3) holds because the elements of J^\perp can be taken as representatives of the elements of $\mathbb{Q}[x_1, x_2, \dots, x_n]/J$.

ORBITS

Let G be a group of permutations of $1, 2, \dots, n$. Given an element $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) \in G$ and a vector $\mathbf{a} = (a_1, a_2, \dots, a_n)$ we set

$$\sigma \mathbf{a} = (a_{\sigma_1}, a_{\sigma_2}, \dots, a_{\sigma_n})$$

This given, we let

$$[\mathbf{a}]_G = \{\mathbf{b} = \sigma \mathbf{a} : \sigma \in G\}$$

and call it the “*orbit*” of \mathbf{a} under G . The subgroup

$$G_{\mathbf{a}} = \{\sigma \in G : \sigma \mathbf{a} = \mathbf{a}\}$$

is called the “*stabilizer*” of \mathbf{a} .

If $G_{\mathbf{a}}$ is trivial then \mathbf{a} is called “*regular*” and the orbit $[\mathbf{a}]_G$ has cardinality $|G|$.

In general we have

$$|[\mathbf{a}]_G| = |G|/|G_{\mathbf{a}}|$$

In any case we set

$$J_{[\mathbf{a}]_G} = (\mathbf{P} : \mathbf{P}(\mathbf{b}) = 0 \quad \forall \quad \mathbf{b} \in [\mathbf{a}]_G)$$

and

$$R_{[\mathbf{a}]_G} = \mathbb{Q}[X_1, X_2, \dots, X_n]/J_{[\mathbf{a}]_G}, \quad \text{gr } R_{[\mathbf{a}]_G} = \mathbb{Q}[X_1, X_2, \dots, X_n]/\text{gr } J_{[\mathbf{a}]_G}$$

We also set

$$H_{[\mathbf{a}]_G} = (\text{gr } J_{[\mathbf{a}]_G})^\perp$$

and call the elements of this space the “*orbit harmonics*”.

Now it follows that

$$\dim R_{[\mathbf{a}]_G} = \dim \text{gr } R_{[\mathbf{a}]_G} = \dim H_{[\mathbf{a}]_G} = |G|/|G_{\mathbf{a}}|$$

next

Conical Orbit Harmonics

(This is why symmetry proves the $n!$ result)

Theorem

Let a be a regular point. Set

$$n_o = \max_{\mathbf{b} \in \mathcal{B}} \deg_{\mathbf{a}} R_{|\mathbf{a}|_{\mathbf{G}}}(\mathbf{b})$$

Suppose that \mathcal{B} is a basis for $R_{|\mathbf{a}|_{\mathbf{G}}}$ with

$$\mathcal{B}^{\leq n_o} = \mathcal{B} \quad \text{and} \quad |\mathcal{B}^{\leq i}| = d_o + d_1 + \dots + d_i \quad (i = 0, 1, \dots, n_o)$$

Then the equalities

$$d_i = d_{n_o - i} \quad (i = 0, 1, \dots, n_o)$$

imply

- (i) $h(\mathcal{B}) = \{h(\mathbf{b}) : \mathbf{b} \in \mathcal{B}\}$ is a basis for $\text{gr } R_{|\mathbf{a}|_{\mathbf{G}}}$
- (ii) $F_{\text{gr } R_{|\mathbf{a}|_{\mathbf{G}}}}(\mathbf{q}) = F_{H_{|\mathbf{a}|_{\mathbf{G}}}}(\mathbf{q}) = \sum_{i=0}^{n_o} d_i \mathbf{q}^i$,
- (iii) $H_{|\mathbf{a}|_{\mathbf{G}}}$ is a cone.

Where $\Delta_\mu[X; Y]$ comes from

Step 1

Fill the Ferrers diagram of μ with $1, 2, \dots, n$ in all possible ways to get the $n!$ tableaux.

Step 2

Construct the corresponding orbit points

Theorem

Call $(a, b) = (a_1, a_2, \dots, a_n; b_1, b_2, \dots, b_n)$ the first point so constructed.

Then

(1) (a, b) together with all the other points are none other than the orbit

$$[(a, b)]_{S_n} = \{(\mathbf{a}_{\sigma_1}, \mathbf{a}_{\sigma_2}, \dots, \mathbf{a}_{\sigma_n}; \mathbf{b}_{\sigma_1}, \mathbf{b}_{\sigma_2}, \dots, \mathbf{b}_{\sigma_n}) : \sigma \in S_n\}$$

$$(2) \Delta_\mu[X, Y] \in H_{[(a, b)]_{S_n}}$$

(3) and since $H_{[(a, b)]_{S_n}}$ is closed under differentiation it follows that

$$M_\mu[X; Y] \subseteq H_{[(a, b)]_{S_n}}$$

This implies

$$\dim M_\mu[X; Y] \leq n!$$

with equality if and only if $H_{[(a, b)]_{S_n}}$ is a cone.

The Algorithm

Step 1

Fill the Ferrers diagram of μ with $1, 2, \dots, n$ in all possible ways to get the $n!$ tableaux.

Step 2

Construct the corresponding orbit points

5040

In this case

3		
6	1	2
4	7	5

Step 4

Compute the kicking statistics.

$$\begin{bmatrix} 1 \\ 12 \\ 75 \\ 306 \\ 807 \\ 1319 \\ 1319 \\ 807 \\ 306 \\ 75 \\ 12 \\ 1 \end{bmatrix}$$

Theorem

If the kicking statistic are symmetric then the top components of the kicking polynomials

give the $n!$ independent derivatives of $\Delta_{\mu}(X, Y) = \det \left\| x_j^{p_i} y_j^{q_i} \right\|_{i,j=1}^n$

MAGICS?

No

Only Algebraic Combinatorics

The Group Harmonics

Let J_G be the ideal generated by the homogeneous G -invariant polynomials and set
It is customary to call the elements of H_G “*G-Harmonics*” .

Note that if $P(\mathbf{x})$ is any homogeneous G -invariant polynomial then for any point
 $\mathbf{a} = (a_1, a_2, \dots, a_n)$ we have

$$P(\sigma\mathbf{a}) - P(\mathbf{a}) = 0 \quad (\forall \sigma \in G) \implies P(\mathbf{x}) - P(\mathbf{a}) \in J_{[\mathbf{a}]_G} \implies P(\mathbf{x}) \in \text{gr} J_{[\mathbf{a}]_G}$$

Thus

$$J_G \subseteq \text{gr} J_G \quad (2)$$

and taking orthogonal complements gives

$$H_{[\mathbf{a}]_G} \subseteq H_G \quad (3)$$

Now if \mathbf{a} is regular we have $|G| = |[a]_G| = \dim H_{[a]_G}$ and thus in full generality we must have

$$|G| \leq \dim H_G$$

It is interesting to know when equality holds

Can you guess?

Groups Generated by Reflections

Theorem

If G is a finite group of $n \times n$ matrices then any of the statements below implies the remaining ones.

- (A) $H_{[a]_G} = H_G$ for all regular points,
- (B) $\dim H_G = |G|$,
- (C) H_G is a cone with summit the product of the reflecting hyperplanes,
- (D) G is generated by reflexions

The classical example

The space H_{S_n} (Harmonics of S_n) is the linear span of derivatives of

$$DD_{1^n}(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

THE END