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**Some New Methods
In The Theory
of**

COHEN-MACCAULLAY ALGEBRAS

by

A. M. Garsia and N. Wallach

Notation

$$\mathbf{X}_n = \{x_1, x_2, \dots, x_n\}$$

$\mathbf{Q}[\mathbf{X}_n] \implies$ The algebra of polynomials in x_1, x_2, \dots, x_n

V a graded vector space

$\mathcal{H}_m(V) \implies$ the subspace of the homogeneous elements of degree m in V .

$$V = \mathcal{H}_0(V) \oplus \mathcal{H}_1(V) \oplus \mathcal{H}_2(V) \oplus \dots \oplus \mathcal{H}_m(V) \oplus \dots$$

The Hilbert series of V

$$F_V(\mathbf{q}) = \sum_{m \geq 0} \dim(\mathcal{H}_m(V)) \mathbf{q}^m$$

Basics

Let \mathbf{A} be a Finitely Generated Graded Algebra.

$$\text{Let } \mathbf{A} = \bigoplus_{m \geq 0} \mathcal{H}_m(\mathbf{A})$$

- (1) \mathbf{n}_1 = the order of “1” as a pole of the Hilbert series $\mathbf{F}_{\mathbf{A}}(\mathbf{q})$.
- (2) \mathbf{n}_2 = the maximum number of algebraically independent elements in \mathbf{A} .
- (3) \mathbf{n}_3 = the minimum number of elements $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n$ such that

$$\dim \mathbf{A}/(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n)_{\mathbf{A}} < \infty \quad (*)$$

Fact: $\mathbf{n}_1 = \mathbf{n}_2 = \mathbf{n}_3 = \mathbf{n}_{\mathbf{A}}$ = the “*Krull dimension*” of \mathbf{A}

If (*) holds with $\mathbf{n} = \mathbf{n}_{\mathbf{A}}$ then $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n$ are called a “*System Of Parameters*”. “**S.O.P.**” in brief.

More Basics

Let $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n \in \mathbf{A}$ be homogeneous of degrees d_1, d_2, \dots, d_n and suppose they constitute a **S.O.P** for \mathbf{A} . Let $\mathbf{B} = \{f_1, f_2, \dots, f_N\}$ be a basis for the quotient

$$\mathbf{A}/(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n)_{\mathbf{A}} < \infty$$

Then every $\mathbf{P} \in \mathbf{A}$ has an expansion of the form

$$\mathbf{P} = \sum_{i=1}^N f_i \mathbf{Q}_i(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n)$$

with $\mathbf{Q}_1, \mathbf{Q}_2, \dots, \mathbf{Q}_N \in \mathbf{Q}[Y_1, \dots, Y_n]$. In other words the collection $\{f_i \mathbf{q}_1^{P_1}, \mathbf{q}_2^{P_2}, \dots, \mathbf{q}_n^{P_n}\}_{1 \leq i \leq N}$

spans \mathbf{A} as a vector space. In particular it follows that

$$\mathbf{F}_{\mathbf{A}}(\mathbf{q}) \ll \frac{\sum_{i=1}^N \mathbf{q}^{\text{degree}(f_i)}}{(\mathbf{1} - \mathbf{q}^{d_1})(\mathbf{1} - \mathbf{q}^{d_2}) \cdots (\mathbf{1} - \mathbf{q}^{d_n})}$$

The Cohen Macaulay Property

If in the expansion

$$\mathbf{P} = \sum_{i=1}^N f_i \mathbf{Q}_i(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n)$$

\mathbf{P} uniquely determines the coefficients $\mathbf{Q}_i(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n)$ then the collection

$$\{f_i \mathbf{q}_1^{\mathbf{P}_1}, \mathbf{q}_2^{\mathbf{P}_2}, \dots, \mathbf{q}_n^{\mathbf{P}_n}\}_{1 \leq i \leq N}$$

is a vector space basis for \mathbf{A} . This holds true **if and only if** we have the equality

$$\mathbf{F}_{\mathbf{A}}(\mathbf{q}) = \frac{\sum_{i=1}^N \mathbf{q}^{\text{degree}(f_i)}}{(1 - \mathbf{q}^{\mathbf{d}_1})(1 - \mathbf{q}^{\mathbf{d}_2}) \cdots (1 - \mathbf{q}^{\mathbf{d}_n})}$$

Then \mathbf{A} is a free module over $\mathbf{Q}[\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n]$ of rank N and \mathbf{A} is said to be “Cohen-Macaulay”.

A useful criterion

Theorem A

Let $q_1, q_2, \dots, q_n \in \mathbf{A}$ be homogeneous of degrees d_1, d_2, \dots, d_n and an

S.O.P. for \mathbf{A} . Let

$$\dim \mathbf{A}/(q_1, q_2, \dots, q_n)_{\mathbf{A}} = \mathbf{N}$$

with basis $\mathbf{B} = \{f_1, f_2, \dots, f_{\mathbf{N}}\}$. Then the condition

$$\lim_{q \rightarrow 1} (1 - q^{d_1})(1 - q^{d_2}) \cdots (1 - q^{d_n}) F_{\mathbf{A}}(q) = \mathbf{N}$$

forces the equality

$$F_{\mathbf{A}}(q) = \frac{\sum_{i=1}^{\mathbf{N}} q^{\text{degree}(f_i)}}{(1 - q^{d_1})(1 - q^{d_2}) \cdots (1 - q^{d_n})}$$

yielding that \mathbf{A} is a free module over $\mathbf{Q}[q_1, q_2, \dots, q_n]$ of rank \mathbf{N} and therefore

\mathbf{A} is a Cohen-Macaulay algebra.

m-Quasi-Invariants

Denote by “ $s_{i,j}$ ” the transposition that interchanges x_i and x_j . This given, we set

$$\mathbf{QI}_m[\mathbf{X}_n] = \left\{ \mathbf{P}(x) \in \mathbf{Q}[\mathbf{X}_n] : \frac{(1 - s_{ij})\mathbf{P}(x)}{(x_i - x_j)^{2m+1}} \in \mathbf{Q}[\mathbf{X}_n] \ \forall \ 1 \leq i < j \leq n \right\}$$

Note that

$$\mathbf{Q}[\mathbf{X}_n] = \mathbf{QI}_0[\mathbf{X}_n] \supset \mathbf{QI}_1[\mathbf{X}_n] \supset \mathbf{QI}_2[\mathbf{X}_n] \supset \cdots \supset \mathbf{QI}_m[\mathbf{X}_n] \supset \cdots \supset \mathbf{QI}_\infty[\mathbf{X}_n] = \mathbf{SYM}[\mathbf{X}_n]$$

Note further that for all $1 \leq i < j \leq n$ we have

$$(1 - s_{ij})\mathbf{P}\mathbf{Q} = ((1 - s_{ij})\mathbf{P})\mathbf{Q} + (s_{ij}\mathbf{P})(1 - s_{ij})\mathbf{Q}$$

Thus each $\mathbf{QI}_m[\mathbf{X}_n]$ is an algebra and in fact also an \mathbf{S}_n -module.

r-Quasi-Symmetric

Recall the Florent Hivert “local” r-action of S_n

$$s_i x_i^a x_{i+1}^b = \begin{cases} x_i^a x_{i+1}^b & \text{if } a, b \geq r \\ x_i^b x_{i+1}^a & \text{otherwise} \end{cases} \tag{*}$$

here $s_i = s_{i,i+1}$. Since we have

$$s_i^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i \quad \forall |i - j| \geq 2$$

(*) defines an action of S_n called the “r-action.” Florent Hivert defines

$$\mathbf{r}\text{-QSym} = \left\{ P(\mathbf{x}) : P(\mathbf{x}) \text{ is invariant under the r-action} \right\}$$

We have a descending chain of algebras

$$\mathbf{Q}[X_n] \supset \mathbf{1}\text{-QSym}[X_n] \supset \mathbf{2}\text{-QSym}[X_n] \supset \dots \supset \mathbf{r}\text{-QSym}[X_n] \supset \dots \supset \mathbf{Sym}[X_n]$$

Note that $\mathbf{1}\text{-QSym}[X_n]$ is the space of Gessel’s “quasi-symmetric functions”.

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$\mathbf{r-QSym}[\mathbf{X}_n]$ is a free module over $\mathbf{Sym}[\mathbf{X}_n]$ for all $\mathbf{r} \geq 1$

(Conjectured by Florent Hivert)

(Still open inspite of some premature announcements)

Towards common Proof

Theorem

Let A be a finitely generated graded algebra such that

- (1) $A \subseteq \mathbf{Q}[\mathbf{X}_n]$
- (2) $q_1, q_2, \dots, q_n \in A$ homogeneous S.O.P. for $\mathbf{Q}[\mathbf{X}_n]$ of degrees d_1, d_2, \dots, d_n
- (3) We have $\Pi(\mathbf{x}) \in \mathbf{Q}[q_1, q_2, \dots, q_n]$ such that

$$\Pi(\mathbf{x})\mathbf{Q}[\mathbf{X}_n] \subseteq A$$

- (4) $\dim A/(q_1, q_2, \dots, q_n)_A \leq d = d_1 d_2 \dots d_n$

This given, A is free over $\mathbf{Q}[q_1, q_2, \dots, q_n]$ of rank d

Sketch of Proof

We begin by choosing a point $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)$ such that

- (1) $\Pi(\mathbf{a}) \neq \mathbf{0}$
- (2) The set $[\mathbf{a}] = \{ \mathbf{q}_1(\mathbf{x}) = \mathbf{q}_1(\mathbf{a}), \mathbf{q}_2(\mathbf{x}) = \mathbf{q}_2(\mathbf{a}), \dots, \mathbf{q}_n(\mathbf{x}) = \mathbf{q}_n(\mathbf{a}) \}$ has cardinality $\mathbf{d} = \mathbf{d}_1 \mathbf{d}_2 \dots \mathbf{d}_n$.

This given, we define

$$\mathbf{J}_{[\mathbf{a}]} = (\mathbf{P} \in \mathbf{A} : \mathbf{P}(\mathbf{x}) = \mathbf{0} \quad \forall \quad \mathbf{x} \in [\mathbf{a}])$$

and set

$$\mathbf{R}_{[\mathbf{a}]} = \mathbf{A} / \mathbf{J}_{[\mathbf{a}]}$$

We show next that

$$\dim \mathbf{R}_{[\mathbf{a}]} = \mathbf{d}$$

The dimension of $\mathbf{R}_{[\mathbf{a}]}$

If $[\mathbf{a}] = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_d\}$ construct $\Phi_i(\mathbf{x}) \in \mathbf{Q}[\mathbf{X}_n]$ so that

$$\Phi_i(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{b}_i \\ 0 & \text{if } \mathbf{x} = \mathbf{b}_j \text{ with } j \neq i \end{cases}$$

Set

$$\Psi_i(\mathbf{x}) = \Phi_i(\mathbf{x}) \frac{\Pi(\mathbf{x})}{\Pi(\mathbf{a})}$$

then $\Psi_i(\mathbf{x}) \in \mathbf{A}$ and

$$\Psi_i(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{b}_i \\ 0 & \text{if } \mathbf{x} = \mathbf{b}_j \text{ with } j \neq i \end{cases}$$

Moreover for all $\mathbf{P}(\mathbf{x}) \in \mathbf{A}$ we have

$$\mathbf{P}(\mathbf{x}) \equiv \sum_{i=1}^d \mathbf{P}(\mathbf{b}_i) \Psi_i(\mathbf{x}) \quad (\text{modulo } \mathbf{J}_{[\mathbf{a}]})$$

Thus $\{\Psi_i(\mathbf{x})\}_{i=1}^d$ is a basis for $\mathbf{R}_{[\mathbf{a}]}$ consequently $\dim \mathbf{R}_{[\mathbf{a}]} = d$

More Quotient Rings

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For $\mathbf{P}(\mathbf{x}) \in \mathbf{Q}[\mathbf{X}_n]$ let $\mathbf{h}(\mathbf{P})$ denote the homogeneous component of highest degree in $\mathbf{P}(\mathbf{x})$ and set

$$\mathbf{gr} \mathbf{J}_{[\mathbf{a}]} = (\mathbf{h}(\mathbf{P}) : \mathbf{P} \in \mathbf{J}_{[\mathbf{a}]})_{\mathbf{A}} \quad \text{and} \quad \mathbf{gr} \mathbf{R}_{[\mathbf{a}]} = \mathbf{A}/\mathbf{gr} \mathbf{J}_{[\mathbf{a}]}$$

We can easily prove that

$$\dim \mathbf{gr} \mathbf{R}_{[\mathbf{a}]} = \dim \mathbf{R}_{[\mathbf{a}]} = \mathbf{d}$$

Now note that

$$\mathbf{q}_i(\mathbf{x}) - \mathbf{q}_i(\mathbf{a}) \in \mathbf{J}_{[\mathbf{a}]} \implies \mathbf{q}_i(\mathbf{x}) \in \mathbf{gr} \mathbf{J}_{[\mathbf{a}]}$$

in particular

$$\mathbf{gr} \mathbf{J}_{[\mathbf{a}]} \supset (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n)_{\mathbf{A}}$$

and thus

$$\mathbf{d} = \dim \mathbf{A}/\mathbf{gr} \mathbf{J}_{[\mathbf{a}]} \leq \dim \mathbf{A}/(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n)_{\mathbf{A}}$$

but then hypothesis (4) gives the equality

$$\dim \mathbf{A}/(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n)_{\mathbf{A}} = \mathbf{d}$$

Conclusions

Hypothesis (3) gives

$$\frac{q^{\text{degree } \Pi(\mathbf{x})}}{(1 - q)^n} \ll \ll \mathbf{F}_A(q) \ll \ll \frac{1}{(1 - q)^n}$$

Thus

$$\lim_{q \rightarrow 1} (1 - q^{d_1})(1 - q^{d_2}) \cdots (1 - q^{d_n}) \mathbf{F}_A(q) = d_1 d_2 \cdots d_n = \mathbf{d}$$

and the Cohen-Macauliness of \mathbf{A} now follows from Theorem A.

It remains to verify that the hypotheses of Theorem B are satisfied for **Quasi-Invariants** as well as **r-Quasi-Symmetric**

- (1) Is trivially satisfied
- (2) In both cases we take $\mathbf{q}_i(\mathbf{x}) = \mathbf{e}_i(\mathbf{x})$
- (3) For $\mathbf{QI}_m[\mathbf{X}_n]$ we take $\Pi(\mathbf{x}) = \prod_{1 \leq i < j \leq n} (\mathbf{x}_i - \mathbf{x}_j)^{2m}$
- (3) For **r-Qsym** we take $\Pi(\mathbf{x}) = (\mathbf{x}_1 \mathbf{x}_2 \cdots \mathbf{x}_n)^r$

The Crucial Inequality

$$(4) \quad \dim A/(q_1, q_2, \dots, q_n)_A \leq d = d_1 d_2 \dots d_n$$

For $\mathbf{1}\text{-}\mathbf{Q}\text{s}\text{ym}$ the proof took a considerable effort.

For $\mathbf{r}\text{-}\mathbf{Q}\text{s}\text{ym}$ when $\mathbf{r} > 1$ it is an open problem.

For $\mathbf{QI}_m[\mathbf{X}_n]$ it is not difficult and we can give here a brief sketch of the argument.

Recall that if we set for two polynomials $P, Q \in \mathbf{Q}[X_n]$

$$\langle P, Q \rangle = P(\partial_x)Q(x) \Big|_{x=0}$$

we get a non degenerate bilinear form. Now there is an \mathbf{m} -deformation of this form which is non degenerate on $\mathbf{QI}_m[\mathbf{X}_n]$.

The m-Bilinear form

For any homogeneous polynomial $\mathbf{P}(\mathbf{x})$ of degree \mathbf{d} we have

$$\mathbf{P}(\partial_{\mathbf{x}}) = \frac{1}{2^{\mathbf{d}}\mathbf{d}!} \sum_{\mathbf{k}=0}^{\mathbf{d}} \binom{\mathbf{d}}{\mathbf{k}} (-1)^{\mathbf{k}} \Delta^{\mathbf{d}-\mathbf{k}} \mathbf{P}(\mathbf{x}) \Delta^{\mathbf{k}} \quad \left(\Delta = \sum_{i=1}^n \partial_{x_i}^2 \right)$$

Define

$$\mathbf{L}_m = \Delta - 2m \sum_{1 \leq i < j \leq n} \frac{1}{x_i - x_j} (\partial_{x_i} - \partial_{x_j})$$

and for each $\mathbf{P} \in \mathbf{QI}_m[\mathbf{X}_n]$, homogeneous of degree \mathbf{d} set

$$\gamma^{\mathbf{P}}(\mathbf{x}, \partial_{\mathbf{x}}) = \frac{1}{2^{\mathbf{d}}\mathbf{d}!} \sum_{\mathbf{k}=0}^{\mathbf{d}} \binom{\mathbf{d}}{\mathbf{k}} (-1)^{\mathbf{k}} \mathbf{L}_m^{\mathbf{d}-\mathbf{k}} \mathbf{P}(\mathbf{x}) \mathbf{L}_m^{\mathbf{k}}$$

For $\mathbf{P}, \mathbf{Q} \in \mathbf{QI}_m[\mathbf{X}_n]$ we set $\langle \mathbf{P}, \mathbf{Q} \rangle_m = \gamma^{\mathbf{P}}(\mathbf{x}, \partial_{\mathbf{x}}) \mathbf{Q}(\mathbf{x}) \Big|_{\mathbf{x}=0}$

The m-Harmonics

Using the m-bilinear form define “*the m-Harmonics of [a]*” by

$$\mathbf{H}_{[a]}(\mathbf{m}) = (\mathbf{gr} \mathbf{J}_{[a]})^{\perp \mathbf{m}}$$

and the “*the m-Harmonics of S_n* ” by

$$\mathbf{H}_{S_n}(\mathbf{m}) = (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)^{\perp \mathbf{m}}$$

we have that $(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n) \subseteq \mathbf{gr} \mathbf{J}_{[a]} \implies \mathbf{H}_{[a]}(\mathbf{m}) \subseteq \mathbf{H}_{S_n}(\mathbf{m})$

and the non degeneracy of the m-bilinear form gives

$$\mathbf{n}! = \dim \mathbf{gr} \mathbf{R}_{[a]} = \dim \mathbf{H}_{[a]}(\mathbf{m}) \leq \dim \mathbf{H}_{S_n}(\mathbf{m}) = \dim \mathbf{QI}_{\mathbf{m}}[\mathbf{X}_n] / (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$$

so we only need to show that $\dim \mathbf{H}_{S_n}(\mathbf{m}) \leq \mathbf{n}!$

The Dimension Bound

It can be shown that

$$\mathbf{H}_{S_n}(\mathbf{m}) = \{ \mathbf{Q} \in \mathbf{QI}_m[\mathbf{X}_n] : \gamma_{\mathbf{e}_i}(\mathbf{x}, \partial_{\mathbf{x}})\mathbf{Q} = \mathbf{0} \text{ for } i = 1, 2, \dots, n \}$$

Now it follows from the definition that

$$\gamma_{\mathbf{e}_i}(\mathbf{x}, \partial_{\mathbf{x}}) = \mathbf{e}_i(\partial_{\mathbf{x}}) + \dots (\text{lower order terms})$$

From this it follows that every derivative of an m -Harmonic $\mathbf{Q}(\mathbf{x})$ at the point \mathbf{a} expressed in terms of the $n!$ derivatives

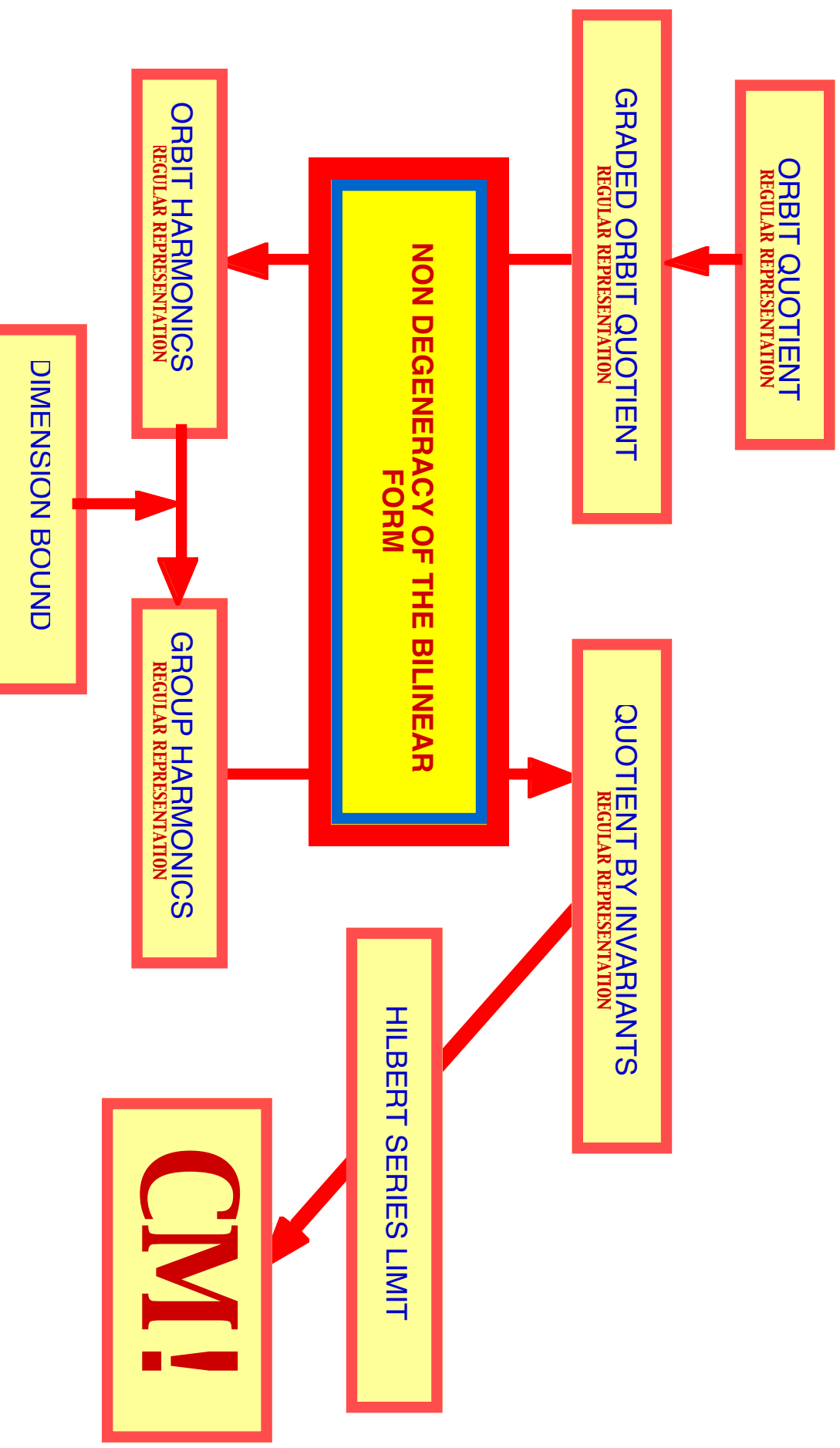
$$\left. \partial_{x_1}^{p_1} \partial_{x_2}^{p_2} \dots \partial_{x_n}^{p_n} \mathbf{Q}(\mathbf{x}) \right|_{\mathbf{x}=\mathbf{a}} \quad \text{for } \mathbf{0} \leq p_i \leq i - 1$$

from which it follows that $\dim \mathbf{H}_{S_n}(\mathbf{m}) \leq n!$

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