Name *Answer Key* **Student ID No.** ______________

Lecture B (MWF 1:25–2:15)
Circle the section in which you are enrolled on Thursdays in WLH 2115:

T.A.: Jason Bell	**T.A.: Rino Sanchez**
B01 12:20-1:10	B03 2:30-3:20
B02 1:25-2:15	B04 3:35-4:25

Do not open this booklet until instructed to do so.

Work alone.

You must show your work; if you type the formula $\sqrt{3^2+4^2}$ into your calculator and get the answer 5, you should write down $\sqrt{3^2+4^2}=5$. If relevant work is not shown, you may not get credit.

Use the space provided. If necessary, write "see other side" and continue working on the back of the same sheet.

Approved calculators are permitted. You are also allowed to have one sheet of notes made according to the specifications explained in class. No other resources are permitted.

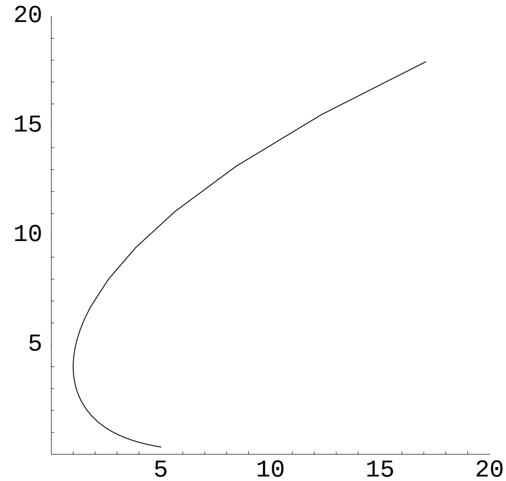
Circle your final answers when relevant.

Time for midterm: 50 minutes.

1	**25**	/25
2	**25**	/25
3	**25**	/25
4	**25**	/25
Total	**100**	/100

1. Both parts concern the parametric curve

$$x = e^t - t$$
$$y = 4e^{t/2}$$
$$-5 \le t \le 3$$



(a) Compute the velocity vector at (version P) $(x, y) = (e^2 - 2, 4e)$; (version Q) $(x, y) = (e - 1, 4e^{1/2})$; (version R) $(x, y) = (e^3 - 3, 4e^{3/2})$.

$$\frac{dx}{dt} = e^t - 1\ , \qquad \frac{dy}{dt} = 4e^{t/2} \cdot \frac{d}{dt}\left(\frac{t}{2}\right) = 4e^{t/2} \cdot \frac{1}{2} = 2e^{t/2}\ , \qquad \vec{v}(t) = \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} e^t - 1 \\ 2e^{t/2} \end{bmatrix}$$

Use the coordinates given to determine what value of t to plug in to this equation:

(version P) $e^t - t = e^2 - 2$ and $4e^{t/2} = 4e$ gives $t = 2$, so $\vec{v}(2) = \boxed{\begin{bmatrix} e^2 - 1 \\ 2e \end{bmatrix}}$

(version Q) $e^t - t = e - 1$ and $4e^{t/2} = 4e^{1/2}$ gives $t = 1$, so $\vec{v}(1) = \boxed{\begin{bmatrix} e - 1 \\ 2e^{1/2} \end{bmatrix}}$

(version R) $e^t - t = e^3 - 3$ and $4e^{t/2} = 4e^{3/2}$ gives $t = 3$, so $\vec{v}(3) = \boxed{\begin{bmatrix} e^3 - 1 \\ 2e^{3/2} \end{bmatrix}}$

(b) Compute the distance traveled along the curve in the given time interval. Fully evaluate the integral. *Hint: it involves the square root of a perfect square.*

Since the velocity vector is $\vec{v}(t) = \begin{bmatrix} e^t - 1 \\ 2e^{t/2} \end{bmatrix}$, the speed is

$$v(t) = |\vec{v}(t)| = \sqrt{(e^t - 1)^2 + (2e^{t/2})^2} = \sqrt{e^{2t} - 2e^t + 1 + 4e^t} = \sqrt{e^{2t} + 2e^t + 1} = (e^t + 1)\ .$$

(Common mistake: "$\sqrt{a^2 + b^2} = \sqrt{a^2} + \sqrt{b^2} = a + b$" is FALSE.)

The distance travelled along the curve is the integral of the speed between the starting and ending times:

$$\int_{-5}^{3} v(t)\, dt = \int_{-5}^{3} (e^t + 1)\, dt = \left(e^t + t\right)\Big|_{-5}^{3} = \boxed{e^3 - e^{-5} + 8}.$$

2. Let $f(x,y) = x^2 - 6x + 9 + y^2$ for all questions on this page.

(a) Compute

(i) $f_x(x,y) = \boxed{2x - 6}$

(ii) (version P) $f_x(7,5) = 2(7) - 6 = \boxed{8}$; (version Q) $f_x(8,3) = 2(8) - 6 = \boxed{10}$; (version R) $f_x(9,4) = 2(9) - 6 = \boxed{12}$

(iii) $\dfrac{\partial^2 f}{\partial y\,\partial x} = \dfrac{\partial}{\partial y}\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial y}(2x-6) = \boxed{0}$

(iv) $\dfrac{\partial}{\partial x}\left(f(x,y)^{100}\right) =$

$= 100 f^{99}\,\dfrac{\partial f}{\partial x} = \boxed{100(x^2 - 6x + 9 + y^2)^{99}\,(2x-6)}$

$= \boxed{200(x^2 - 6x + 9 + y^2)^{99}\,(x-3)} = \boxed{(200x - 600)\,(x^2 - 6x + 9 + y^2)^{99}}$

(b) Make a single contour map showing the level curves $f(x,y) = k$ for (version P) $k = 4$ and $k = 16$; (version Q) $k = 4$ and $k = 25$; (version R) $k = 16$ and $k = 25$. Include relevant axes, measurements, and coordinates. Do not include any other level curves. *Hint: $f(x,y)$ is the sum of two perfect squares.*

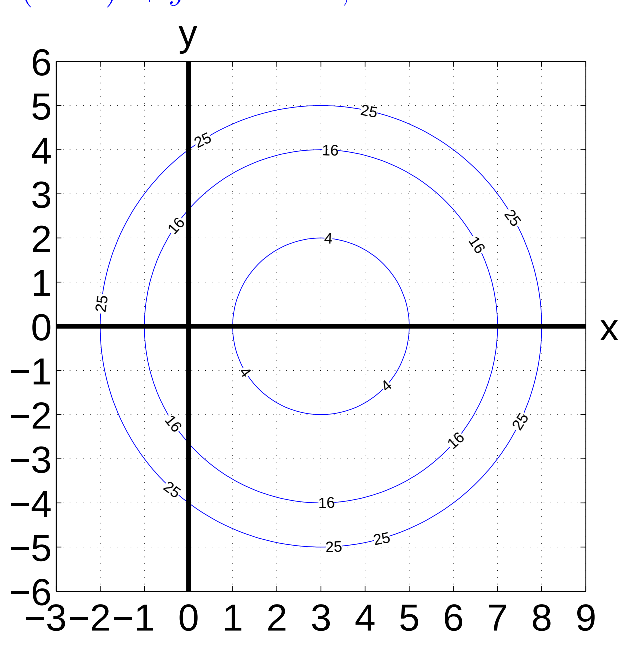
$x^2 - 6x + 9 + y^2 = k$ so $(x-3)^2 + y^2 = k$, a circle of radius $\sqrt{k}$ centered at $(3,0)$.

Each version of the test has two of the following on it:

$k = 4$ gives $(x-3)^2 + y^2 = 4 = 2^2$, a circle of radius 2 and center $(3,0)$.

$k = 16$ gives $(x-3)^2 + y^2 = 16 = 4^2$, a circle of radius 4 and center $(3,0)$.

$k = 25$ gives $(x-3)^2 + y^2 = 25 = 5^2$, a circle of radius 5 and center $(3,0)$.



3. Let $f(x,y) = x^2y^3$ for all questions on this page.

(a) Compute the directional derivative of $f(x,y)$ at the point $(2,-1)$ in the direction $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

$D_{\vec{u}}f = \vec{u} \cdot \nabla f$

The direction vector has length $\sqrt{5^2 + (-1)^2} = \sqrt{26}$.

A unit vector in that direction is $\vec{u} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \Big/ \left| \begin{bmatrix} 5 \\ -1 \end{bmatrix} \right| = \frac{1}{\sqrt{26}} \begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

The gradient of f is $\nabla f = \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{bmatrix} = \begin{bmatrix} 2xy^3 \\ 3x^2y^2 \end{bmatrix}$.

The gradient vector at this point is $\nabla f(2,-1) = \begin{bmatrix} 2(2)(-1)^3 \\ 3(2)^2(-1)^2 \end{bmatrix} = \begin{bmatrix} -4 \\ +12 \end{bmatrix}$.

The directional derivative is

$$D_{\vec{u}}f = \vec{u} \cdot \nabla f = \frac{5(-4) - 1(12)}{\sqrt{26}} = \boxed{-\frac{32}{\sqrt{26}}}$$

(b) At the point $(2,-1)$, determine a unit vector $\vec{u}$ pointing in the direction of maximum increase in f.

$\vec{u}$ should be a unit vector in the same direction as ∇f.

$|\nabla f| = \sqrt{(-4)^2 + 12^2} = \sqrt{16 + 144} = \sqrt{160} = 4\sqrt{10}$ so

$$\vec{u} = \frac{\nabla f}{|\nabla f|} = \boxed{\frac{1}{\sqrt{160}} \begin{bmatrix} -4 \\ 12 \end{bmatrix}} = \frac{1}{4\sqrt{10}} \begin{bmatrix} -4 \\ 12 \end{bmatrix} = \boxed{\frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 3 \end{bmatrix}}$$

Note: the maximum rate of increase is $|\nabla f| = \sqrt{160} = 4\sqrt{10}$.

(c) Suppose x and y are both functions of time t, and that at $t = 60$, we have $x = 2$, $y = -1$, (version P) $dx/dt = 2$; (version Q) $dx/dt = 6$; (version R) $dx/dt = 8$, and $dy/dt = 10$. Compute the value of df/dt at $t = 60$.

Apply the chain rule:

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ \text{(version P)} &= (-4)(2) + (12)(10) = -8 + 120 = \boxed{112} \\ \text{(version Q)} &= (-4)(6) + (12)(10) = -24 + 120 = \boxed{96} \\ \text{(version R)} &= (-4)(8) + (12)(10) = -32 + 120 = \boxed{88} \end{aligned}$$

4. Let $f(x,y) = x^3 + \cos(x^2 - y^2)$ for all questions on this page.

(a) When $x = 2$ and $y = -2$, what is $f(x,y)$?

$2^3 + \cos(2^2 - (-2)^2) = 8 + \cos(4-4) = 8 + \cos(0) = 8 + 1 = 9.$

(b) Compute the tangent plane to the surface $z = f(x,y)$ at the point $(2, -2, z_0)$ (where for z_0 you should plug in the correct answer to the previous part of the question).

The tangent plane formula is $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$. Here, $(x_0, y_0, z_0) = (2, -2, 9)$.

$$f_x(x,y) = 3x^2 - 2x\sin(x^2 - y^2)$$
$$f_x(2,-2) = 3(2)^2 - 2(2)\sin(2^2 - (-2)^2) = 3(4) + 4\sin(0) = 12$$
$$f_y(x,y) = 2y\sin(x^2 - y^2)$$
$$f_y(2,-2) = 2(-2)\sin(2^2 - (-2)^2) = -4\sin(0) = 0$$

so $\boxed{z - 9 = 12(x-2) + 0(y - (-2))}$.

This can be rearranged into various other ways of expressing planes: $z - 9 = 12x - 24 \quad \Rightarrow \quad z = 12x - 24 + 9 = 12x - 15 \quad \Rightarrow \quad \boxed{12x - z - 15 = 0}$ or $\boxed{12x - z = 15}$, etc.

(c) Use the linear approximation to $f(x,y)$ at $(2,-2)$ to estimate the value of $f(2.1, -2.3)$. *Hint: this makes use of the answer to part (b), and is* NOT *the same thing as computing $f(2.1, -2.3)$ on your calculator.*

Shortest way: $z - 9 = 12(x-2) = 12(2.1-2) = 12(.1) = 1.2$ so $z = 9 + 1.2 = \boxed{10.2}$

Longer way: replace z by $L(x,y)$ in the answer to (b), and then compute $L(2.1, -2.3)$. Here it is for two versions of the answer to (b):

$L(x,y) = 9 + 12(x-2) \quad \Rightarrow \quad L(2.1, -2.3) = 9 + 12(2.1 - 2) = 9 + 12(.1) = 9 + 1.2 = \boxed{10.2}$;

$L(x,y) = 12x - 15 \quad \Rightarrow \quad L(2.1, -2.3) = 12(2.1) - 15 = 25.2 - 15 = \boxed{10.2}$.

WARNING: the calculator answer is

$$2.1^3 + \cos(2.1^2 - (-2.3)^2) = 9.261 + \cos(-.88) \approx 9.898151144$$

if it's in radians mode, and ≈ 10.26088205 in degree mode. Both of these are incorrect answers.

Did you remember to

- **Put your name, student ID number, and section number on the front?**
- **Circle your final answers?**
- **Check your work?**

Exam booklets will be collected promptly when time is called.