Semi-primary Lattices and Tableau Algorithms

by

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Abstract

We develop the theory of semi-primary lattices, a class of modular lattices, including abelian subgroup lattices and invariant subspace lattices, in which an integer partition is assigned to every element and every interval. Flags in these lattices give rise to chains of partitions, which may be encoded as tableaux. In certain of these lattices, Steinberg and van Leeuwen respectively have shown that relative positions and co-types, which describe configurations of elements in flags, are generically computed by the well known Robinson-Schensted and evacuation algorithms on standard tableaux. We explore extensions of this to semi-primary lattices: we consider the nongeneric configurations, leading to nondeterministic variations of the Robinson-Schensted and evacuation tableau games, and consider exact and asymptotic enumeration of the number of ways to achieve certain configurations. We also introduce other configuration questions leading to new tableau games, and develop a number of deterministic and nondeterministic tableau operators that can be combined to describe the generic and degenerate configurations of flags undergoing various transformations.

We also look at similar problems in the class of modular lattices whose complemented intervals have height at most 2, such as Stanley's Fibonacci lattice $Z(r)$. Here the generic relative position is related to Fomin's analogue of the Robinson-Schensted correspondence in $Z(1)$.

Thesis Supervisor: Richard P. Stanley
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Notation

Symbol	Description	Page
$\star$	a hole in a tableau during a game	72
$\bullet$	in a tableau of shape λ/μ, the "entry" in cells of μ; also, a cell in a Ferrers diagram	20, 22
$\#S$	cardinality of set S	
$\emptyset$	empty partition, composition, sequence, or tableau	20, 21
$\hat{0}$	minimum element of poset	16
$\hat{1}$	maximum element of poset	16
$x \vee y$	least upper bound, or "join," of x and y	16
$x \wedge y$	greatest lower bound, or "meet," of x and y	16
x^*	inclusion of x into the dual lattice	17
$[x,y][k]$	join of all cycles in $[x,y]$ of rank $\leq k$; $x[k] = [\hat{0},x][k]$	35
$[x,y](k)$	meet of all cocycles in $[x,y]$ of corank $\leq k$; $x(k) = [x,\hat{1}](k)$	35
$\vec{x}[[\lambda]]$	$x_1[\lambda_1] \vee x_2[\lambda_2] \vee \cdots$	51
$\vec{x}((\lambda))$	$x_1(\lambda_1) \wedge x_2(\lambda_2) \vee \cdots$	51
$x \lessdot y$	cover relation: $x < y$ and nothing is between them	16
$x \mathrel{\dot{\leq}} y$	$x \lessdot y$ or $x = y$	16
$i \lessdot j$	$i = j - 1$ in $\mathbb{Z}$	16
$\mu \leq \lambda$	$\mu_i \leq \lambda_i$ for all $i > 0$	21
$\mu \leq_{\boxed{e}} \lambda$	$\mu \leq \lambda$, and fill λ/μ with e	75
$\mu \leq_{\vee} \lambda$	$\mu \leq \lambda$ and λ/μ is a vertical strip	56
$\mu \lessdot_{\lvert r \rvert} \lambda$	$\lambda'_r \gtrdot \mu'_r$ and $\lambda'_i = \mu'_i$ for $i \neq r$	77
$\mu \lessdot_x \lambda$	$\lambda \gtrdot \mu$ and $\lambda = \mu \oplus x$	75
$\lambda \vdash n$	$\sum_i \lambda_i = n$	20
$\mu \sqsubseteq \lambda$	subcolumn partition: $m_i(\mu') \leq m_i(\lambda')$ for $i > 0$	53
λ'	conjugate partition: $\lambda'_j = \#\{\, i : \lambda_i \geq j \,\}$	20
$\boldsymbol{A} + \boldsymbol{B}$	append $\boldsymbol{B}$ to $\boldsymbol{A}$ rowwise	60
$\lambda + \mu$	$(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots)$	21
$\lambda \cup \mu$	$(\lambda'_1 + \mu'_1, \lambda'_2 + \mu'_2, \ldots)'$	21
$\lambda \oplus \mu$	union of subsets of $\mathbb{P} \times \mathbb{P}$	21
$\lambda \ominus \mu$	set difference of subsets of $\mathbb{P} \times \mathbb{P}$	21
λ/μ	a skew partition or composition; an ordered pair with $\mu \leq \lambda$	21
$\lvert\lambda\rvert$	$\sum_i \lambda_i$	20
$\binom{n}{k}$	binomial coefficient $n!/(k!(n-k)!)$	
$\langle n \rangle$	$(q^n - 1)/(q - 1) = 1 + q + \cdots + q^{n-1}$	44
$\langle n \rangle!$	$\langle 1 \rangle \cdot \cdots \cdot \langle n \rangle$, $\langle 0 \rangle ! = 1$	44
Ax	join of all weak upper covers of x	25
$A_y x$	join of all weak upper covers of x in $[x,y]$	25
$\mathcal{A}_\lambda \mu$	partition with ith part $\min\{\mu_i + 1, \lambda_i\}$	52
ALR	Littlewood-Richardson sequence of an element	63
$\mathrm{arank}[x,y]$	atomic rank of $[x,y]$	30
$\mathcal{C}_\mu \lambda$	partition with ith part $\max\{\lambda_i - 1, \mu_i\}$	52
$\mathcal{C}\lambda$	$(\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_k - 1)$ where $\lambda_k > \lambda_{k+1} = 0$	40
Cy	meet of all weak lower covers of y	25

$C_x y$	meet of all weak lower covers of y in $[x, y]$	25
CLR	Littlewood-Richardson sequence of an element	63
$\mathrm{col}(\boldsymbol{P}, k)$	column of entry k in $\boldsymbol{P}$	23
$\mathrm{entry}(\boldsymbol{P}, c)$	entry at cell c of $\boldsymbol{P}$	64
$F^{\boldsymbol{P}}(q)$	number of flags of type $\boldsymbol{P}$	45
$f_\Lambda(q)$	number of flags with $\mathrm{itype}\, f = \Lambda$	89
$\Gamma_{\mathcal{L}}(x)$	complements to x in $\mathcal{L}$	49
$g^\lambda_{\mu\nu}(q)$	Hall polynomial	62
$\mathrm{high}(f)$	high index used in a flag or tableau	16, 23
λ	lowercase greek letters are partitions or compositions	20
Λ	interval type table, meet type table, A-statistics	73, 85, 131, 161, 173, 178
Λ'	transpose, dependent on structure	87, 131
$\mathcal{L}$	lattice	
$L(G)$	lattice of subgroups of G	24
$L(V, N)$	lattice of N-invariant subspaces of V	24
$\mathrm{LR}_*, \mathrm{LR}^*$	extremal Littlewood-Richardson sequences	67
$\mathrm{low}(f)$	low index used in a flag or tableau	16, 23
$M_\lambda(q)$	$\prod_i \langle m_i(\lambda)\rangle!$	46
$m_i(\lambda)$	multiplicity: number of parts of λ of size i	20
$\mathbb{N}$	$\{0, 1, 2, \ldots\}$	
$n(\lambda)$	$\sum_{i>0}(i-1)\lambda_i = \sum_i \binom{\lambda'_i}{2}$	21
$n(\lambda/\mu)$	$n(\lambda) - n(\mu)$	21
$n_0(\lambda)$	$\sum_{i>0} \binom{\lambda'_i - \lambda'_{i+1}}{2}$	46
$\mathrm{osh}\, \boldsymbol{P}$	outer shape of tableau $\boldsymbol{P}$	22
$O(q^{-1})$	a function whose product with q is bounded by a constant as $q \to \infty$	
$\mathbb{P}$	$\{1, 2, 3, \ldots\}$	
$\boldsymbol{P}$	uppercase bold italic letters are tableau	22
$\rho(x)$	rank of x	16
$\bar\rho(x)$	corank of x	16
$\mathrm{row}(c)$	row of cell c	23
$\mathrm{row}(\boldsymbol{P}, k)$	row of entry k in $\boldsymbol{P}$	23
$S(\lambda)$	straightening: arrange parts of λ in decreasing order	67
$S(\boldsymbol{P})$	straightening: arrange entries of each column in order, and remove gaps	67
$\mathrm{sh}\, \boldsymbol{P}$	skew shape of tableau $\boldsymbol{P}$	22
$\mathrm{cotype}\, x$	$\mathrm{type}[x, \hat{1}]$	34
$\mathrm{Atype}[x, y]$	$(\rho(A^{k-1}_y x, A^k_y x))_{k>0}$	28
$\mathrm{Ctype}[x, y]$	$(\rho(C^k_x y, C^{k-1}_x y))_{k>0}$	28
$\mathrm{ftype}\, f$	$(\mathrm{type}[f_l, f_l], \mathrm{type}[f_l, f_{l+1}], \ldots, \mathrm{type}[f_l, f_h])$	45
$\mathrm{ftype}_x\, f$	$(\mathrm{type}[x, f_l], \mathrm{type}[x, f_{l+1}], \ldots, \mathrm{type}[x, f_h])$	45
$\mathrm{itype}\, f$	$(\mathrm{type}[f_i, f_j])_{0 \le i \le j \le n}$	76
$\mathrm{itype}\, \boldsymbol{P}$	$(S(\lambda^{(j)} - \lambda^{(i)}))_{l \le i \le j \le h}$	102
$\mathrm{type}[x, y]$	type of interval in a semi-primary lattice	34
$\mathrm{type}\, x$	$\mathrm{type}[\hat{0}, x]$	34

vlen	vertical strip length	152
$\mathbb{Z}$	$\{0, \pm 1, \pm 2, \ldots\}$	

Fibonacci lattice notation.

$x \wedge y$	greatest lower bound in $Z(r)$	169
$x \vee y$	least upper bound in $Z(r)$	169
$x \leq_F y$	$x \leq y$ in $Fib(r)$	165
$x \leq_Z y$	$x \leq y$ in $Z(r)$	165
$x \lessdot_1 y$	$x = 2^k\alpha$ and $y = 2^k 1_i \alpha$ in $Z(r)$	165
$x \lessdot_2 y$	$x = 2^k 1_i \alpha$ and $y = 2^k 2\alpha$ in $Z(r)$	165
$x \lessdot_{1_*} y$	$y = x1$ in Z_*	172
$x \lessdot_{2_*} y$	$x = \alpha 11^m$ and $y = \alpha 21^m$ in Z_*	172
$\kappa^+(x)$	number of upper covers of x	165
$\kappa^-(x)$	number of lower covers of x	165
μ, ν, τ	decomposition of a pair of words	166
$Fib(r)$	one of the Fibonacci lattices	165
$Z(r)$	one of the Fibonacci lattices	165
Z_*	restriction of $Fib(1)$	172

Operations on flags.

flag g	low(g)	high(g)
$f=(f_l, f_{l+1}, \ldots, f_h)$	l	h
$\partial f=(f_{l+1}, \ldots, f_h)$	$l+1$	h
$df=(f_l, \ldots, f_{h-1})$	l	$h-1$
$Cf=(C_{f_l} f_l, \ldots, C_{f_l} f_h)$	l	h
$C_x f=(C_x f_l, \ldots, C_x f_h)$	l	h
$Af=(A_{f_h} f_l, \ldots, A_{f_h} f_h)$	l	h
$A_x f=(A_x f_l, \ldots, A_x f_h)$	l	h
$f^*=(f_h^*, \ldots, f_l^*)$	$n-h$	$n-l$
$f \vee x=(f_l \vee x, \ldots, f_h \vee x)$	l	h
$f \wedge x=(f_l \wedge x, \ldots, f_h \wedge x)$	l	h
$f^s=$ saturate f	l	h

Tableau games.

	LR-sequences of adjacent elements	66
$\mathcal{A}$	delete rowends, prepend $\bullet$'s	103
$\mathcal{C}: \ \mathrm{ftype}\, f \longmapsto \mathrm{ftype}\, Cf$	delete first column	103, 117
$\mathcal{C}: \ \mathrm{ftype}_{\hat{0}}\, f \longmapsto \mathrm{ftype}_{\hat{0}}\, C_{\hat{0}} f$		
$\mathcal{CL}, \mathcal{C}\widetilde{\mathcal{L}}: \ \mathrm{ftype}\, f \longmapsto \mathrm{ftype}\, Af$		157
$\Delta, \widetilde{\Delta}: \ \mathrm{ftype}\, f \longmapsto \mathrm{ftype}\, \partial f$	elementary evacuation step	72, 76, 116
$\partial: \ \mathrm{ftype}_{\hat{0}}\, f \longmapsto \mathrm{ftype}_{\hat{0}}\, \partial f$	$l+1 \mapsto \bullet$	116
$\partial: \ \mathrm{ftype}\, f \longmapsto \mathrm{ftype}_{f_l}\, \partial f$		
$\mathcal{D}_c, \widetilde{\mathcal{D}}_c: \ \mathrm{ftype}\, f \longmapsto \mathrm{ftype}(f \wedge x)$	Robinson-Schensted deletion step	126
	$f_l \leq x \lessdot f_h$ and $c = \lambda/(\mathrm{type}[f_l, x])$	
$\mathcal{D}_c, \widetilde{\mathcal{D}}_c: \ \mathrm{ftype}_{\hat{0}}\, f \longmapsto \mathrm{ftype}_{\hat{0}}(f \wedge x)$	$x \lessdot f_h$ and $c = \lambda/\,\mathrm{type}\, x$	
$\mathcal{D}^*_c, \widetilde{\mathcal{D}^*_c}: \ \mathrm{ftype}\, f \longmapsto \mathrm{ftype}(f \vee x)$	transposed R-S deletion step	137
	$x \gtrdot f_l$ and $c = \lambda/(\mathrm{type}[f_l \vee x, f_h \vee x])$	
$\mathcal{D}^*_c, \widetilde{\mathcal{D}^*_c}: \ \mathrm{ftype}_{\hat{0}}\, f \longmapsto \mathrm{ftype}_x(f \vee x)$	$x \gtrdot \hat{0}$ and $c = \lambda/\,\mathrm{type}[x, f_h \vee x]$	
$d: \ \mathrm{ftype}\, f \longmapsto \mathrm{ftype}\, df$	remove cells with h	103, 117
$d: \ \mathrm{ftype}_{\hat{0}}\, f \longmapsto \mathrm{ftype}_{\hat{0}}\, df$		
$\mathrm{ev}, \widetilde{\mathrm{ev}}: \ \mathrm{ftype}\, f \longmapsto \mathrm{ftype}\, f^*$	Schützenberger's evacuation	121
$\mathrm{ev}^\lambda, \widetilde{\mathrm{ev}}^\lambda: \ \mathrm{ftype}_{\hat{0}}\, f \longmapsto \mathrm{ftype}_{\hat{0}^*}\, f^*$		121
$\mathcal{I}_c, \widetilde{\mathcal{I}}_c: \ \mathrm{ftype}_{\hat{0}}\, f \longmapsto \mathrm{ftype}_{\hat{0}}(f \vee x)$	internal insertion	140
	$x \gtrdot \hat{0}$ and $c = \mathrm{type}\, x/\,\mathrm{type}\, f_l$	
$j, \tilde{j}: \ \mathrm{ftype}_{\hat{0}}\, f \longmapsto \mathrm{ftype}\, f$	*jeu de taquin*	115
$j^c, \tilde{j}^c: \ \mathrm{ftype}_{\hat{0}}\, f \longmapsto \mathrm{ftype}_x\, f$	*jeu de taquin* slide	112
	$x \gtrdot \hat{0}$ and $c = \mathrm{type}[\hat{0}, f_l]/\,\mathrm{type}[x, f_l]$	
$\mathcal{L}, \widetilde{\mathcal{L}}: \ \mathrm{ftype}_{\hat{0}}\, f \longmapsto \mathrm{ftype}_{\hat{0}}\, Af$	leftward vertical strip game	145
$\mathcal{R}, \widetilde{\mathcal{R}}: \ \mathrm{ftype}_{\hat{0}}\, f \longmapsto \mathrm{ftype}_{\hat{0}}\, Cf$	rightward vertical strip game	150
$\mathcal{S}, \widetilde{\mathcal{S}}: \ \mathrm{ftype}\, f \longmapsto \mathrm{ftype}\, f^s$	saturate a row and column weak tableau	
$\mathcal{S}, \widetilde{\mathcal{S}}: \ \mathrm{ftype}_{\hat{0}}\, f \longmapsto \mathrm{ftype}_{\hat{0}}\, f^s$		118
$\mathcal{S}^{-1}: \ \mathrm{ftype}\, f^s \longmapsto \mathrm{ftype}\, f$	unsaturate	120
$\mathcal{S}^{-1}: \mathrm{ftype}_{\hat{0}}\, f^s \longmapsto \mathrm{ftype}_{\hat{0}}\, f$		

Games $\mathcal{G}$ are deterministic, and follow the same rules as the ordinary game by that name when such a game was already known. Games $\widetilde{\mathcal{G}}$ are nondeterministic, with many possible outcomes, among which is $\mathcal{G}$. Most maps are uniform (see Section 5.3), so composition of flag operators induces composition of the deterministic games. The only exceptions are $\mathcal{S}$ (similar to uniform, see Section 6.8); $\mathcal{D}_c$ and $\widetilde{\mathcal{D}}_c$ (generic but not uniform); $\mathcal{I}_c$ (neither generic nor uniform); and $\mathcal{A}$ and LR-sequence transformations (tableau games describing transformations of a different nature than the others).

Part I. Background

1. Introduction

An **integer partition** of n is a weakly decreasing sequence of nonnegative integers, $\lambda_1 \geq \cdots \geq \lambda_k$, whose sum is n. It may be represented by a diagram consisting of left justified rows of boxes, where the ith row from the top has λ_i boxes. Young's lattice is the set of all partitions ordered by $\mu \leq \lambda$ iff the diagram of μ is contained in the diagram of λ. A **standard tableau** of shape λ is a filling of the boxes of the diagram of λ with the numbers $1, \ldots, n$ so that each row increases from left to right, and each column increases from top to bottom. Robinson, Schensted, and others created an algorithmic bijection between ordered pairs of standard tableaux of the same shape with entries $1, \ldots, n$, and permutations of $1, \ldots, n$. Schützenberger introduced an involution ev on standard tableaux of any given shape, also algorithmic in nature; a good exposition of these algorithms may be found in Sagan [**22**]. The purely algorithmic Robinson-Schensted map was found by Steinberg [**29**] to describe naturally certain properties of the flag variety of all subspaces of V fixed by a given nilpotent transformation N, where V is a vector space over an infinite field. Hesselink [**10**] and van Leeuwen [**31**] found that ev naturally describes other properties in the same flag variety.

To any pair $W_1 \subset W_2$ of N-invariant subspaces, we may associate an integer partition called the **type** of the quotient W_2/W_1, by examining the Jordan canonical form of the action of N on the quotient W_2/W_1; it is a block sum of matrices with 1's on the main diagonal and the diagonal just above it. The lengths of the main diagonals, sorted into decreasing order, form an integer partition, denoted type(W_2/W_1) or in lattice terms, type$[W_1, W_2]$. Any subinterval $[W_1', W_2']$ (i.e., W_1', W_2' are N-invariant and $W_1 \subseteq W_1' \subseteq W_2' \subseteq W_2$) has a smaller type in Young's lattice. Given a saturated flag $0 = W_0 \subset W_1 \subset \cdots \subset W_n = V$ of N-invariant subspaces of V, form a chain of increasing partitions in Young's lattice: type$(W_0/0), \ldots,$ type$(W_n/0)$. This chain can be encoded as a standard Young tableau by filling the square added in the ith term with i. The tableau so obtained is called the type of the flag. The cotype of the flag is the tableau encoding the chain of partitions type$(V/W_n), \ldots,$ type(V/W_0); van Leeuwen showed that, in the Zariski topology, the generic cotype of a flag of type P is ev(P). The **relative position** of two flags in a modular lattice is a permutation that measures the distance between two flags; the higher the permutation is in the Bruhat order, the farther apart the flags are. Steinberg showed that, in the Zariski topology, the generic relative position of flags of type P and Q is the same permutation that the Robinson-Schensted algorithm associates with the pair (P, Q).

A **semi-primary lattice** is a modular lattice of finite height in which, for every join-irreducible x, the interval $[\hat{0}, x]$ is a chain, and for every meet-irreducible x, the interval $[x, \hat{1}]$ is a chain; see Jónsson and Monk [**13**] (for background on lattices see Stanley [**27**]). A **q-regular semi-primary lattice** is a semi-primary lattice in which every interval of length 2 is either a chain or has exactly $q+1$ atoms. The lattice of N-invariant subspaces of a vector space over any field (not only the complex numbers) is a semi-primary lattice, and if the field is finite of order q, it is q-regular. The lattice of subgroups of a finite abelian p-group is another semi-primary lattice, and is p-regular. As with the first example, an integer partition may be assigned to any interval in the lattice; the quotient of two groups is isomorphic to a product of cyclic p-groups, $\mathbb{Z}/p^{\lambda_1} \times \cdots \times \mathbb{Z}/p^{\lambda_k}$, and if the λ_i are arranged in weakly decreasing order, they form the integer partition corresponding to the type of the quotient. There is a lattice theoretic definition of this partition for all semi-primary lattices, coinciding with the partitions given in these two cases: for an interval $[x, y]$, express y as the join of independent join-irreducibles over x (the meet of any one with the join of the others is x). Sort the ranks (as measured in $[x, y]$) of these irreducibles into weakly decreasing order to obtain a partition. This partition turns out to be independent of what independent join-irreducibles were chosen, and is taken as the definition of $\text{type}[x, y]$.

Lattices and Combinatorial Algorithms. Although objects that are instances of semi-primary lattices have been studied at length, they have not generally been considered as semi-primary lattices, and the results about them do not always immediately apply to other semi-primary lattices. The generic flag cotype and relative positions described previously do not apply if the field is not infinite, because the notion of "genericity" and the proofs of these results are critically dependent on the Zariski topology. We must reformulate the problems and find new methods of proof if we are to solve them in the wider context of semi-primary lattices. We consider probabilistic and enumerative analogues and degenerate cases.

For q-regular semi-primary lattices, a probabilistic analogue of Hesselink and van Leeuwen's result is that all but a fraction $O(q^{-1})$ of the flags of type P have cotype $\text{ev}(P)$ (see Theorem 5.14). An enumerative extension is the following empirical observation, verified for certain cases: the number of flags of type P whose cotype is Q is often a polynomial in q, and the degree of this polynomial is maximized when $Q = \text{ev}(P)$. We describe a refinement of these polynomials that allows us to analyze the degenerate behavior of flags in semi-primary lattices. Denote a saturated flag by $f = (f_0 \lessdot \cdots \lessdot f_n)$. Given partitions $\Lambda = (\lambda^{(ij)})$ (where $0 \le i \le j \le n$) the number of flags in a q-regular semi-primary lattice whose **interval type table** is $\text{type}[f_i, f_j] = \lambda^{(ij)}$ is often a polynomial in q depending only on the partitions Λ, and on no further structure of the lattice; in particular, the polynomials are the same for abelian subgroup lattices and invariant subspace lattices. (In the case of invariant subspaces over an infinite field, the dimension of the closure of the set of flags achieving this type table equals the degree of this polynomial, but any meaning the lower order terms might have is lost.) This polynomial generalizes Hall polynomials (see Macdonald [**16**] for a comprehensive treatment), which may be used to count the number of flags in which only certain intervals' types are specified. These polynomials have been computed for all arrays Λ with $n \le 8$ (see Appendix A), and for a class of arrays that includes all those occurring for lattices whose type has two columns (see Section 5.6).

The Fibonacci Lattice. Fomin [**4**, **5**] has generalized the Robinson-Schensted algorithm from standard Young tableaux, which encode chains in Young's lattice, to chains in certain other lattices. One such lattice is the **Fibonacci Lattice** $Z(r)$, introduced by Stanley [**28**]. It consists of words from the alphabet $\{1_1, \ldots, 1_r, 2\}$, in which y covers x iff some 1_i in x preceded only by 2's is changed to a 2, or a 1_i is inserted in a position preceded only by 2's. Fomin bijectively associates permutations in S_n to pairs of saturated chains in $Z(1)$ from $\hat{0}$ to equal elements of rank n.

This is a direct analogue of the Robinson-Schensted algorithm in Young's lattice. In Section 9, we introduce **strongly modular lattices**, the class of modular lattices whose complemented intervals all have height at most 2, of which $Z(r)$ is an example. We study this in Section 9. Steinberg's result about the Robinson-Schensted algorithm for invariant subspace lattices over infinite fields has an analogue for strongly modular lattices: the relative position of two flags is bounded above in the Bruhat order by a variation of Fomin's Robinson-Schensted algorithm for $Z(1)$ applied to the flags' types, where the types are chains of integer compositions given by a generalized lattice interval type defined in Section 3.3. For $Z(r)$, the number of pairs of flags of prescribed types in a prescribed relative position is a polynomial in r, whose degree is maximized when the relative position is the same permutation assigned given by the modified Robinson-Schensted algorithm. Fomin does not have an analogue of Schützenberger's evacuation for $Z(r)$, but there is an analogue of the interval type table for strongly modular lattices. In $Z(r)$, the number of ways of achieving a particular interval type table is a polynomial in r, and there is a generic interval type table for flags of a given type.

Overview. Part I includes this introduction, and continues with a thorough grounding in the required background on lattices, partitions, and tableaux.

In Part II, we develop semi-primary lattices, describe their basic properties, representation, enumeration, and prior work on them. In Section 3, we introduce new lattice operators A and C that generalize $\ker N$ and NV in invariant subspace lattices, or $\operatorname{socle} G$ and pG for p-subgroup lattices, to discrete modular lattices whose complemented intervals have finite length. This includes semi-primary lattices and the Fibonacci lattice. Integer compositions, coinciding with the interval type previously defined for semi-primary lattices, can be assigned to any interval in terms of these operators A and C. These generalized interval types, and the new operators, are studied in a general setting.

In Section 4, we develop the fundamentals of semi-primary lattices. We begin in Section 4.1 by developing them essentially as Jónsson and Monk [**13**] did, but with ties to the new lattice operators. We also establish that the type of an interval is larger (in Young's lattice) than the type of any subinterval. In Section 4.2, we introduce several new characterizations of semi-primary lattices: they may be characterized by lattice polynomials, rank equations, and local conditions in the lattice, in much the same way that modular lattices can. They also may be characterized in terms of properties of the generalized lattice type. In Section 4.3, we discuss what is known about representing many semi-primary lattices as submodule lattices.

In Sections 4.4–4.5, we introduce enumeration in semi-primary lattices. The number of ways to achieve various configurations of elements is often a polynomial in the parameter q described previously. In Section 4.6, we completely characterize and enumerate complemented elements in semi-primary lattices, generalizing what was already known for particular cases.

The notion of a basis in a vector space or matroid has a counterpart in semi-primary lattices. In Sections 4.7–4.8, we enumerate such bases, determine when different elements have similar bases, and analyze lattice polynomials whose variables are elements of a basis.

The Littlewood-Richardson sequence of an element is a statistic introduced by Green [**8**] as a means of computing Hall polynomials. In Section 4.9, we describe Hall polynomials, and then in Sections 4.10–4.11, we describe new properties of these statistics: how these statistics change for adjacent elements; using them to determine the most and least likely cotype of an element given its type; and using them to determine the types between certain pairs of elements in a flag.

The type of a flag is the tableau encoding the chain of partitions that are the types of the elements in the flag. In Part III we discuss how various operators transform flag types. First, in Section 5, we look at intervals between elements of a flag. We describe Schützenberger's evacuation algorithm, the connection Hesselink and van Leeuwen established between it and the generic cotype of a flag of specified type in invariant subspace lattices over infinite fields, and then establish the probabilistic counterpart for it. We then describe an algorithm for computing how many flags have particular interval type tables (computer runs of which are summarized in Appendix A), and conclude with a specific formula for this polynomial for a large class of interval type tables, including all those occurring in lattices whose type has two columns.

In Section 6, we consider many more operations on flags. Given a flag of type $\boldsymbol{P}$ and an operator ϕ, the flag $\phi(f)$ can usually have many types, but a particular one is the most common. The ordinary evacuation, *jeu de taquin*, and Robinson-Schensted games are among those that describe the most common types associated with certain operators. Nondeterministic variants of these games describe other possible values of the type: the ordinary rules of these games are modified to allow choices of several moves at each step, one of which is the usual move. We enumerate the number of pairs of flags $(f, \phi(f))$ where f and $\phi(f)$ have prescribed types. We show how composition of flag operators leads to composition of tableaux games, and determine when composition of the deterministic games yields the most likely value of the composition of the corresponding flag operators. All these games are for saturated flags, so in Section 6.8, we describe a canonical way to transform games on standard tableaux to games on row and column weak tableaux, reflecting how the corresponding operators behave on nonsaturated flags. This transformation generalizes that of Knuth [**15**] for the Robinson-Schensted algorithm on semistandard tableaux.

In Section 7, we describe the Robinson-Schensted correspondence and its relation to relative positions of flags. We are not as successful in providing probabilistic and enumerative counterparts of Steinberg's results as we were for Hesselink and van Leeuwen's in Section 5, but we present partial results and conjectures.

In Section 8, we introduce new tableau games, similar in spirit to the Robinson-Schensted and evacuation algorithms. Instead of sliding or bumping numbers entries along a single path as is done in the other games, a vertical strip of holes is formed in a tableau and numbers slide into and out of it. These games and their nondeterministic versions describe how the new operators A and C behave on flags. A connection between these games and evacuation and the *jeu de taquin* is discussed in Section 8.3. A generalization of the interval type table polynomials based on these games is described in Section 8.5.

Finally, in Part IV, we discuss strongly modular lattices, including Fibonacci lattices and semi-primary lattices whose type has two rows. The Fibonacci lattice $Z(r)$

was defined by Stanley [**28**] in terms of its cover relations. In Section 9.1, we give a simple direct test, apparently new, to determine if two elements are comparable, without having to exhibit a chain of elements from one to the other satisfying this cover relation. We then compute the type of any interval in $Z(r)$; our generalized lattice type yields two different integer compositions, whose parts are 1's and 2's, as types of intervals in strongly modular lattices.

While Young's lattice describes the relation between the type of an interval and the type of its subintervals in a semi-primary lattice, $Z(1)$ and a restriction of $Fib(1)$ describe the analogous problem in strongly modular lattices: lowering the upper boundary of an interval causes the type to decrease in one of these, and raising the lower boundary causes the type to decrease in the other. We describe this in Section 9.2. Next, in Section 9.3, we develop interval type tables for strongly modular lattices, by considering these properties of types of subintervals. We determine the "generic" interval type table of a flag given its type in a strongly modular lattice, and we quantify this for $Z(r)$ by computing how many flags achieve each possible interval type table. It is a polynomial in r, and the polynomials for the "generic" interval type tables have the maximal degrees.

We conclude with an analysis of relative positions of flags in strongly modular lattices. A variation of Fomin's version of the Robinson-Schensted game in $Z(1)$ describes the maximum relative position of flags in any strongly modular lattice, and a nondeterministic variant of the game gives the other possible relative positions. As a consequence, the conjectured maximum relative position for semi-primary lattices does in fact hold for two row semi-primary lattices. In $Z(r)$, we compute polynomials for how many pairs of flags have specified types and relative positions, and these polynomials have maximum degree precisely when the relative position is the permutation given by this modified Robinson-Schensted algorithm on the types of the flags.

2. Review of combinatorial objects

2.1. Posets. Our notation and definitions for partially ordered set and lattice notions are those in Stanley [**27**]; consult that or Dilworth [**2**] for further information.

A **partially ordered set** P, or **poset**, is a set (also denoted P by abuse of notation) with a binary relation $\leq$ satisfying three axioms.

Reflexive: For all $x \in P$, we have $x \leq x$.
Antisymmetric: If $x \leq y$ and $y \leq x$, then $x = y$.
Transitive: If $x \leq y$ and $y \leq z$, then $x \leq z$.

The notation $x < y$ means $x \leq y$ and $x \neq y$, while $y \geq x$ means $x \leq y$, and $y > x$ means $x < y$. When dealing with multiple posets, or multiple partial orders on the same ground set, use subscripts, such as $\leq_P$, or other special symbols, to specify which order is to be used. The relation $\leq$ is **weak** while $<$ is **strict**. Elements x and y of P are **comparable** if either $x \leq y$ or $y \leq x$, and are **incomparable** otherwise.

Two posets P and Q are **isomorphic** if there is a bijection $\phi : P \to Q$ such that $x \leq y$ in P iff $\phi(x) \leq \phi(y)$ in Q.

An **induced subposet** of P (often abbreviated to **subposet**) is a subset Q of the elements of P with the order, if x and y are in Q then $x \leq y$ in Q iff $x \leq y$ in P. A particular subposet of interest is the **closed interval** $[x, y] = \{\, z \in P : x \leq z \leq y \,\}$ with the order induced by P, defined for all x, y with $x \leq y$ in P.

We say that y **covers** x in P if $x < y$ and no element $z \in P$ satisfies $x < z < y$. Equivalently, we say x **is covered by** y, that x **is a lower cover of** y, and that y **is an upper cover of** x. This may be written as $y \gtrdot x$ or $x \lessdot y$. The notations $y \geq x$ and $x \leq y$ mean that either $x = y$ or $x \lessdot y$. For integers x, y we have $x \lessdot y$ iff $x = y - 1$, and $x \leq y$ iff $x = y$ or $x = y - 1$.

A poset P has a **minimum element** $\hat{0}$ if there exists an element $\hat{0} \in P$ such that $x \geq \hat{0}$ for all $x \in P$. Dually, P has a **maximum element** $\hat{1}$ if there exists an element $\hat{1} \in P$ such that $x \leq \hat{1}$ for all $x \in P$. When dealing with multiple posets, use subscripts, $\hat{0}_P$ and $\hat{1}_P$, to differentiate these elements. For example, $\hat{0}_{[x,y]} = x$ and $\hat{1}_{[x,y]} = y$. When dealing with subposets Q of P, denote the minimum and maximum elements of P by $\hat{0}$ and $\hat{1}$, and those of Q by $\hat{0}_Q$ and $\hat{1}_Q$. An element x is **minimal** if for no y does $y < x$ hold, and an element x is **maximal** if for no y does $y > x$ hold.

A **flag** in a poset is a sequence of elements $f = (f_l, f_{l+1}, \ldots, f_h)$ with $f_l \leq f_{l+1} \leq \cdots \leq f_h$. To indicate a more specific relationship between consecutive elements in a flag, such as equality, strictly less than, or weakly or strongly covered by, replace the commas with this relationship. A **saturated flag** has the form $f = (f_l \lessdot f_{l+1} \lessdot \cdots \lessdot f_h)$ and a **multisaturated flag** has the form $f = (f_l \leq f_{l+1} \leq \cdots \leq f_h)$. We may need to be concerned with the indices of the elements of a flag, so define $\mathrm{low}(f) = l$ and $\mathrm{high}(f) = h$. Define operators that remove the bottom and top elements of a flag by $\partial f = (f_{l+1}, f_{l+2}, \ldots, f_h)$ and $df = (f_l, f_{l+1}, \ldots, f_{h-1})$. A **chain** is synonymous with a flag.

The **length of a saturated chain** $(f_l, \ldots, f_h)$ is $h - l$. A poset is **graded** if whenever $x \leq y$, all saturated chains from x to y have the same length, which is denoted $\rho(x, y)$. In posets with $\hat{0}$, the **rank of** x is $\rho(x) = \rho(\hat{0}, x)$, and in posets with $\hat{1}$, the **corank of** x is $\bar{\rho}(x) = \rho(x, \hat{1})$. A graded poset has **finite length** if all saturated chains have finite length. This does not require the poset to have a finite number of elements; the subspaces of a finite dimensional vector space, ordered by subspace inclusion, form a poset (actually a modular lattice, defined below) of finite length, but if the field is infinite, there will be an infinite number of subspaces (in dimensions larger than 1). From now on, we will only consider graded posets in which every closed interval has finite length, even if the poset itself is not of finite length.

2.2. Lattices. If x and y are in a poset P, an **upper bound** is some $z \in P$ such that $z \geq x$ and $z \geq y$. A **least upper bound** of x and y is an upper bound z such that every upper bound w satisfies $z \leq w$. In general, neither an upper bound nor a least upper bound exists, but if a least upper bound exists, it is unique, and is denoted $x \vee y$, which is read "x join y." Dually, the greatest lower bound, if it exists, is denoted $x \wedge y$, which is read "x meet y." A lattice is a poset for which every pair of elements has a least upper bound and a greatest lower bound. Alternately, we may define a lattice as a set with a pair of binary operations $\wedge$ and $\vee$, obeying the following laws.

Associative: $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$.
Commutative: $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$.
Idempotent: $x \vee x = x \wedge x = x$.
Absorption: $x \wedge (x \vee y) = x = x \vee (x \wedge y)$.

The partial order description and the binary operation description can be recovered from each other via $x \wedge y = x$ iff $x \vee y = y$ iff $x \leq y$. Because meet and join are associative and commutative, any finite set of elements in a lattice has a meet and a join. If all infinite sets of elements also have meets and joins, a lattice is called **complete**. In particular, all lattices of finite length are complete.

All lattices of finite length have $\hat{0}$ and $\hat{1}$.

The **dual** of a poset $(P, \leq)$ is $(P^*, \leq^*)$, with the same ground set as P but the reverse order. For $x \in P$, let x^* be inclusion of x into P^*. The order is $x^* \leq^* y^*$ iff $y \leq x$. We have $\hat{0}^* = \hat{1}$ when either exists, and $\hat{1}^* = \hat{0}$ when either exists. In a lattice, $x^* \vee^* y^* = (x \wedge y)^*$ and $x^* \wedge^* y^* = (x \vee y)^*$. Applying an operator, definition, or other notion to the dual of a poset is referred to as dualizing that notion, and often, we can dualize theorems by replacing every notion in its statement and proof with the dual notion.

In any lattice, a nonzero element x is **join-irreducible** if it cannot be expressed as $x = y \vee z$ with $y < x$ and $z < x$. The element $\hat{0}$ is not considered to be a join-irreducible. A local condition for an element to be join-irreducible is that it covers precisely one element. **Meet-irreducibles** are defined dually as elements $x \neq \hat{1}$ which cannot be expressed as $x = y \wedge z$ with $y > x$ and $z > x$. A local condition for an element to be meet-irreducible is that it is covered by precisely one element.

A special type of join-irreducible is an **atom**, which is an element that covers $\hat{0}$. The atoms of an interval $[x, y]$ are the elements of the interval that cover x. Dually, a **coatom** is any element covered by $\hat{1}$, and a coatom of an interval $[x, y]$ is any element of the interval that is covered by y. A lattice is **atomic** if every element can be expressed as the join of atoms, and dually, is **coatomic** if every element can be expressed as the meet of coatoms.

A **complement** to an element x in a lattice is some y with $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$. While $\hat{0}$ and $\hat{1}$ are always unique complements to each other, it is not always true that elements have complements, or that complements are unique when they do exist. A **complemented lattice** is a lattice in which all elements possess complements.

There are several classes of lattices of interest in combinatorics. An **upper semimodular** lattice is a lattice in which all closed intervals have finite length, and the following local condition is satisfied: if x and y both cover $x \wedge y$, then $x \vee y$ covers both x and y. If a lattice is of finite length, it is upper semimodular iff it is graded and its rank function satisfies $\rho(x) + \rho(y) \geq \rho(x \wedge y) + \rho(x \vee y)$ for all x, y in the lattice. Lower semimodular lattices are defined dually by inverting the meets, joins, and inequalities.

A **modular** lattice is a lattice that is both upper and lower semimodular. That is, $x \vee y$ covers both x and y iff both x and y cover $x \wedge y$. A lattice of finite length is modular iff it is graded and its rank function satisfies $\rho(x)+\rho(y) = \rho(x \wedge y)+\rho(x \vee y)$. In addition to this local condition and the rank equation, there is also a lattice identity that applies to all lattices without any finiteness restrictions: a lattice is modular iff $x \leq z$ implies $x \vee (y \wedge z) = (x \vee y) \wedge z$.

A **distributive** lattice is a lattice in which all x, y, z satisfy either of the equivalent equations $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Every distributive lattice is modular, but not conversely. For example, the lattice of all subsets of a set, ordered by set inclusion, has union and intersection as meet and join, and is distributive. However, the lattice of all subgroups of an abelian group, ordered by subgroup inclusion, is modular but usually not distributive.

We hope to add semi-primary lattices to this list. We will show that in addition to the classical definitions of semi-primary lattices, they, too, may be defined by local conditions, rank equations, and lattice identities similar to the ones used for the preceding classes of lattices.

2.3. Properties of modular lattices. The following facts about modular lattices will be frequently used.

Lemma 2.1. *For any x, y in a modular lattice $\mathcal{L}$, there is an isomorphism from $[x \wedge y, y]$ to $[x, x \vee y]$ preserving meets and joins. The intervals are said to be* ***perspective****. The isomorphism is given by $z \mapsto z \vee x$, and its inverse is $z \mapsto z \wedge y$. Conversely, any lattice of finite length in which these maps are inverse for all x, y is modular.*

Proof. Let x, y be in a modular lattice $\mathcal{L}$. For any $z \in [x \wedge y, y]$, the composition of the two maps given is $(z \vee x) \wedge y$. By the modular law, this equals $z \vee (x \wedge y)$, which is z, since everything in this interval is at least $x \wedge y$. Similarly, for $z \in [x, x \vee y]$, the composition of the maps in the other order is the identity.

For the detailed proof of the converse, consult any reference on lattice theory, such as [**2**, pp. 22–23] (which actually proves it in a class of lattices called **compactly generated lattices** that is broader than finite length lattices). $\square$

Lemma 2.2. *Let $[x, y]$ be an interval of finite length in a modular lattice $\mathcal{L}$. For any $z \in \mathcal{L}$ such that x, y, z lie in an interval of finite length, $\rho(x \wedge z, y \wedge z) \leq \rho(x, y)$, and if equality holds, $[x \wedge z, y \wedge z]$ and $[x, y]$ are perspective. Dually, the same statements apply with meets replaced by joins.*

Proof. Let $w = y \wedge z$, so that $[x \wedge z, y \wedge z] = [x \wedge w, w] \cong [x, x \vee w]$. Then $x \vee w \leq y$ because $x, w \leq y$, so the interval $[x, x \vee w]$ is contained in $[x, y]$ and hence has weakly smaller length. If the lengths are the same then $x \vee w = y$ so $[x \wedge z, y \wedge z]$ and $[x, y]$ are the perspective pair $[x \wedge w, w] \cong [x, x \vee w]$.

The dual statements are proved similarly. $\square$

Lemma 2.3. *In a modular lattice, if $y \gtrdot x$, then for any z such that x, y, z are contained in an interval of finite length, either*

(1) *$y \wedge z \gtrdot x \wedge z$ and $y \vee z = x \vee z$, or*
(2) *$y \wedge z = x \wedge z$ and $y \vee z \gtrdot x \vee z$.*

Proof. All ranks are relative to the minimal element of a finite length interval containing x, y, z. By the modular law,

$$\begin{array}{rcccl} \rho(y \wedge z) & + & \rho(y \vee z) & = & \rho(y) + \rho(z) \\ \rho(x \wedge z) & + & \rho(x \vee z) & = & \rho(x) + \rho(z) \\ \text{so} \quad (\rho(y \wedge z) - \rho(x \wedge z)) & + & (\rho(y \vee z) - \rho(x \vee z)) & = & 1. \end{array}$$

The two parenthesized expressions are nonnegative, so one is 0 and the other is 1. $\square$

Lemma 2.4. *A closed interval in a modular lattice is itself a modular lattice.*

Proof. The equation defining a modular lattice,

$$x \vee (y \wedge z) = (x \vee y) \wedge z \text{ when } x \leq z,$$

is obeyed for x, y, z in any subinterval of the lattice, because the values of $\wedge$ and $\vee$ do not change. $\square$

Theorem 2.5. *The following are equivalent for modular lattices $\mathcal{L}$ of finite length.*

(1) $\hat{1}$ *is the join of atoms.*
(2) $\hat{0}$ *is the meet of coatoms.*
(3) $\mathcal{L}$ *is atomic.*
(4) $\mathcal{L}$ *is coatomic.*
(5) $\mathcal{L}$ *is complemented.*

Proof. Let $x \in \mathcal{L}$. Any set $\{a_1, \ldots, a_n\}$ of atoms of minimal cardinality such that $a_1 \vee \cdots \vee a_n = x$ has $n = \rho(x)$, because in the successive joins $\hat{0} \lessdot a_1 \leq a_1 \vee a_2 \leq \cdots \leq a_1 \vee \cdots \vee a_n = x$, if any $\leq$ is $=$, we can eliminate the atom introduced in the join to its right without affecting the value of any joins. Such a set of S atoms is called an atomic basis of x. If $Y \subseteq S$ and y is the join of all elements of Y, then $\rho(y) = \#Y$, for if it is smaller we can again eliminate atoms from S without affecting its join. If Y and Z are subsets of S with joins y and z, then $\rho(y \vee z) = \#(Y \cup Z)$ so $\rho(y \wedge z) = \rho(y) + \rho(z) - \rho(y \vee z) = \#(Y \cap Z)$, and $y \wedge z$ is the join of the atoms of $Y \cap Z$ (for it is bounded below by that join, and has the same rank).

(1) ⇒ (2): Let $a_1, \ldots, a_n$ be an atomic basis of $\hat{1}$. Then $a_1 \vee \cdots \vee a_{i-1} \vee a_{i+1} \vee \cdots \vee a_n$ (with $i = 1, \ldots, n$) are coatoms whose meet is $\hat{0}$.

(1) ⇒ (3): Assume (3) fails, and let $x > \hat{0}$ be a minimal element not expressible as the join of atoms. If x covers at least two elements, it is the join of the atoms of which those two elements are joins, so x can only cover one element, and thus is a join-irreducible; denote the unique lower cover y.

Let $a_1, \ldots, a_k$ be an atomic basis of y, and $b_1, \ldots, b_n$ be an atomic basis of $\hat{1}$. Let $y_m = y \vee b_1 \vee \cdots \vee b_m$, so that $y = y_0 \leq \cdots \leq y_n = \hat{1}$. Extend the sequence $a_1, \ldots, a_k$ to an atomic basis $a_1, \ldots, a_n$ of $\hat{1}$ by including those b_m with $y_{m-1} \lessdot y_m$. Let $z = a_{k+1} \vee \cdots \vee a_n$, so $y \wedge z = \hat{0}$.

Since $x \vee z \geq y \vee z = \hat{1}$, in fact $x \vee z = y \vee z$. Since $x \gtrdot y$, it follows that $x \wedge z \gtrdot y \wedge z = \hat{0}$ by Lemma 2.3. Since the only element less than or equal to x not also less than or equal to y is x itself, $x = x \wedge z$ and $y = y \wedge z = \hat{0}$. Thus, x is an atom, contradicting the assumption that it is not a join of atoms.

(3) ⇒ (5): The maximum element $\hat{1}$ has complement $\hat{0}$. Now let $x < \hat{1}$ have an atomic basis $a_1, a_2, \ldots, a_k$. Choose atoms $a_{k+1}, \ldots, a_m$ with m maximal so that $x \lessdot x \vee a_{k+1} \lessdot \cdots x \vee a_k \vee \cdots \vee a_m$, and call the final join y. If $y \neq \hat{1}$, then since the lattice is atomic, $\hat{1}$ is the join of all atoms of the lattice, so there is some atom $a_{m+1} \not\leq y$ extending the sequence, contradicting maximality of m. Thus, $y = \hat{1}$, $m = n$, and x has complement $a_{k+1} \vee \cdots \vee a_n$.

(5) ⇒ (1): Suppose $\mathcal{L}$ is complemented and has length n. Form a maximal sequence of atoms $a_1, a_2, \ldots, a_k$ so that $a_1 \lessdot a_1 \vee a_2 \lessdot a_1 \vee a_2 \vee a_3 \lessdot \cdots$. Since the lattice has length n, we must have $k \leq n$. Suppose $k < n$. Let $x = a_1 \vee \cdots \vee a_k$, and choose a complement x' of x. Then x' has rank $n - k \geq 1$, so there is an atom $a_{k+1} \leq x'$. Then $x \wedge a_{k+1} \leq x \wedge x' = \hat{0}$, so $x \wedge a_{k+1} = \hat{0}$ and

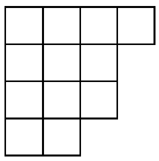
$\rho(x \vee a_{k+1}) = \rho(x) + \rho(a_{k+1}) - \rho(x \wedge a_{k+1}) = k + 1 - 0 = k + 1$, contradicting maximality of the sequence. Thus $k = n$, so $\hat{1}$ is the join of atoms.

(2) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (2) $\Rightarrow$ (1): Dualize (1) $\Rightarrow$ (3) $\Rightarrow$ (5) $\Rightarrow$ (1) $\Rightarrow$ (2). $\square$

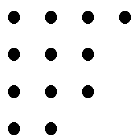
2.4. Compositions, partitions, and tableaux. A (strong) **composition** of a positive integer n is a sequence of positive integers whose sum is n. For example, the compositions of 3 are $(1,1,1)$, $(1,2)$, $(2,1)$, and (3). A **weak composition** may have parts that are 0, and so we may insert any number of zeros into these sequences.

A **partition** of a nonnegative integer n is a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of weakly decreasing nonnegative integers whose sum is n. We write $\lambda \vdash n$ or $n = |\lambda| = \sum_i \lambda_i$. For example, the partitions of 3 are $(1,1,1)$, $(2,1)$, and (3). For convenience, we allow both strong compositions and partitions to have any number, finite or infinite, of zeros appended to them, and we consider two strong compositions or partitions with the same sequence of nonzero entries but differing in the number of zeros at the end to be equivalent; for example, $(4,3,3,2) = (4,3,3,2,0) = (4,3,3,2,0,0) = \cdots$. In instances where all entries are single digits, we may also drop parentheses and commas, yielding 4332 in this example. An exponent indicates that an entry is repeated that number of times consecutively, so that $(4, 3^2, 2, 1^0) = (4,3,3,2)$. The number of parts of λ equal to i is called the **multiplicity** of i, and is denoted $m_i(\lambda)$, or m_i when the partition λ is known from context. The nonzero terms are called the **parts** of λ, and the number of nonzero terms is called the **length** of λ. For $\lambda = (4,3,3,2)$, we have length 4; multiplicities $m_2 = m_4 = 1$, $m_3 = 2$, and $m_i = 0$ otherwise; and $|\lambda| = 12$. We apply the same notational conventions concerning parentheses, commas, exponents, trailing zeroes, and the definition of length to arbitrary sequences and to strong compositions. The unique partition of 0, strong composition of 0, and empty sequence, are denoted $\emptyset$.

The **Young diagram** of a composition or partition λ of n is a left-justified array of n squares (also called **cells**) with λ_i squares in the ith row, while the **Ferrers diagram** has dots in place of the squares. It may be viewed as a subset of $\mathbb{P} \times \mathbb{P}$ (where $\mathbb{P}$ is the set of positive integers) using matrix coordinates, where (i,j) is the cell at the ith row from the top, jth column from the left; the diagram consists of all $(i,j) \in \mathbb{P} \times \mathbb{P}$ with $j \le \lambda_i$. For example, the Young diagram and Ferrers diagram of $(4,3,3,2)$ are respectively



and



The same symbol λ will be used to denote both a sequence of integers and its corresponding diagram.

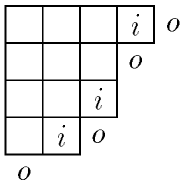
The **conjugate** λ' of a partition λ is the partition obtained by reflecting its diagram about the main diagonal. For example, the conjugate of $(4,3,3,2)$ is $(4,4,3,1)$. Clearly $\lambda'' = \lambda$. While λ_i is the number of squares in the ith row of λ, the number of squares in the jth column is λ'_j. The rows that have squares in the jth column are precisely those whose length is at least j, so $\lambda'_j = \#\{\, i : \lambda_i \ge j \,\}$ (where $\#S$ is the cardinality of set S). Hence, $m_j(\lambda) = \lambda'_j - \lambda'_{j+1}$. Note that λ'_1 is the length of λ.

We define a partial order on compositions and partitions by $\mu \le \nu$ iff $\mu_i \le \nu_i$ for all i, or equivalently, if the the cells of the diagram of μ are a subset of the cells of the diagram of ν. We have $(\mu \vee \nu)_i = \max\{\mu_i, \nu_i\}$ and $(\mu \wedge \nu)_i = \min\{\mu_i, \nu_i\}$. In the case of partitions, this partial order forms a lattice called **Young's lattice**. From the diagram interpretation of the order, it is clear that in Young's lattice, $\mu \le \nu$ iff $\mu' \le \nu'$.

The sum of two partitions or compositions μ and ν is $\mu + \nu = (\mu_1 + \nu_1, \mu_2 + \nu_2, \ldots)$. It is obtained by appending the rows of ν to those of μ. The difference of two compositions, $\mu - \nu = (\mu_1 - \nu_1, \mu_2 - \nu_2, \ldots)$, is well defined when $\nu \le \mu$. The union $\mu \cup \nu$ of two partitions is obtained by adding together the multiplicities of the parts, so $m_i(\mu \cup \nu) = m_i(\mu) + m_i(\nu)$; equivalently, it is obtained by appending columns of ν to the bottoms of those of μ, so for partitions we obtain $\mu \cup \nu = (\mu' + \nu')'$. Beware that this is completely different from the join $\mu \vee \nu$. The concatenation of two strong compositions or finite sequences $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_m)$, is $xy = (x_1, \ldots, x_k, y_1, \ldots, y_m)$.

For $\lambda, \mu \subset \mathbb{P} \times \mathbb{P}$, we define $\lambda \oplus \mu = \{\, c : c \in \lambda \text{ or } c \in \mu \,\}$ as the ordinary set union of these, and $\lambda \ominus \mu = \{\, c : c \in \lambda \text{ and } c \notin \mu \,\}$ as the ordinary set difference. Typically, λ will be a partition or composition and μ will be a single cell.

An **inner corner** of a partition λ is a cell of λ, which if removed would yield a valid partition, and an **outer corner** is a cell not in λ, which if added to λ would yield a valid partition. The inner and outer corners of $(4, 3, 3, 2)$ are marked i and o below.



For a partition λ, we define

$$n(\lambda) = \sum_{i \ge 1} (i-1)\lambda_i = \sum_{i \ge 1} \binom{\lambda'_i}{2}$$

as the number obtained by filling each square in the top row of λ with 0, each square in the next row with 1, and so on, and then adding together all the numbers. The middle expression is obtained by adding the numbers in each row and then adding the row totals, while the right expression is obtained by adding the numbers in each column, and then adding the column totals.

For partitions $\mu \le \lambda$, we define a **skew partition** or **skew shape** λ/μ. It is simply a pair of partitions. Its diagram is $\{\, (i,j) \in \mathbb{P} \times \mathbb{P} : \mu_i < j \le \lambda_i \,\}$. The diagram is ambiguous because different pairs of partitions may yield the same diagram; for example, λ/λ is always empty. The skew partition $\lambda/\emptyset$ is identified with the ordinary partition λ. For a skew partition λ/μ, define $n(\lambda/\mu) = n(\lambda) - n(\mu)$.

A **tableau** of (partition or composition) shape λ is obtained from the Ferrers diagram of λ by replacing the dots with numbers, or from the Young diagram by writing numbers in the boxes. Occasionally we may fill the squares with other symbols. The tableau of shape $\emptyset$ is denoted $\emptyset$. A skew tableau is similarly defined by filling in a skew partition. The plural of "tableau" is "tableaux." Denote tableaux with uppercase bold italic letters, $\boldsymbol{P}$, and denote the shape λ or skew shape λ/μ by sh $\boldsymbol{P}$. Also denote the outer shape λ by osh $\boldsymbol{P}$.

A **row strict** tableau has the numbers strictly increasing from left to right on each row: if the entry at cell (i,j) is a_{ij} then $a_{ij} < a_{i,j+1}$, provided cells $(i,j),(i,j+1) \in \lambda/\mu$. A **row weak** tableau is similar with $<$ replaced by $\leq$, and **column strict** and **column weak** are similar but for columns. A **standard Young tableau** is row and column strict with all numerical entries distinct; if the entries aren't specified, assume $1,\ldots,n$, where n is the number of cells in the diagram. A **semistandard tableau** is row weak and column strict. Here is a standard tableau and a semistandard tableau:

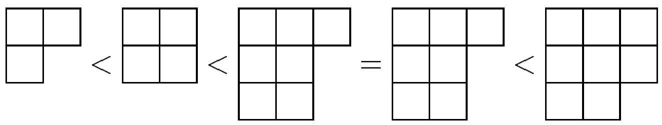
$$\begin{array}{|c|c|c|c|}\hline 1&2&5&6\\\hline 3&7&8\\\cline{1-3} 4\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 2&2&3\\\cline{1-3} 3&4\\\cline{1-2}\end{array}$$

The **hook length** of cell (i,j) in a partition λ is $h(i,j) = (\lambda_i - j) + (\lambda'_j - i) + 1$, which is the number of cells of λ weakly right of (i,j) on the same row, or weakly below it in the same column. The number of standard tableaux of shape λ is denoted f^λ, and may be computed by the well known **Hook Formula**, introduced by Frame, Robinson, and Thrall [**7**, pp. 317–318]:

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)}.$$

See also Sagan [**22**, pp. 91-97] for a survey of derivations of this formula. The number of standard skew tableaux of shape λ/μ is denoted $f^{\lambda/\mu}$, and there is no similarly elegant formula known for it.

A finite or infinite chain of partitions $\lambda^{(l)} \leq \lambda^{(l+1)} \leq \cdots$ may be represented by a row and column weak skew tableau where the cells of $\lambda^{(k)}/\lambda^{(k-1)}$ are filled with k, for each $k > l$. The cells of the inner partition $\lambda^{(l)}$ are filled with $\bullet$ in the Ferrers diagram, and either omitted or filled with $\bullet$ in the Young diagram. Whichever representation is used, the entries in the cells of λl are considered to be $\bullet$. For example, the chain $\lambda^{(3)},\ldots,\lambda^{(7)}$



is represented by the Young tableau or the Ferrers tableau, respectively,

$$\begin{array}{ccc}\cline{3-3} &&\multicolumn{1}{|c|}{5}\\\cline{2-3} &\multicolumn{1}{|c|}{4}&\multicolumn{1}{c|}{7}\\\cline{1-3} \multicolumn{1}{|c|}{5}&\multicolumn{1}{c|}{5}\\\cline{1-2}\end{array}\qquad\text{and}\qquad\begin{array}{ccc}\bullet&\bullet&5\\\bullet&4&7\\5&5\end{array}$$

A standard skew tableau on entries $l+1, l+2, \ldots, h$ is uniquely associated with a saturated chain of partitions $\lambda^{(l)} \lessdot \cdots \lessdot \lambda^{(h)}$ and conversely.

We will often view tableaux and chains of partitions interchangeably, writing $\boldsymbol{P} = (\lambda^{(l)},\ldots,\lambda^{(h)})$, because almost all the tableaux we consider will arise from such chains. This is a skew tableau of shape $\lambda^{(h)}/\lambda^{(l)}$. There is a minor ambiguity in this representation. If $\lambda^{(l)} = \lambda^{(l+1)}$ (respectively, $\lambda^{(h-1)} = \lambda^{(h)}$), the number $l+1$ (respectively, h) will not appear in $\boldsymbol{P}$, so the tableau representation of a chain in Young's lattice does not unambiguously determine the upper and lower indices used in the chain.

When this is an issue, we define $\mathrm{low}(\boldsymbol{P}) = l$ and $\mathrm{high}(\boldsymbol{P}) = h$ (or $\mathrm{high}(\boldsymbol{P}) = \infty$ if the indices are unbounded).

In a tableau $\boldsymbol{P}$ where e occurs just once, let $\mathrm{row}(\boldsymbol{P}, e)$ be the number of the row containing e. Alternately, if $\boldsymbol{P} = (\lambda^{(l)} \leq \cdots \leq \lambda^{(h)})$, and $\lambda^{(e-1)} \lessdot \lambda^{(e)}$, define $\mathrm{row}(\boldsymbol{P}, e)$ as the unique value r for which $\lambda_r^{(e-1)} \lessdot \lambda_r^{(e)}$. Also define $\mathrm{row}_k(\boldsymbol{P}, e) = \lambda_{k+r}^{(e)\prime}$, which is the length of the column k columns right of e once all entries larger than e are removed. Similarly, define $\mathrm{col}(\boldsymbol{P}, e)$ to be the number of the column containing e, and $\mathrm{col}_k(\boldsymbol{P}, e)$ to be the length of the row k rows below e when all entries larger than e are removed. For example, in the tableau

$$\boldsymbol{P} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 7 \\ \hline 3 & 6 & 8 \\ \cline{1-3} 4 & 9 & 10 \\ \cline{1-3} \end{array}$$

where $\lambda^{(8)} = (4, 3, 1) = (3, 2, 2, 1)'$, we have $\mathrm{row}(\boldsymbol{P}, 8) = \mathrm{row}_0(\boldsymbol{P}, 8) = 2$, and $\mathrm{row}_{-2}(\boldsymbol{P}, 8) = 3$; $\mathrm{row}_{-1}(\boldsymbol{P}, 8) = 2$; $\mathrm{row}_1(\boldsymbol{P}, 8) = 1$; and $\mathrm{row}_k(\boldsymbol{P}, 8) = 0$ for $k > 1$. Also, $\mathrm{col}(\boldsymbol{P}, 8) = 3$; the values $\mathrm{col}_k(\boldsymbol{P}, 8)$ for $k = -1, 0, 1$ are $4, 3, 1$; and $\mathrm{col}_k(\boldsymbol{P}, 8) = 0$ for $k > 1$.

Part II. Semi-primary lattices

3. Lattice types

3.1. Subgroup lattices, subspace lattices, and the Fibonacci lattice. We now introduce examples of the lattices that will be studied in depth in this manuscript.

(1) **Subgroup lattice.** Let G be a finite abelian group. The collection $L(G)$ of subgroups of G, with the order $H \leq K$ iff H is a subgroup of K, forms a modular lattice. The meet of two subgroups is their intersection, and the join is the group generated by the two.

Any finite abelian p-group is isomorphic to a product of cyclic p-groups,

$$\mathbb{Z}/p^{\lambda_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{\lambda_k}\mathbb{Z}.$$

Sort the λ_i into weakly decreasing order to form a partition called the **type of** G. Any quotient K/H of finite abelian p-groups is itself a finite abelian p-group, and the interval $[H, K]$ of $L(G)$ is isomorphic to $L(K/H)$. The type of the interval $[H, K]$ is the type of the group K/H, and this is less than the type of G in Young's lattice. The length of an interval of type μ is $|\mu|$.

(2) **Invariant subspace lattice.** Let V be a finite dimensional vector space, and N be a nilpotent transformation, that is, a linear transformation such that for all vectors $\vec{v} \in V$, we have $N^k\vec{v} = 0$ for sufficiently large k. A subspace W of V is N-invariant iff NW is a subspace of W. The collection $L(V, N)$ of all N-invariant subspaces of V, ordered by subspace inclusion, forms a modular lattice. The meet of two N-invariant subspaces is their intersection, and the join is their span.

Because N is nilpotent, all the roots of its characteristic equation are 0. The Jordan canonical form of N is a matrix that is the block sum of blocks with the characteristic root 0 on the diagonal; 1 just above each entry on the main diagonal; and 0 everywhere else. For example,

$$\left[\begin{array}{cccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

is the sum of two such blocks, one of length 4 and the other of length 3. Different bases of V might put these blocks in a different order, but the partition formed by sorting these lengths into weakly decreasing order is an invariant of N, called the **type of** N. This is the type of the lattice $L(V, N)$.

If $W \leq X$ are two N-invariant subspaces of V, their quotient X/W has an action induced by N. The lattice $L(X/W, N)$ is isomorphic to the interval $[W, X]$ in $L(V, N)$. The type of the interval $[W, X]$ is the type of the lattice $L(X/W, N)$, and is a subpartition of the type of $L(V, N)$ in Young's lattice. The length of an interval of type μ is $|\mu|$.

Both $L(G)$ and $L(V, N)$ are instances of semi-primary lattices. We will prove facts about semi-primary lattices through latticial means, without assuming they have a represention as a subgroup lattice or a subspace lattice. Such proofs are called **synthetic**. The lattice structure admits no knowledge of the elements of the group G, or the elements of the space V; only the subgroups and subspaces are known. We will sometimes provide alternate coordinate-based, or **analytic**, proofs where such elements are known. The analytic proofs will be based on the space $L(V, N)$. When the type of $L(V, N)$ is λ, we can choose a basis of V with elements

$$e_{ij} \text{ with } (i, j) \in \lambda, \text{ such that } Ne_{i1} = 0 \text{ and } Ne_{ij} = e_{i,j-1} \text{ when } j > 1.$$

Since N is linear, this defines the action N on the whole space.

(3) **The Fibonacci Lattice $Z(r)$ (see [28]).** Let $r \in \mathbb{P}$. Form a poset $Z(r)$ on the set of all words from the alphabet $\{1_1, \ldots, 1_r, 2\}$, in which v covers u, denoted $v >_Z u$, if u is obtained from v by either deleting the leftmost 1 of v (with any subscript) or by changing a 2 of v preceeded only by 2's to a 1 (with any subscript). So in $Z(2)$, we have $221_2 21_1 >_Z 2221_1$, $1_1 21_2 21_1$, $1_2 21_2 21_1$, $21_1 1_2 21_1$, and $21_2 1_2 21_1$.

This forms a modular lattice, which will be studied in Section 9.1. The rank of an element in the lattice is the sum of the digits in the word, ignoring the subscripts. Every complemented interval in this lattice has length at most 2: we see that for any word x, all upper covers of x are also lower covers of $2x$, so all atomic intervals have length at most 2, and a modular interval of finite length is atomic iff it is complemented.

3.2. New lattice operations A and C.

Definition 3.1. Let $\mathcal{L}$ be a lattice and $x \in \mathcal{L}$. Let

$$Ax = \bigvee_{y \gtrdot x} y$$

be the join of all elements covering x, or x if x is maximal, and let

$$Cx = \bigwedge_{y \lessdot x} y$$

be the meet of all elements covered by x, or x if x is minimal. If $x \leq z$ are both in $\mathcal{L}$, let

$$A_z x = \bigvee_{y:\, x \lessdot y \leq z} y$$

be the join of all atoms in the interval $[x, z]$, and

$$C_x z = \bigwedge_{y:\, x \leq y \lessdot z} y$$

be the meet of all coatoms in $[x, z]$. These are well defined in any discrete lattice where all complemented intervals have finite length. The operators A and C are dual to each other.

We also define iterates of A and C: $A_y^0 x = x$ and $A_y^{k+1} x = A_y(A_y^k x)$, and similarly for the unary form of A, and for both forms of C.

This coincides with the **Frattini element** of a complete lattice [**32**, p. 214], which is $C\hat{1}$ in our notation. However, we will consider properties of A and C as binary operators, and the restriction that all complemented intervals have finite length will yield certain properties not present in other complete lattices.

We now consider our three sample lattices.

(1) **Subgroup lattice.** Let

$$G = \mathbb{Z}/p^{\lambda_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{\lambda_k}\mathbb{Z}.$$

The atoms of $L(G)$ are nonzero subgroups of G with no proper subgroups; they are isomorphic to $\mathbb{Z}/pZ$. The **socle** of G is the maximum elementary subgroup, that is, the unique subgroup isomorphic to one of the form $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \cdots$. It is the join of all the atomic subgroups. Explicitly, it is

$$\mathrm{socle}(G) = p^{\lambda_1 - 1}\mathbb{Z}/p^{\lambda_1}\mathbb{Z} \times \cdots \times p^{\lambda_k - 1}\mathbb{Z}/p^{\lambda_k}\mathbb{Z}.$$

We have $A\hat{0}$ is the socle of G, and for any subgroups $H \leq K$ of G, we have $A_K H$ is the image in $L(G)$ of the socle of K/H under the natural isomorphism from $L(K/H)$ to $[H, K]$. Thus,

$$A_K^r H = \{\, g \in H \;:\; p^r g \in K \,\}.$$

The coatoms of $L(G)$ are those K for which $G/K \cong \mathbb{Z}/p\mathbb{Z}$. Their meet is $CG = pG$. For any subgroups $H \leq K$ of G, we have $C_H K$ is the image in $L(G)$ of pK/H under the natural isomorphism from $L(K/H)$ to $[H, K]$. So

$$C_H^r K = H + p^r K.$$

(2) **Invariant subspace lattice.** The atoms of $L(V, N)$ are the nonzero N-invariant subspaces of V with no nonzero proper subspaces that are N-invariant. If $W \in L(V, N)$ and $W \neq 0$, then NW is a proper subspace of W because the chain $W \supseteq NW \supseteq N^2W \supseteq \cdots$ is 0 after a finite number of steps, and once a $\supseteq$ is $=$, so are all further ones to the right. Thus, if W is atomic, it is a subspace of the kernel of N. The atoms of $L(V, N)$ are 1-dimensional subspaces of V spanned by some vector from the kernel of N, and the join of all the atoms is the kernel of N.

Let $W \leq X$ be N-invariant subspaces of V. Then $A_X W$ is the image in $L(V, N)$ of the kernel of N on X/W, and

$$A_X^r W = \{\, \vec{v} \in W \;:\; N^r \vec{v} \in X \,\}.$$

The coatoms of $L(V, N)$ are maximal proper subspaces of V. Their meet is $CV = NV$. For N-invariant subspaces $W \leq X$ of V, we have $C_W X = W + NX$ is the image in $L(V, N)$ of NX/W under the natural isomorphism from $L(X/W)$ to $[W, X]$, and

$$C_W^r X = W + N^r X.$$

(3) **The Fibonacci Lattice $Z(r)$.** The operators A and C have no counterpart in the prior theory of these lattices. Their values will be derived in full in Section 9.1. We derive the values of the unary operators Ax and Cx here. For any word x, if we insert a 1_i in the leading sequence of 2's, we can then change it to a 2, while if we change the first 1_i to a 2, we can then insert a 1_i

following that change. So all upper covers of x are lower covers of $2x$. As long as either $r > 1$ or $x > \hat{0}$, there is more than one upper cover, and so the join of all upper covers of x is $Ax = 2x$. When $r = 1$ and $x = \hat{0}$, the only upper cover of x is 1_1, so $Ax = 1_1$.

Now consider Cx. If x has the form $1_i y$, then the only lower cover is y, so $Cx = y$. If x has the form $2y$, we see that y is a lower cover of all lower covers of x, for we either delete the first 1_i and then change the 2 preceding it to 1_i, or we change some initial 2 to a 1_i and then delete that 1_i, to obtain y. There is always more than one lower cover unless $r = 1$ and $x = 2$, so $Cx = y$ unless $r = 1$ and $x = 2$, when $Cx = 1$. In short, Cx is obtained from x by deleting the first digit, with the exceptions $C\emptyset = \emptyset$ for all r, and $C2 = 1$ for $r = 1$.

We will compute $A_y x$ and $C_x y$ in Section 9.1.

When we develop the theory of semi-primary lattices, the elements of a group G or a vector space V will not be available; only the lattice structure of $L(G)$ or $L(V, N)$ will be known. By examining only the lattice structure and the operators A and C, rather than exhibiting elements of a group or a vector space, the results will be applicable to any semi-primary lattice, and not just ones with particular representations.

Theorem 3.2. *If $\mathcal{L}$ is upper semimodular and $x \leq x'$, then $Ax \leq Ax'$. If $\mathcal{L}$ is lower semimodular and $x \leq x'$, then $Cx \leq Cx'$.*

Proof. It suffices to prove these for $x \lessdot x'$, and since the proofs are the same but with all lattice operations dualized, only the first will be shown. If $y \gtrdot x$, then by upper semimodularity, $y \vee x' \gtrdot x'$. Since Ax is the join of all $y \gtrdot x$, which includes x', it's also the join of all $y \vee x'$ with $y \gtrdot x$. But Ax' is the join of all these and possibly more, so Ax' is at least as large as Ax. $\square$

Theorem 3.3. *Consider a modular lattice of finite length.*

(1) $C_x Ax = A_x Cx = x$.
(2) $A^k C^k x \geq x$ *and* $C^k A^k x \leq x$ *for all nonnegative integers* k.
(3) $A^k C^k A^k x = A^k x$ *and* $C^k A^k C^k x = C^k x$.
(4) *If* $x \leq y$ *then* $A_y x = Ax \wedge y$ *and* $C_x y = Cy \vee x$.
(5) $A(x \wedge y) = Ax \wedge Ay$ *and* $C(x \vee y) = Cx \vee Cy$.
(6) $C^k x \leq y$ *iff* $x \leq A^k y$.

Note. Items (4) and (5) will be used extensively, and so should be ingrained now as basic arithmetic.

Proof.

(1) The interval $[x, Ax]$ is atomic, hence coatomic by Theorem 2.5, so the meet $C_x Ax$ of all its coatoms is x. The dual argument is similar.
(2) First, $A^0 C^0 x = x$ and $ACx \geq A_x Cx = x$. Next, let k be a positive integer, and assume that the theorem holds for smaller nonnegative k. Then $AC(C^{k-1}x) \geq C^{k-1}x$ so $A^{k-1}(AC(C^{k-1}x)) \geq A^{k-1}(C^{k-1}x)$, or $A^k C^k x \geq A^{k-1}C^{k-1}x \geq x$. Similarly $C^k A^k x \leq x$.
(3) On the one hand, $C^k A^k x \leq x$ by (2), so $A^k(C^k A^k x) \leq A^k x$ by Theorem 3.2. On the other hand, $A^k C^k(A^k x) \geq (A^k x)$ by (2). So in fact $A^k C^k A^k x = A^k x$. The other equality is proven dually.

(4) Since $A_y x$ is a join of a subset of the elements of which Ax is a join, $A_y x \leq Ax$, and since all the elements of which $A_y x$ is a join are less than or equal to y, also $A_y x \leq y$. Thus $A_y x \leq Ax \wedge y$.

Conversely, the interval $[x, Ax]$ is atomic and hence coatomic (by Theorem 2.5), and the meet of any of its coatoms with y is either $Ax \wedge y$ or a coatom of $[x, Ax \wedge y]$ (by Lemma 2.3). So

$$x \leq C_x(Ax \wedge y) \leq \bigwedge_{v:\, x \leq v \lessdot Ax} (v \wedge y) = C_x Ax \wedge y = x \wedge y = x.$$

So $[x, Ax \wedge y]$ is coatomic, hence atomic, so $Ax \wedge y$ is the join of some of the elements that cover x but are less than y, and hence is less than or equal to the join $A_y x$ of all elements that cover x and are less than or equal to y.

(5) Since $x \wedge y \leq x, y$, it follows that $A(x \wedge y) \leq Ax, Ay$ and hence $A(x \wedge y) \leq Ax \wedge Ay$.

Now we show that $[x \wedge y, Ax \wedge Ay]$ is coatomic. The intervals $[x, Ax]$ and $[y, Ay]$ are atomic and hence coatomic. The meet of any coatom of $[x, Ax]$ with Ay, and the meet of any coatom of $[y, Ay]$ with Ax, is either $Ax \wedge Ay$ or is covered by $Ax \wedge Ay$. So in $(Ay \wedge C_x Ax) \wedge (Ax \wedge C_y Ay) = x \wedge y$, the left side can be expressed as the meet of coatoms of $[x \wedge y, Ax \wedge Ay]$, whence this interval is coatomic, and hence atomic. Thus $Ax \wedge Ay \leq A(x \wedge y)$; the reverse inequality was already shown, so $Ax \wedge Ay = A(x \wedge y)$.

(6) If $C^k x \leq y$ then $x \leq A^k(C^k x) \leq A^k y$; the first inequality is by (2), and the second is by Theorem 3.2. Dually, if $A^k y \geq x$ then $y \geq C^k(A^k y) \geq x$. $\square$

Corollary 3.4 (Monotonicity of A, C lattice polynomials). *In modular lattices of finite length, all lattice polynomials constructed from variables, meets, joins, and the unary or binary forms of the operators A and C, weakly increase as the variables weakly increase.*

This extends the monotonicity of lattice polynomials constructed from variables, meets, and joins in arbitrary lattices.

3.3. Interval types in modular lattices. To each finite length interval $[x, y]$ in a lattice, we associate two strong compositions. Let

$$\operatorname{Atype}[x,y] \stackrel{\text{def}}{=} \Big(\rho(x, A_y x), \rho(A_y x, A_y^2 x), \rho(A_y^2 x, A_y^3 x), \ldots\Big)$$
$$\operatorname{Ctype}[x,y] \stackrel{\text{def}}{=} \Big(\rho(C_x y, y), \rho(C_x^2 y, C_x y), \rho(C_x^3 y, C_x^2 y), \ldots\Big)$$

We consider evaluating these in our three example lattices.

(1) **Subgroup lattice.** We compute $\operatorname{Ctype}[0, G]$, where

$$G = \mathbb{Z}/p^{\lambda_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{\lambda_k}\mathbb{Z}.$$

The rank of G in the lattice is $\lambda_1 + \ldots + \lambda_k$, and

$$C^r G = p^r \mathbb{Z}/p^{\lambda_1}\mathbb{Z} \times \cdots \times p^r \mathbb{Z}/p^{\lambda_k}\mathbb{Z}.$$

When $r \geq \lambda_i$, the factor $p^r\mathbb{Z}/p^{\lambda_i}\mathbb{Z}$ is 0, and when $r < \lambda_i$, it has rank $\lambda_i - r$, so the rank of C^rG is

$$\sum_{i=1}^{k} \max\{\lambda_i - r, 0\} = \sum_{j>r} \lambda'_j.$$

The successive differences in ranks as r increases are thus $\lambda'_1, \lambda'_2, \ldots$. So $\mathrm{Ctype}[0, G] = \lambda'$.

Now compute $\mathrm{Atype}[0, G]$. We have

$$A^r 0 = \{ g \in H \,:\, p^r g = 0 \} = p^{\max\{\lambda_1 - r, 0\}}\mathbb{Z}/p^{\lambda_1} \times \cdots \times p^{\max\{\lambda_k - r, 0\}}\mathbb{Z}/p^{\lambda_k}.$$

The successive differences in ranks are $\lambda'_1, \lambda'_2, \ldots$ so $\mathrm{Atype}[0, G] = \lambda'$.

(2) **Invariant subspace lattice.** Consider the lattice $L(V, N)$ of type λ. Choose a basis of V,

$$e_{ij} \text{ with } (i, j) \in \lambda, \text{ such that } Ne_{i1} = 0 \text{ and } Ne_{ij} = e_{i,j-1} \text{ when } j > 1.$$

We have

$$C^rV = N^rV = \mathrm{span}\,\{ e_{ij} \,:\, (i, j) \in \text{ and } (i, j + r) \in \lambda \},$$

which has rank $\lambda'_{r+1} + \lambda'_{r+2} + \cdots$. Thus, the successive differences in ranks again yield the partition $\mathrm{Ctype}[0, V] = \lambda'$.

Now

$$A^rV = \ker N^r = \mathrm{span}\,\{ e_{ij} \,:\, (i, j) \in \lambda \text{ and } j \leq r \}$$

has rank $\lambda'_1 + \cdots + \lambda'_r$. Thus, the successive differences in ranks yet again yield the same partition, $\mathrm{Atype}[0, V] = \lambda'$.

We will later see that in all semi-primary lattices, Atype and Ctype are the same, and are the conjugate partition of the natural notion of type. This is unique to semi-primary lattices.

(3) **The Fibonacci Lattice $Z(r)$.** We will fully compute Atype and Ctype in $Z(r)$ in Section 9.1; for now we consider an example. Let $x = 221_2 21_1$ in $Z(2)$. We compute $\mathrm{Ctype}[\hat{0}, x]$. We showed before that when $r > 1$, Cx is evaluated by stripping off the first digit of x, so the successive drops in rank are $2, 2, 1, 2, 1$, and $\mathrm{Ctype}[\hat{0}, x] = (2, 2, 1, 2, 1)$. Now observe that $A^3\hat{0} = 222 \leq x$, and that the interval $[222, x]$ is the chain $\{222 \lessdot 2221_1 \lessdot 221_2 21_1\}$. So $\mathrm{Atype}[\hat{0}, x] = (2, 2, 2, 1, 1)$, which is different from $\mathrm{Ctype}[\hat{0}, x]$.

Theorem 3.5. *The sequences* $\mathrm{Atype}[x, y]$ *and* $\mathrm{Ctype}[x, y]$ *are strong compositions. In a modular lattice, they both have the same number of parts.*

Proof. The first is a strong composition because if r is the minimum nonnegative integer with $A^r_y x = y$, then for $0 < i \leq r$, the ith part of $\mathrm{Atype}[x, y]$ is nonzero as $A_y(A^{i-1}_y x) > A^{i-1}_y x$, while for $i > r$, the ith part is 0. Similarly, if s is the minimum nonnegative integer with $C^s_x y = x$, then $\mathrm{Ctype}[x, y]$ is a strong composition with s parts.

By Theorem 3.3(2), we have $y \leq A^s_y C^s_x y$. Also $y \geq A^s_y C^s_x y$, since A_y yields elements at most y. Thus, $y = A^s_y C^s_x y = A^s_y x$, so $s \geq r$. By a dual argument, $s \leq r$, so $s = r$. $\square$

The number of parts of $\mathrm{Atype}[x, y]$, or equivalently of $\mathrm{Ctype}[x, y]$, is the **atomic rank** of the interval $[x, y]$, and will be denoted $\mathrm{arank}[x, y]$. It is also the minimum nonnegative r satisfying any of the following equivalent conditions: $A^r_y x = y$; $A^r x \geq y$; $C^r_x y = x$; and $C^r y \leq x$. In a modular lattice with $\hat{0}$, define $\mathrm{arank}\, x = \mathrm{arank}[\hat{0}, x]$, and in a finite height modular lattice $\mathcal{L}$, define $\mathrm{arank}\, \mathcal{L} = \mathrm{arank}[\hat{0}, \hat{1}]$. The function $\mathrm{arank}[x, y]$ weakly increases as x decreases or y increases, and when the decrease in x or increase in y is by a single cover relation, the function stays the same or goes up by 1. Also see [**32**, p. 215], which defines the ascending and descending Loewy series in a complete modular lattice, which in our notation are $\hat{0} < A\hat{0} < \cdots < A^r\hat{0} = \hat{1}$ and $\hat{0} = C^s\hat{1} < \cdots < C\hat{1} < \hat{1}$, and shows $r = s$.

Theorem 3.6. *In a modular lattice, if $x \leq y \leq z$ then $\mathrm{Atype}[x, y] \leq \mathrm{Atype}[x, z]$ and $\mathrm{Ctype}[y, z] \leq \mathrm{Ctype}[x, z]$ as compositions.*

Proof. The ith part of $\mathrm{Atype}[x, y]$ is the length of the interval $[A^{i-1}_y x, A^i_y x]$. The ith part of $\mathrm{Atype}[x, v]$ is the length of the interval $[A^{i-1}_v x, A^i_v x] = [v \wedge A^{i-1}_y x, v \wedge A^i_y x]$, and is hence weakly smaller by Lemma 2.2. $\square$

When $x \leq y \leq z$, the relationship between $\mathrm{Atype}[x, z]$ and $\mathrm{Atype}[y, z]$, or between $\mathrm{Ctype}[x, z]$ and $\mathrm{Ctype}[x, y]$, is not as nicely described, in general, but we describe it here for completeness. In the theorem, let $|\mu_{\leq k}|$ denote $\mu_1 + \mu_2 + \cdots + \mu_k$.

Theorem 3.7. *Let $x \lessdot y \leq z$, and set $\mu = \mathrm{Atype}[x, z]$ and $\nu = \mathrm{Atype}[y, z]$. Dually, let $x \leq y \lessdot z$, and set $\mu = \mathrm{Ctype}[x, z]$ and $\nu = \mathrm{Ctype}[x, y]$. In each case, for all $i \geq 0$ we have $|\mu_{\leq i}| \leq 1 + |\nu_{\leq i}| \leq |\mu_{\leq i+1}|$. If either inequality is equality for a particular i, it is equality for all larger i.*

Proof. We have $x \lessdot y$ so $A^0_z x \leq A^0_z y \leq A^1_z x$. Apply A^i_z to obtain $A^i_z x \leq A^i_z y \leq A^{i+1}_z x$. If either inequality is equality for some i, it is equality for all larger i, as we are applying a power of A_z to two equal elements. Take the ranks of these elements in the interval $[x, z]$ to obtain $\rho(x, A^i_z x) \leq \rho(x, A^i_z y) = 1 + \rho(y, A^i_z y) \leq \rho(x, A^{i+1}_z x)$. Finally, express this in terms of μ and ν as $|\mu_{\leq i}| \leq 1 + |\nu_{\leq i}| \leq |\mu_{\leq i+1}|$. $\square$

Theorem 3.8. *In a graded lattice, let $\mathrm{Atype}[a, b] = \lambda = (\lambda_1, \lambda_2, \ldots)$. For each k, the element $A^k_b a$ is the unique element c of $[a, b]$ with $\mathrm{Atype}[a, c] = (\lambda_1, \ldots, \lambda_k)$. Dually, if $\mathrm{Ctype}[a, b] = (\lambda_1, \lambda_2, \ldots)$, then for each k, the unique element c of $[a, b]$ with $\mathrm{Ctype}[c, b] = (\lambda_1, \ldots, \lambda_k)$ is $C^k_a b$.*

Proof. We prove the first statement; the second statement may be proved dually. Let $c \in [a, b]$ with $\mathrm{Atype}[a, c] = (\lambda_1, \ldots, \lambda_k)$. The ith part of $\mathrm{Atype}[a, c]$ is $\rho(A^{i-1}_c a, A^i_c a)$, and of $\mathrm{Atype}[a, b]$ is $\rho(A^{i-1}_b a, A^i_b a)$. We have $A^0_b a = A^0_c a = a$. We have $A^1_b a \geq A^1_c a$, since the right side is the join of a subset of the elements of which the left side is a join. Since the first part of $\mathrm{Atype}[a, b]$ and $\mathrm{Atype}[a, c]$ are equal, $\rho(A^1_b a) = \rho(A^1_c a)$, so in fact $A^1_b a = A^1_c a$. By iterating this, $A^i_b a = A^i_c a$ for $i = 0, \ldots, k$, since the first k parts of $\mathrm{Atype}[a, b]$ and $\mathrm{Atype}[a, c]$ agree. For further parts of $\mathrm{Atype}[a, c]$ to be 0 requires that $A^{i-1}_c a = A^i_c a$ when $i > k$, which requires $A^i_c a = c$ for $i \geq k$. So $c = A^k_c a = A^k_b a$. $\square$

Another way to compute Atype and Ctype is the following.

Theorem 3.9. *Let $\mathcal{L}$ be a modular lattice of finite height. Let $x \in \mathcal{L}$. Let $\lambda = \operatorname{Atype}\mathcal{L}$ and $\nu = \operatorname{Atype}[\hat{0}, x]$. Let $\mu_j = \rho(x \vee A^{j-1}\hat{0}, x \vee A^j\hat{0})$. Then $\mu + \nu = \lambda$.*

Dually, let $\lambda = \operatorname{Ctype}\mathcal{L}$ and $\nu = \operatorname{Ctype}[x, \hat{1}]$. Let $\mu_j = \rho(x \wedge C^{j-1}\hat{1}, x \wedge C^j\hat{1})$. Then $\mu + \nu = \lambda$.

Proof. We prove the statement for Atype. By the modular rank equation,

$$\rho(x \vee A^j\hat{0}) = \rho(x) + \rho(A^j\hat{0}) - \rho(x \wedge A^j\hat{0}) = \rho(x) + \rho(A^j\hat{0}) - \rho(A^j_x a).$$

Subtracting the like equation for $j-1$ from this yields

$$\mu_j = \rho(A^{j-1}\hat{0}, A^j\hat{0}) - \rho(A^{j-1}_x\hat{0}, A^j_x\hat{0}) = \lambda_j - \nu_j. \quad \square$$

Corollary 3.10. *Let $\mathcal{L}$ be a finite height modular lattice. Let $x \in \mathcal{L}$. Let $\lambda = \operatorname{Atype}\mathcal{L}$ and $\tilde{\nu} = \operatorname{Atype}[\hat{0}, x]$. Then*

$$\operatorname{Atype}[\hat{0}, x \vee A^k\hat{0}] = (\lambda_1, \ldots, \lambda_k, \tilde{\nu}_{k+1}, \tilde{\nu}_{k+2}, \ldots).$$

Dually, let $x \in \mathcal{L}$. Let $\lambda = \operatorname{Ctype}\mathcal{L}$ and $\tilde{\nu} = \operatorname{Atype}[x, \hat{1}]$. Then

$$\operatorname{Ctype}[x \wedge A^k_a b, x] = (\lambda_1, \ldots, \lambda_k, \tilde{\nu}_{k+1}, \tilde{\nu}_{k+2}, \ldots).$$

Proof. We prove the statement for Atype. Let $z = x \vee A^k\hat{0}$. Let $\nu = \operatorname{Atype}[\hat{0}, z]$ and for all $j \geq 1$ let $\mu_j = \rho(z \vee A^{j-1}\hat{0}, z \vee A^j\hat{0})$. When $j \leq k$,

$$z \vee A^j\hat{0} = x \vee A^k\hat{0} \vee A^j\hat{0} = x \vee A^k\hat{0}$$

so $\mu_1 = \cdots = \mu_j = 0$. Thus the first k parts of ν equal the first k parts of λ. When $j \geq k$ we have $z \vee A^j\hat{0} = x \vee A^j\hat{0}$, so that the remaining parts of ν equal the remaining parts of $\tilde{\nu}$. $\square$

4. Semi-primary lattices

4.1. Basic concepts.

Definition 4.1. An element c of a finite length lattice is called a **cycle** if $[\hat{0}, c]$ is a chain and a **cocycle** if $[c, \hat{1}]$ is a chain [**13**, 4.1]. An r-cycle is a cycle of rank r, and an r-cocycle is a cocycle of corank r.

Definition 4.2. A lattice is **semi-primary** if it is modular of finite length, all join-irreducibles are cycles, and all meet-irreducibles are cocycles [**13**, 4.2].

We consider our three sample lattices.

(1) **Subgroup lattice.** Consider $L(G)$. If a subgroup H of G is the product of two or more nonzero groups, then H is the join of those smaller groups, and hence is not a join-irreducible. So the join-irreducibles are the nonzero cyclic subgroups of G, that is, those isomorphic to $\mathbb{Z}/p^r\mathbb{Z}$ for some $r > 0$. The subgroups of this group form a chain

$$p^r\mathbb{Z}/p^r\mathbb{Z} < p^{r-1}\mathbb{Z}/p^r\mathbb{Z} < \cdots < \mathbb{Z}/p^r\mathbb{Z},$$

so any join-irreducible in $L(G)$ is a cycle.

Dually, the meet-irreducibles are those proper subgroups H of G for which G/H is a cyclic group, and a similar argument applies. Thus, $L(G)$ is a semi-primary lattice.

(2) **Invariant subspace lattice.** Consider $L(V,N)$. For any nonzero vector $\vec{v} \in V$, let $S(\vec{v}) = \operatorname{span}\{\vec{v}, N\vec{v}, N^2\vec{v}, \ldots\}$. We see that for any N-invariant subspace W of V, if $\vec{v} \in W$ then $S(\vec{v}) \leq W$. If we cannot express W as $W = S(\vec{v})$, we must choose several vectors and express W as the span of the spaces $S(\vec{v_1}), S(\vec{v_2}), \ldots$. Each of these spaces is a proper N-invariant subspace of V, so W is not join-irreducible. Hence, the join-irreducibles are those spaces of the form $S(\vec{v})$. The subspaces of $S(\vec{v})$ form the chain

$$\hat{0} = S(N^r\vec{v}) \lessdot S(N^{r-1}\vec{v}) \lessdot \cdots \lessdot S(\vec{v})$$

for some r. Thus the join-irreducibles of $L(V,N)$ are cycles. By a dual argument, the meet-irreducibles are cocycles.

(3) **The Fibonacci Lattice $Z(r)$.** These are not semi-primary. The only lower cover of 122 is 22, so 122 is a join-irreducible. However, the interval $[\emptyset, 122]$ has incomparable elements 12 and 21, so it is not a chain.

The lattice $Z(r)$ has infinite length, but even the finite length closed intervals in it are not semi-primary, except in trivial cases.

Now we develop the theory of semi-primary lattices. The development is based on that of Jónsson and Monk [**13**], but we introduce the operators A and C; the partitions Atype and Ctype; and we show the connection the types have with Young's lattice.

Theorem 4.3. *Every closed interval in a semi-primary lattice $\mathcal{L}$ is semi-primary [**13**, 4.3–4.4].*

Proof. Every interval in a modular lattice is modular. Let $[a,b]$ be a closed interval and c a join-irreducible of the induced subposet $[a,b]$ of $\mathcal{L}$. We must show $[a,c]$ is a chain. We do not consider a to be join-irreducible in $[a,c]$, so $c > a$, and there is a unique coatom v of $[a,c]$. Let z be a join-irreducible of $\mathcal{L}$ with $z \leq c$ but $z \not\leq v$; since every element is the join of join-irreducibles and $c \gtrdot v$, such a join-irreducible exists. Then $a \vee z$ is in $[a,c]$ but not $[a,v]$; these intervals only differ in the element c, so $a \vee z = c$. So $[a,c] = [a, a\vee z] \cong [a \wedge z, z]$, and since z is a join-irreducible of $\mathcal{L}$ and hence a cycle, the last interval is a chain. Thus, c is a cycle in $[a,b]$. The dual argument shows that the meet-irreducibles of $[a,b]$ are cocycles. $\square$

Recall that one way of defining $\operatorname{arank}\mathcal{L}$ is as the minimum r for which $A^r\hat{0} = \hat{1}$.

Theorem 4.4. *The maximum of $\rho(c)$ as c ranges over the cycles of a semi-primary lattice $\mathcal{L}$ is* $\operatorname{arank}\mathcal{L}$ *(cf. [**13**, 4.6]).*

Proof. Let $r = \operatorname{arank}\mathcal{L}$. If $\mathcal{L}$ is a chain then the maximum rank cycle is $\hat{1}$ and indeed $\operatorname{arank}\mathcal{L} = r = \rho(\hat{1})$. So assume $\mathcal{L}$ is not a chain.

For any cycle c, we have $c = c \wedge \hat{1} = c \wedge A^r\hat{0} = A_c^r\hat{0}$, so all cycles have $\rho(c) \leq r$. For any coatom v of $\mathcal{L}$, we have $\operatorname{arank} v \leq r$. If some coatom has $\operatorname{arank} v = r$, then any r-cycle in $[\hat{0}, v]$ is also an r-cycle in $\mathcal{L}$, so the upper bound is achieved. So assume all coatoms have $\operatorname{arank} vler - 1$. Then $v \leq A^{r-1}\hat{0}$ for all coatoms, so $A^{r-1}\hat{0}$ is weakly greater than the join of all coatoms. Since $\mathcal{L}$ is not a chain, $\hat{1}$ is not a join-irreducible, so there are multiple coatoms, and the join of all coatoms is $\hat{1}$. Thus $\hat{1} \leq A^{r-1}\hat{0}$, so $\operatorname{arank}\mathcal{L} \leq r - 1$, contradicting $\operatorname{arank}\mathcal{L} = r$. $\square$

Theorem 4.5. *If $\mathcal{L}$ is a semi-primary lattice and c is a cycle of maximum possible rank $r = \operatorname{arank}\mathcal{L}$, then c has a complement in $\mathcal{L}$. Any such complement is a cocycle of corank r (cf. [**13**, 4.8]).*

Note. See Theorem 4.44 for an enumeration of complemented cycles of each rank; Theorem 4.43 for a complete characterization of complemented elements; and Theorem 4.47 for an enumeration of the number of complements an element has.

Proof. We induct on intervals ordered by inclusion. The trivial case $c = \hat{1}$ has complement $\hat{0}$. Assume $c \neq \hat{1}$. There must be an atom v of $\mathcal{L}$ besides the one in $[\hat{0}, c]$, because if that is the only atom, then $\hat{0}$ is a meet-irreducible and hence a cocyle, so that $[\hat{0}, \hat{1}]$ is a chain, so that the maximum cycle is $c = \hat{1}$. The element $c \vee v$ is an r-cycle in $[v, \hat{1}]$ because $[v, c \vee v] \cong [c \wedge v, c] = [\hat{0}, c]$, so $c \vee v$ is a maximal cycle in $[v, \hat{1}]$ (where maximality follows from the preceding theorem and monotonicity of arank on intervals). By induction, the current theorem applies to this smaller interval, whence there is $c' \in [v, \hat{1}]$ with $(c \vee v) \vee c' = \hat{1}$ and $(c \vee v) \wedge c' = v$. So $c \vee c' = c \vee (v \vee c') = \hat{1}$, and $v \vee (c \wedge c') = v$, so $c \wedge c' \leq v$, so $c \wedge c' \leq c \wedge v = \hat{0}$. Thus c' is a complement to c in $\mathcal{L}$.

For any complement c' of c, we have $[c', \hat{1}] = [c', c' \vee c] \cong [c' \wedge c, c] = [\hat{0}, c]$, so c' is a cocycle of corank r. $\square$

Definition 4.6. A sequence $x_1, \ldots, x_k$ of elements of a finite length modular lattice are **(join) independent** if any of the following equivalent conditions hold [**13**, §2].

(1) $\rho(x_1 \vee \cdots \vee x_k) = \rho(x_1) + \cdots + \rho(x_k)$.
(2) $(x_1 \vee \cdots \vee x_{i-1}) \wedge x_i = \hat{0}$ for $i = 2, \ldots, k$.
(3) For any subsets I and J of $\{1, \ldots, n\}$,

$$\left(\bigvee_{i \in I} x_i\right) \wedge \left(\bigvee_{i \in J} x_i\right) = \left(\bigvee_{i \in I \cap J} x_i\right),$$

so in fact, $\{x_1, \ldots, x_k\}$ generates a Boolean subalgebra of the lattice.

There is also a dual notion of meet independence.

Theorem 4.7. *Every element b of a semi-primary lattice $\mathcal{L}$ is the join of independent cycles [**13**, 4.9]. If $b = x_1 \vee \cdots \vee x_k$ is such a join and the x_i are arranged so that $\rho(x_1) \geq \cdots \geq \rho(x_k)$, then the partition $(\rho(x_1), \ldots, \rho(x_k))$ is $(\operatorname{Ctype}[\hat{0}, b])'$.*

Note. See Theorem 4.48 for an enumeration of the ways to choose $x_1, \ldots, x_k$.

Proof. We induct on $\rho(b)$. If $b = \hat{0}$ the join is empty. By the last two theorems, we may choose a cycle x_1 of maximum rank in b, and then a complement y_1 to it in b; since $\rho(y_1) = \rho(b) - \rho(x_1) < \rho(b)$, we inductively have y_1 as the join of independent cycles $x_2, \ldots, x_k$, and so b is the join of independent cycles $x_1, \ldots, x_k$.

Given $b = x_1 \vee \cdots \vee x_k$ with the x_i independent cycles, for any $r \geq 0$, the cycles $C^r x_1, \ldots, C^r x_k$ are independent, so $C^r b = C^r x_1 \vee \cdots \vee C^r x_k$ has rank $\rho(C^r b) = \rho(C^r x_1) + \cdots + \rho(C^r x_k)$. Since $\rho(C^r x_i) = \max\{\rho(x_i) - r, 0\}$, we have for $r > 0$ that $\rho(C^{r-1} b) - \rho(C^r b)$ is the number of i with $\rho(x_i) = r$. $\square$

Definition 4.8. Although the independent cycles in the last theorem are not uniquely determined by b, the partition determined by their ranks is unique. This partition is called the **type of** b, denoted $\operatorname{type} b$. The **type of a semi-primary lattice** is $\operatorname{type} \mathcal{L} = \operatorname{type} \hat{1}$. The **type of an interval** $[a, b]$, denoted $\operatorname{type}[a, b]$, is the type of the semi-primary lattice $[a, b]$, so it may be expressed as the partition of decreasing ranks of independent join-irreducibles in $[a, b]$ whose join is b, and as $(\operatorname{Ctype}[a, b])'$. The **cotype of** b is $\operatorname{cotype} b = \operatorname{type}[b, \hat{1}]$.

An alternative approach to this last theorem is taken in [**13**, pp. 97,103]. They do not compute anything analogous to $\operatorname{type} b = (\operatorname{Ctype}[\hat{0}, b])'$. Instead, they apply a theorem of Ore [**19**]: If an element b of a finite length modular lattice has two representations as a join of independent join-irreducibles,

$$b = x_1 \vee \cdots \vee x_k = y_1 \vee \cdots \vee y_m,$$

then $k = m$ and there is a permutation σ of $1, \ldots, k$ such that

$$b = y_{\sigma(1)} \vee \cdots \vee y_{\sigma(i)} \vee x_{i+1} \vee \cdots \vee x_k$$

for $i = 1, \ldots, k$. Thus, x_i and $y_{\sigma(i)}$ have the same rank for each i. A complete proof of Ore's theorem in this form can be found in [**2**, pp. 58–60]. If we apply this theorem to a semi-primary lattice, we see that any two expressions of b as the join of independent join-irreducibles have the same number of cycles and the same multiset of ranks of these cycles, so that $\operatorname{type} b = (\rho(x_1), \ldots, \rho(x_k))$ is well defined.

Theorem 4.9. *The type of a semi-primary lattice $\mathcal{L}$ and the type of its dual are equal [**13**, 4.11].*

Proof. Let $\hat{1} = x_1 \vee \cdots \vee x_k$ be the join of independent join-irreducibles with $\rho(x_k) \geq \cdots \geq \rho(x_1)$. The x_i's are independent, so lattice polynomials in them form a boolean algebra. The complement of x_i is $y_i = \vee_{j \neq i} x_i$, the y_i's are independent in the dual lattice to $\mathcal{L}$, and their meet is $\hat{0}$. Further, $[y_i, \hat{1}] = [y_i, x_i \vee y_i] \cong [x_i \wedge y_i, x_i] = [\hat{0}, x_i]$, so y_i is a meet-irreducible in $\mathcal{L}$ whose corank is the rank of x_i. Thus, the partition $(\bar{\rho}(y_1), \ldots, \bar{\rho}(y_k))$ giving the type of the dual of $\mathcal{L}$, and the partition $(\rho(x_1), \ldots, \rho(x_k))$ giving the type of $\mathcal{L}$, are the same. □

Note. A semi-primary lattice need not be isomorphic to its dual. See Figure 1 for a minimal example of an irregular (Definition 4.24) semi-primary lattice that's not self-dual. For regular semi-primary lattices, the smallest examples are provided by non-Desarguean projective planes. There are four projective planes of order 9, two of which are self-dual, and the other two of which are dual to each other; see Stevenson [**30**, p. 80]. Thus, there are two 9-regular semi-primary lattices of type $(1, 1, 1)$ that are not self-dual.

Corollary 4.10. *The type of an interval $[a, b]$ of a semi-primary lattice may be computed as any of the following.*

1. $(\rho(a, x_1), \ldots, \rho(a, x_k))$, *where $x_1, \ldots, x_k$ are independent join-irreducibles of weakly decreasing rank in $[a, b]$ whose join is b;*
2. $(\rho(y_1, b), \ldots, \rho(y_k, b))$, *where $y_1, \ldots, y_k$ are independent meet-irreducibles of weakly decreasing corank in $[a, b]$ whose meet is a;*
3. $(\operatorname{Ctype}[a, b])'$;

(4) $(\mathrm{Atype}[a,b])'$.

Proof. (1) and (3) are equal by Theorem 4.7, while (2) and (4) are equal by a dual argument, and (1) and (2) are equal by Theorem 4.9. $\square$

Inaba [**12**, Theorem 34] proved that if $a \geq b$, the number of parts in type a is weakly larger than in type b; that type a is lexicographically weakly larger than type b; and for each $k > 0$, the number of parts of size at least k in type a weakly exceeds that in type b. This is a cryptic description of Young's lattice. It generalizes to intervals as follows.

Theorem 4.11. *Let $a \leq a' \leq b' \leq b$. Then* $\mathrm{type}[a,b] \geq \mathrm{type}[a',b']$ *in Young's lattice.*

Proof. Both types are partitions in Young's lattice. We have

$$(\mathrm{type}[a,b])' = \mathrm{Atype}[a,b] \geq \mathrm{Atype}[a,b'] = (\mathrm{type}[a,b'])',$$

where the equalities hold by the preceding theorem and the inequality holds by Theorem 3.6 (considering the partitions to be compositions, but noting the order is the same). Similarly,

$$(\mathrm{type}[a,b'])' = \mathrm{Ctype}[a,b'] \geq \mathrm{Ctype}[a',b'] = (\mathrm{type}[a',b'])'.$$

Thus $(\mathrm{type}[a,b])' \geq (\mathrm{type}[a',b'])'$, so $\mathrm{type}[a,b] \geq \mathrm{type}[a',b']$. $\square$

Definition 4.12. For an interval $[a,b]$ in a semi-primary lattice, let $[a,b][k]$ be the join of all cycles of rank at most k, and $[a,b](k)$ be the meet of all cocycles of corank at most k. This extends the notation $x[k] = [\hat{0},x][k]$ defined in [**13**, 4.5]. We also define $x(k) = [x,\hat{1}](k)$.

For any partition λ and $k \geq 0$, let

$$\lambda[k] = (\min\{\lambda_1,k\}, \min\{\lambda_2,k\}, \ldots) = (\lambda'_1,\ldots,\lambda'_k)'.$$

Theorem 4.13. *Let $\mathcal{L}$ be a semi-primary lattice of type λ. There is a unique element of type $\lambda[k]$, and it may be expressed as $\hat{1}[k]$ and $A^k\hat{0}$. In an interval $[a,b]$ of type λ, there is a unique element z with* $\mathrm{type}[a,z] = \lambda[k]$*, and it may be expressed $[a,b][k] = A^k_b a$. Dually, $[a,b](k) = C^k_a b$ is the unique element of cotype $\lambda[k]$ in $[a,b]$.*

Proof. We prove the first sentence, and the others follow by applying it to intervals, which are semi-primary, and by dualizing. By Theorem 3.8, $A^k\hat{0}$ is the unique element with this type. Suppose c is a cycle of rank at most k. Then $c = A^k_c\hat{0} = c \wedge A^k\hat{0}$, so $c \leq A^k\hat{0}$, so the join of all such cycles satisfies $\hat{1}[k] \leq A^k\hat{0}$, whence $\mathrm{type}\,\hat{1}[k] \leq \lambda[k]$. Conversely, given any decomposition of $\hat{1}$ into independent cycles, $\hat{1} = x_1 \vee \cdots \vee x_m$, we have that $x_1[k],\ldots,x_m[k]$ are independent cycles of rank at most k, whose ranks are precisely the row lengths of $\lambda[k]$. Thus, their join has type $\lambda[k]$; this join is a lower bound of $\hat{1}[k]$ because it is a join of only some of the cycles of rank at most k, so $\lambda[k] \leq \mathrm{type}\,\hat{1}[k]$, whence in fact $\mathrm{type}\,\hat{1}[k] = \lambda[k]$. Since there is a unique element of this type, $\hat{1}[k] = A^k\hat{0}$. $\square$

Corollary 4.14. *If a and b are independent elements in a semi-primary lattice, then $(a \vee b)[k] = a[k] \vee b[k]$.*

Proof. (cf. [**13**, 4.13]) Since a and b are independent, $\mathrm{type}(a \vee b) = (\mathrm{type}\, a) \cup (\mathrm{type}\, b)$, and

$$\begin{aligned} \mathrm{type}\Big((a \vee b)[k]\Big) &= \Big((\mathrm{type}\, a) \cup (\mathrm{type}\, b)\Big)\,[k] \;=\; (\mathrm{type}\, a)[k] \cup (\mathrm{type}\, b)[k] \\ &= \Big(\mathrm{type}\, a[k]\Big) \cup \Big(\mathrm{type}\, b[k]\Big). \end{aligned}$$

Since $a[k] \leq a$ and $b[k] \leq b$, the elements $a[k], b[k]$ are independent, so the type of $a[k] \vee b[k]$ is precisely the last expression in the above equation. By the previous theorem, $(a \vee b)[k]$ is the unique element of $[\hat{0}, a \vee b]$ whose type is the first expression in this equation. Since $a[k] \vee b[k]$ is in this interval, it must equal $(a \vee b)[k]$. □

4.2. Alternate characterizations of semi-primary lattices. We present some new characterizations of semi-primary lattices. Theorem 4.15(1)–(3) is the classical definition of semi-primary lattices, and all the other characterizations in this section are new. In particular, Theorem 4.15(4) is a local characterization of semi-primary lattices; Theorem 4.16 is a rank equation characterization; and Theorem 4.17 is analogous to interval perspectivity in modular lattices, and is essentially a lattice equation characterization of semi-primary lattices.

Theorem 4.15. *Let $\mathcal{L}$ be a modular lattice of finite length. Then the following are equivalent.*

(1) *$\mathcal{L}$ is semi-primary.*
(2) *Every element is the join of cycles and the meet of cocycles.*
(3) *Every join-irreducible is a cycle and every meet-irreducible is a cocycle.*
(4) *Whenever x covers precisely one element, that element is either $\hat{0}$ or in turn covers precisely one element. Whenever x is covered by precisely one element, that element is either $\hat{1}$ or is covered by precisely one element.*
(5) *For every interval $[x, y]$, we have $\mathrm{Atype}[x, y] = \mathrm{Ctype}[x, y]$.*
(6) *For every interval $[x, y]$, both $\mathrm{Atype}[x, y]$ and $\mathrm{Ctype}[x, y]$ are partitions.*
(7) *Whenever $x \lessdot y$, we have $Cx \leq Cy$ and $Ax \leq Ay$.*

Proof.

(1)⇔(2): This is the definition of a semi-primary lattice.

(2)⇔(3): In any lattice, every element is the join of join-irreducibles and the meet of meet-irreducibles, and all cycles are join-irreducibles and all cocycles are meet-irreducibles.

(3)⇔(4): The elements with unique lower covers are precisely the join-irreducibles, and the elements with unique upper covers are precisely the meet-irreducibles.

(1)⇒(5): This is Theorem 4.9.

(5)⇒(4): Suppose x is a join-irreducible. Then $\mathrm{Ctype}[\hat{0}, x]$ begins with a 1, so by (5), $\mathrm{Atype}[\hat{0}, x]$ does too. Let $v = Cx$. If $v = \hat{0}$, then x is an atom and hence a cycle. Otherwise, $\hat{0} \leq A_v\hat{0} \leq A_x\hat{0}$, but $A_x\hat{0}$ is an atom because $\mathrm{Atype}[\hat{0}, x]$ begins with a 1, and $A_v\hat{0}$ is nonzero because $v > \hat{0}$. So $A_v\hat{0} = A_x\hat{0}$, so $\mathrm{Atype}[\hat{0}, v]$ begins with a 1, so $\mathrm{Ctype}[\hat{0}, v]$ begins with a 1, so v is also a join-irreducible. Thus, (4) holds.

(1)⇒(6): This follows from Theorem 4.7.

(6)⇒(3): Let x be a join-irreducible. Then $\mathrm{Ctype}[\hat{0}, x]$ begins with 1, and the entries are weakly decreasing, so in fact, all the nonzero entries are 1. So $[\hat{0}, x]$ is a chain, whence x is a cycle. Similarly, meet-irreducibles are cocycles, so $\mathcal{L}$ is semi-primary.

(1)⇒(7): Since $\mathrm{Ctype}[\hat{0}, x] \leq \mathrm{Ctype}[\hat{0}, y]$ in Young's lattice (by Theorem 4.11) and the right side has one more square than the left, in fact the two partitions are equal in all but one part, in which the second partition exceeds the first by 1. Now $Cx \leq Cy$ by Theorem 3.2, so if the partitions are equal in the first part, $\rho(Cx) \lessdot \rho(Cy)$ and $Cx \lessdot Cy$, while if the partitions differ by 1 in the first part, $Cx = Cy$. By a dual argument, $Ax \leq Ay$.

(7)⇒(4): Let y be a join-irreducible, so $Cy \lessdot y$. Then $C^2y \leq Cy$, so Cy is a join-irreducible or $\hat{0}$. Dually, the unique element covering a meet-irreducible is also a meet-irreducible or $\hat{1}$. □

Note. (4) characterizes semi-primary lattices by a local condition similar to the local condition that can be used to describe modular and semimodular lattices. While the preceding characterizations are in terms of cycles and cocycles, and hence require there to be a $\hat{0}$ and $\hat{1}$ in the lattice, this characterization does not require either, and so may possibly be used to extend the definition of semi-primary lattices to discrete modular lattices of infinite length.

Theorem 4.16. *Let $\mathcal{L}$ be a finite length graded lattice. Then $\mathcal{L}$ is semi-primary iff the* ***semi-primary equation***

$$\rho(A^k x \wedge y) + \rho(x \vee C^k y) = \rho(x) + \rho(y)$$

holds for all nonnegative integers k and all x, y in $\mathcal{L}$, iff it holds for $k = 0, 1$.

Proof. Let $\mathcal{L}$ be semi-primary. If $x \leq y$, this equation is equivalent to the fact that $\mathrm{Atype}[x, y] = \mathrm{Ctype}[x, y]$, since the sum of the first k parts of $\mathrm{Atype}[x, y]$ is $\rho(A^k x \wedge y) - \rho(x)$ and the sum of the first k parts of $\mathrm{Ctype}[x, y]$ is $\rho(y) - \rho(x \vee C^k y)$. For arbitrary x and y, let $x' = x \wedge y \leq y$, and apply this equation to x', y to get

$$\rho(A^k x' \wedge y) + \rho(x' \vee C^k y) = \rho(x') + \rho(y).$$

The left-hand side is

$$\begin{aligned}
\rho(A^k x \wedge A^k y \wedge y) + \rho((y \wedge x) \vee C^k y) &= \\
\rho(A^k x \wedge y) + \rho(y \wedge (x \vee C^k y)) &= \\
\rho(A^k x \wedge y) + \rho(x \vee C^k y) - \rho(y \vee x \vee C^k y) + \rho(y) &= \\
\rho(A^k x \wedge y) + \rho(x \vee C^k y) + (\rho(y) - \rho(x \vee y)) &= \\
\rho(A^k x \wedge y) + \rho(x \vee C^k y) + (\rho(x \wedge y) - \rho(x))
\end{aligned}$$

Equate it with the right side $\rho(x \wedge y) + \rho(y)$ and rearrange the terms to obtain the semi-primary equation.

Now suppose $\mathcal{L}$ is a finite length graded lattice that satisfies the semi-primary equation for $k = 0, 1$. The $k = 0$ case is precisely the rank identity for modular lattices, so $\mathcal{L}$ is modular. Let $x = \hat{0}$ and let y be a join-irreducible. Then the $k = 1$ equation yields $\rho(A\hat{0} \wedge y) = \rho(y) - \rho(Cy) = 1$, so that $[\hat{0}, y]$ has a unique atom, v. Now y is join-irreducible in $\mathcal{L}$, hence in the sublattice $[v, y]$ as well (or $y = v$), so inducting on the length of the lattice, we have that $[v, y]$ is a chain, so $[\hat{0}, y]$ is too. Meet-irreducibles are cocycles by a dual argument, so $\mathcal{L}$ is semi-primary. $\square$

One way to characterize modular lattices is by perspectivity of intervals, introduced in Lemma 2.1. A lattice is modular iff for each pair of elements x, y, the functions ϕ and ψ below are lattice isomorphisms

$$\begin{array}{rcl} [x, x \vee y] & \longrightarrow & [x \wedge y, y] \\ t & \overset{\phi}{\longmapsto} & t \wedge y \\ x \vee t' & \overset{\psi}{\longleftarrow\!\shortmid} & t' \end{array}$$

iff the functions ϕ' and ψ' obtained by swapping all occurrences of x and y are lattice isomorphisms. There is a partial analogue for semi-primary lattices, which provides an algebraic counterpart to the semi-primary equation.

Theorem 4.17. *Consider*

$$\begin{array}{rcl} I_k = [x, x \vee C^k y] & \longrightarrow & [A^k x \wedge y, y] = J_k \\ t & \overset{\phi_k}{\longmapsto} & A^k t \wedge y \\ x \vee C^k t' & \overset{\psi_k}{\longleftarrow\!\shortmid} & t' \end{array}$$

A graded lattice of finite length is semi-primary iff ϕ_k and ψ_k are inverse lattice isomorphisms for all x, y in the lattice and all $k \geq 0$, iff they are inverse for $k = 0, 1$.

Proof. Suppose these are inverse for $k = 0, 1$. The $k = 0$ maps are precisely the characterization of modular lattices in terms of perspective intervals, so the lattice is modular. Now let y be a join-irreducible and $x = \hat{0}$. We must show that Cy is a join-irreducible (or $\hat{0}$) as well. By the $k = 1$ equation, $[\hat{0}, Cy]$ and $[A_y\hat{0}, y]$ are isomorphic. Since y is join-irreducible in $[\hat{0}, y]$, it also is join-irreducible (or minimal) in the subinterval $[A_y\hat{0}, y]$, whence Cy is join-irreducible (or $\hat{0}$) in $[\hat{0}, Cy]$ and hence in the whole lattice. By a dual argument, the unique element covering each meet-irreducible is a meet-irreducible or $\hat{1}$, so by Theorem 4.15(4), the lattice is semi-primary.

Conversely, suppose a lattice is semi-primary. The maps ϕ_k and ψ_k are order-preserving in any modular lattice. For any $t \in I_k$ we have $C^k(A^k t \wedge y) \leq C^k A^k t \leq t$ and $x \leq t$, so that $\psi_k(\phi_k(t)) = x \vee C^k(A^k t \wedge y) \leq t$. If $t, u \in I_k$ and $t \lessdot u$ then $\phi_k(t) \leq \phi_k(u)$ by Theorem 4.15(7) and Lemma 2.3; since ϕ_k maps the endpoints of I_k to those of J_k and both I_k and J_k have the same length by the semi-primary equation, in fact $\phi_k(t) \lessdot \phi_k(u)$. Similarly, for $t', u' \in J_k$ with $t' \lessdot u'$, we have $\psi_k(t') \lessdot \psi_k(u')$. So ϕ_k and ψ_k preserve the rank of an element within the interval. Thus for $t \in I_1$, both sides of $\psi_k(\phi_k(t)) \leq t$ have the same rank, and so are equal. Similarly, $\phi_k(\psi_k(t')) = t'$ when $t' \in J_k$, so ψ_k and ϕ_k are inverses. $\square$

The analogue with modular lattices is not complete. The transposed pair of intervals to consider is $I'_k = [x \vee C^k y, y]$ and $J'_k = [x, A^k x \wedge y]$. Unless $x \leq y$, these intervals are both empty, since they are subintervals of $[x, y]$. When $x \leq y$ does hold, the intervals do not need to be isomorphic, though they both have the same type because Atype$[x, y]$ = Ctype$[x, y]$. Figure 1 is the Hasse diagram of a minimal example with these intervals not isomorphic, with $k = 1$. The vertices are the elements of the lattice, and the edges are the cover relations in the lattice.

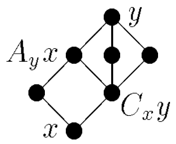


FIGURE 1. An irregular, non self-dual, semi-primary lattice.

Note. Theorem 4.17 can be transformed to equations satisfied by all x, y, t in a semi-primary lattice by projecting any t into the intervals I_k or J_k, as follows. A lattice of finite length is semi-primary iff for all x, y, t,

$$\begin{aligned} \psi_k(\phi_k(x \vee (t \wedge (x \vee C^k y)))) &= x \vee (t \wedge (x \vee C^k y)) \\ \phi_k(\psi_k(y \wedge (t' \vee (A^k x \wedge y)))) &= y \wedge (t' \vee (A^k x \wedge y)) \end{aligned}$$

hold for all nonnegative k, iff they hold for $k = 0, 1$.

Definition 4.18. If P and Q are graded posets, a map $\phi : P \to Q$ is called a **growth** if $x \leq y \Rightarrow \phi(x) \leq \phi(y)$.

Theorem 4.19. *Let $\mathcal{L}$ be a semi-primary lattice. Let $\phi(t)$ be a lattice polynomial constructed from meets, joins, A, C, elements of $\mathcal{L}$, and one occurrence of the variable t. Then ϕ is a growth on $\mathcal{L}$. Further, if intervals $[x, y]$ and $[\phi(x), \phi(y)]$ have the same length, they are isomorphic.*

Proof. For any $x \in \mathcal{L}$, both $t \wedge x$ and $t \vee x$ are growths in t by Lemma 2.2, and if they preserve interval lengths, they induce isomorphisms by Lemma 2.3.

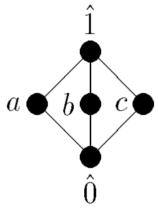
The function At is a growth by Theorem 4.15(7). Suppose $[a, b]$ and $[Aa, Ab]$ have the same length. Let $c = Ab$ and $\lambda = \text{type}[a, c]$. Then $\rho(a, Aa) = \rho(a, Aa \wedge c) = \lambda'_1 = \rho(a \vee Cc, c)$ so $\rho(a, a \vee Cc) = \rho(Aa, c) = \rho(Aa, Ab)$. Also $a \vee Cc = a \vee CAb \leq b$ so $[a, a \vee Cc] \subseteq [a, b]$. Since these have equal lengths, in fact $a \vee Cc = b$. Finally, $[a, b] = [a, a \vee Cc] \cong [Aa \wedge c, c] = [Aa, Ab]$. The proof for Ct is similar.

Finally, all lattice polynomials of the form in the theorem are compositions of the forms just considered, and hence are growths that induce isomorphisms when they preserve interval lengths. $\square$

Note. Isomorphisms of the form in the above theorem may be useful in determining conditions under which intervals of the same type in a lattice are in fact isomorphic. Also see [**1**] for a study of rank and order preserving maps from certain semi-primary lattices of one type to another type.

Example 4.20. Note that not all lattice polynomials are growths, and not all growths expressible as lattice polynomials induce isomorphisms when they preserve an interval's length.

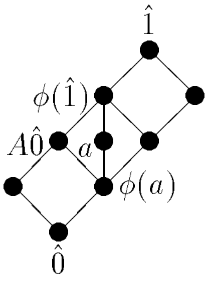
(1) S.V. Fomin (personal communication) notes that all nondistributive modular lattices contain a sublattice isomorphic to



The polynomial $\phi(t) = (t \vee b) \wedge (t \vee c)$ has values $\phi(\hat{0}) = \hat{0}$ and $\phi(a) = \hat{1}$, so it is not a growth. Thus, for all nondistributive modular lattices, some lattice polynomial is not a growth.

(2) In any semi-primary lattice $\mathcal{L}$, choose $x \in \mathcal{L}$ and consider the polynomial $\phi(t) = t \wedge (x \vee Ct)$. It monotonically increases in t. We have $\rho(\phi(t)) = \rho(t) + \rho(x \vee Ct) - \rho(x \vee t)$ (since $t \vee x \vee Ct = x \vee t$). If $t \lessdot t'$, the increase in rank from $\phi(t)$ to $\phi(t')$ is at most 2 because upon substituting t' for t, each rank on the right of the rank equation can go up by 1. The only way to achieve an increase of 2 is if $t \lessdot t'$ and $x \vee Ct \lessdot x \vee Ct'$ but $x \vee t = x \vee t'$. However, applying C_x to both sides of $x \vee t = x \vee t'$ yields $x \vee Ct = x \vee Ct'$, so this cannot occur. Thus, $\phi(t)$ is a growth.

Now consider $\phi(t) = t \wedge (A\hat{0} \vee Ct)$ applied to the length 2 interval $[a, \hat{1}]$ in the following lattice. We have type$[a, \hat{1}] = \square\square$ and type$[\phi(a), \phi(\hat{1})] = \begin{smallmatrix}\square\\\square\end{smallmatrix}$, so while the interval length is preserved, the intervals are not isomorphic. A similar construction can be done in any semi-primary lattice of type $(3, 1)$ with at least three elements of rank two by choosing a with type a = cotype a = $\square\square$. Any semi-primary lattice whose type contains $(3, 1)$ has intervals of type $(3, 1)$ in which such a construction can be done as well, provided there are at least three elements on the middle level.



We now provide a strengthening of Theorem 3.3(2) for semi-primary lattices.

Definition 4.21. For a partition λ, let $\mathcal{C}^k\lambda = (\lambda'_{k+1}, \lambda'_{k+2}, \ldots)'$ be obtained from λ by deleting the first k columns.

Theorem 4.22. *In a semi-primary lattice, $A^kC^kx = x \vee A^k\hat{0}$ and $C^kA^kx = x \wedge C^k\hat{1}$.*

Proof. We prove the first statement, and the second is proven dually. Since $A^kC^kx \geq x$ and $A^kC^kx \geq A^k\hat{0}$, we have $A^kC^kx \geq x \vee A^k\hat{0}$. Let λ = type $\mathcal{L}$ and μ = type x. Then type $C^kx = \mathcal{C}^k\mu$. Since $C^kx \in [\hat{0}, C^k\hat{1}]$, we have type$[A^k\hat{0}, A^kC^kx] = \mathcal{C}^k\mu$ as well, by Theorem 4.17. Thus,

$$\begin{aligned}\operatorname{type}[\hat{0}, A^k C^k x] = (\operatorname{Atype}[\hat{0}, A^k C^k x])' &= (\operatorname{Atype}[\hat{0}, A^k \hat{0}])' + (\operatorname{Atype}[A^k \hat{0}, A^k C^k x])' \\ &= \lambda[k] + \mathcal{C}^k \mu = \mu \vee \lambda[k] = \operatorname{type}(x \vee A^k \hat{0}).\end{aligned}$$

The only tricky equality is the last, which follows from Corollary 3.10. Since $A^k C^k x$ is bounded below by $x \vee A^k \hat{0}$, and they both have the same type, they are equal. $\square$

4.3. Classification and representation of semi-primary lattices. We summarize what is known about representing semi-primary lattices. See Jónsson and Monk [**13**] for a detailed history of the development of this representation.

Definition 4.23 (see [13, 6.1,6.2]**).** A lattice is **primary** if it is semi-primary and no interval of length 2 has precisely 2 elements on the middle level, i.e., all intervals of type $\boxminus$ have at least 3 atoms. Clearly the dual of a primary lattice is also primary.

Definition 4.24. A **q-regular lattice** is a lattice with the property that all intervals of length 2 are either chains or have precisely $q+1$ atoms. A **q-regular semi-primary lattice** is, therefore, a semi-primary lattice in which every interval of type $\boxminus$ has $q+1$ atoms. A **regular lattice** is a lattice with the property that all intervals of length 2 that aren't chains have the same cardinal number of atoms, whether a finite $q+1$, or an infinite cardinal. All other lattices are **irregular**.

Example 4.25. We provide examples of different lattices and their classifications.

(1) If G is a finite abelian p-group, then $L(G)$ is p-regular and primary. Every interval of type $\boxminus$ is a quotient isomorphic to $L((\mathbb{Z}/p\mathbb{Z})^2)$. The subgroups of $(\mathbb{Z}/p\mathbb{Z})^2$ are itself; 0; and the groups with generators $(1, i)$ (where $i \in \mathbb{Z}/p\mathbb{Z}$) and the group with generator $(0, 1)$. So there are $p+1$ atoms in the interval. Because $p + 1 \geq 3$ for all primes, it is primary.
(2) The lattice $L(V, N)$ over a field K is $\#K$-regular and primary. Let V be a vector space over a field K and N be a nilpotent transformation. Every interval of $L(V, N)$ of type $\boxminus$ is isomorphic to the lattice of subspaces of a two dimensional vector space over K. There are $\#K + 1$ atomic subspaces (with essentially the same construction as in the subgroup lattice), so when $\#K$ is finite, $L(V, N)$ is $\#K$-regular, and when it is infinite, it is regular. In either case, all fields have at least two elements, so $\#K + 1 \geq 3$, so $L(V, N)$ is primary.
(3) The 1-regular semi-primary lattices are products of chains. Nontrivial products of chains (at least two nonempty factors) are not primary.
(4) A direct product of semi-primary lattices is a semi-primary lattice, but in nontrivial products, is neither regular (unless all factors are chains) nor primary. Intervals of type $\boxminus$ are either wholly contained in one factor, or are the product of a chain of length 1 from one factor and a chain of length 1 from another factor. Intervals of the latter sort have exactly 2 atoms, so the product isn't primary. If all intervals of the former sort have exactly 2 atoms, this is a product of chains.
(5) If λ is a partition with two rows, we may construct an infinite number of primary irregular lattices of type λ. Take any primary lattice $\mathcal{L}$ of type λ. We require an element x which is both a cycle and a cocyle. One way to obtain such

an x is to decompose $\hat{1} = x \vee y$ where x and y are independent join-irreducibles. Then x is both a cycle and a cocycle (as $[x, \hat{1}] = [x, x \vee y] \cong [x \wedge y, y] = [\hat{0}, y]$).

The cover Ax of x is not join-irreducible, because if it is, then Ax and x are both complements to y, but have two different ranks. Thus, Ax has at least one lower cover besides x, and the meet of any such lower cover with x must be Cx by modularity. So $[Cx, Ax]$ is an interval of type $\boxminus$, and it contains an element x which has no cover relation with any other element in the lattice. Form a lattice $\mathcal{L}'$ from $\mathcal{L}$ by inserting additional atoms into $[Cx, Ax]$; the additional atoms cover Cx and are covered by Ax, and also have no other cover relations with any elements of $\mathcal{L}'$. By the local characterizations of modularity and semi-primariness, this lattice is semi-primary, for if we consider the local part of the lattice with the additional atoms, they behave exactly as x did in the original lattice $\mathcal{L}$. Each interval has at least 3 atoms, so $\mathcal{L}'$ is primary and irregular.

While two-rowed lattices usually contain a plethora of elements that are both cycles and cocycles, lattices whose type has more than two rows do not have such elements. Suppose $x > \hat{0}$ is a cycle. Then $Ax \geq x \vee A\hat{0}$ so $\operatorname{type} Ax \geq (\operatorname{type} x) \vee (1^{\lambda_1'})$, whence $\rho(Ax) \geq \rho(x) + \lambda_1' - 1$. When $\lambda_1' > 2$, this implies $\rho(Ax) - \rho(x) > 1$, so x isn't a cocycle. So this construction doesn't work when λ has at least 3 rows.

(6) The parameters q and λ are insufficient to classify all the q-regular semi-primary lattices. Let $G = (\mathbb{Z}/p^2\mathbb{Z})^3$. Let $\mathbb{F}_p$ be the finite field of order p; V be a 6-dimensional vector space over $\mathbb{F}_p$; and N be a nilpotent transformation on V of type $(2,2,2)$. Both $L(G)$ and $L(V,N)$ are p-regular semi-primary lattices of type $(2,2,2)$, and Butler and Hales [**1**, p. 9] prove that they are not isomorphic. Thus, for any partition $\lambda \geq (2,2,2)$, the p-subgroup lattice of type λ and the invariant subspace lattice of type λ over the field $\mathbb{F}_p$ are not isomorphic, because their subintervals of type $(2,2,2)$ are not isomorphic.

Theorem 4.26 (see [13, 6.3]). *In a primary lattice with at least three independent atoms, any two intervals of type* $\boxminus$ *are projective.*

Corollary 4.27. *Every primary lattice of type λ with $\lambda_1' \geq 3$ is regular.*

Definition 4.28 (see [13, 6.6]). A ring R with unit is **completely primary and uniserial** if there is a two-sided ideal P of R such that every left or right ideal of R is of the form P^k, where $P^0 = R$, and $P^k = 0$ for some finite k.

Definition 4.29 (see [13, 2.1]). A lattice is **Arguesian** if it satisfies the following lattice-theoretic analogue of Desargues' Law: for any elements $a_0, a_1, a_2, b_0, b_1, b_2$, if

$$y = (a_0 \vee a_1) \wedge (b_0 \vee b_1) \wedge \Big[\big((a_0 \vee a_2) \wedge (b_0 \vee b_2)\big) \ \vee \ \big((a_1 \vee a_2) \wedge (b_1 \vee b_2)\big)\Big],$$

then

$$(a_0 \vee b_0) \wedge (a_1 \vee b_1) \wedge (a_2 \vee b_2) \leq (a_0 \wedge (a_1 \vee y)) \ \vee \ (b_0 \wedge (b_1 \vee y)).$$

Jónsson and Monk [**13**] show that every Arguesian lattice is modular, and that every primary lattice of type λ where the first part has multiplicity at least 4 is Arguesian. Not all primary lattices with $m_{\lambda_1} < 4$ are Arguesian, because there are non-Desarguean projective planes. Any projective plane of order q gives rise to a

q-regular primary lattice of type (1^3) by taking the points p to be the atoms and the lines l to be the coatoms, with $p < l$ in the lattice iff p is on the line l in the plane. We also attach a minimum element $\hat{0}$ and a maximum element $\hat{1}$. For examples of non-Desarguean projective planes, see, for example, [**11**, Chapter IX], or [**30**, Chapter 2.4 and Part 3].

There is a representation theorem for certain primary lattices, which generalizes the coordinatization of projective geometries.

Theorem 4.30 (see [13, 6.7,12.3]**).** *The lattice of all submodules of a finitely generated module over a completely primary and uniserial ring is primary. Conversely, any primary Arguesian lattice with $m_{\lambda_1} \geq 3$ (which includes all primary lattices with $m_{\lambda_1} \geq 4$), is isomorphic to the lattice of submodules of a finitely generated module over a completely primary and uniserial ring.*

These lattices include $L(G)$, $L(V, N)$, and the lattice of submodules of a module of finite length over a discrete valuation ring; see Macdonald [**16**, p. 85]. The residue field of the ring plays a role similar to the field in the lattice $L(V, N)$; in particular, if q is the order of the residue field, the lattice of the theorem is q-regular.

Regonatti has a representation theorem for modular lattices.

Theorem 4.31 (see [20, pp. 45–47]**).** *Let $\mathcal{L}$ be a modular lattice. If the number of atoms equals the number of coatoms in every interval of length 3, then $\mathcal{L}$ is the direct product of primary q-lattices (with the different factors possibly having different values of q).*

There is also a converse of sorts to this theorem.

Theorem 4.32 (see [13, 6.2]**).** *Every primary lattice that is not a chain is* ***simple****, that is, has no nontrivial homomorphic images.*

4.4. Enumeration in q-regular semi-primary lattices. In a q-regular semi-primary lattice $\mathcal{L}$, we often can count how many solutions $(x_1, x_2, \ldots)$ there are to a system of equations $\rho(p_i(x_1, x_2, \ldots)) = r_i$, where the p_i are lattice polynomials in the operators A and C; meets and joins; constants from the lattice; and the variables x_i, while the r_i are integers. These counts often turn out to be polynomials in q depending on the parameters r_i and the type of the lattice, but not on any further structure of the lattice. The polynomials are the same for abelian subgroup lattices, invariant subspace lattices, and any other q-regular semi-primary lattice.

The equations we consider will be derived from type specifications. For example, in this section, we will count how many atoms of cotype μ are in a q-regular semi-primary lattice of type λ. We have type$[a, \hat{1}] = \mu$ iff $\rho(a, A^k a) = \mu'_1 + \cdots + \mu'_k$, so such a are the solutions of $\rho(A^k a) = 1 + \mu'_1 + \cdots + \mu'_k$ for all k.

We will also count how many flags $f = (f_0 \leq f_1 \leq \cdots)$ there are in which type $f_i = \lambda^{(i)}$ is specified for each i. In this case, the type equations are $\rho(A^k\hat{0} \wedge f_i) = \lambda^{(i)\prime}_1 + \cdots + \lambda^{(i)\prime}_k$ for all i and k. The fact that f is a flag can also be expressed as a rank equation: since $f_i \leq f_{i+1}$, we have $f_i \wedge f_{i+1} = f_i$, so $\rho(f_i \wedge f_{i+1}) = |\lambda^{(i)}|$. Or, we could use $\rho(f_i \vee f_{i+1}) = |\lambda^{(i+1)}|$.

The counts we obtain for q-regular semi-primary lattices also provide information about regular lattices with a variety structure. The proofs of the counts generally entail selecting an element within a certain interval but excluded from a lower dimension subinterval. In the variety case, we would instead select an element of the interval generic with respect to a certain property that excludes the subinterval. The closure of the solution space would have the same dimension as the degree of the polynomial specifying the number of choices in the interval, which equals the degree of the polynomial specifying the number of choices in the interval less the smaller dimension subinterval. We obtain the following principle.

Principle 4.33. *Consider a system of rank equations which has $f(q)$ solutions in all q-regular semi-primary lattices of specified type for sufficiently large q. In a semi-primary lattice of that type with a variety structure, the dimension of the closure of the set of solutions to the rank equations equals the degree of the polynomial $f(q)$.*

Definition 4.34. For a nonnegative integer n, let

$$\begin{aligned} \langle n\rangle_q &\stackrel{\text{def}}{=} (q^n - q^1)/(q-1) = 1 + q + \cdots + q^{n-1} \\ \langle n\rangle_q! &\stackrel{\text{def}}{=} \langle n\rangle_q \langle n-1\rangle_q \cdots \langle 1\rangle_q \\ \langle 0\rangle_q! &\stackrel{\text{def}}{=} 1. \end{aligned}$$

When the subscript is omitted, implicitly assume it is q, so $\langle n\rangle = \langle n\rangle_q$. We will commonly use $\langle n\rangle_{q^{-1}}$, and it should be noted that $\langle n\rangle_{q^{-1}} = q^{1-n}\langle n\rangle$ and $\langle n\rangle_{q^{-1}}! = q^{-\binom{n}{2}}\langle n\rangle!$. The common notation $(\boldsymbol{n})$ for $\langle n\rangle$ is not used here because often in this manuscript, n will be a long expression with parentheses, and the former notation is difficult to read.

Theorem 4.35. *The number of atoms in a q-regular semi-primary lattice $\mathcal{L}$ of type λ is $\langle\lambda_1'\rangle$. Dually, the number of coatoms of such a lattice is also $\langle\lambda_1'\rangle$.*

Proof. The atoms of $\mathcal{L}$ and the atoms of $[\hat{0}, A\hat{0}]$ are identical, and the type of the latter is $(1^{\lambda_1'})$, so it suffices to consider lattices of type $\lambda = (1^n)$. When $n = 0$, there are no atoms; when $n = 1$, there is one atom, $\hat{1}$, and when $n = 2$, there are $q+1$ atoms since the lattice is q-regular. So the theorem holds for $n = 0, 1, 2$. Now assume it holds for integers smaller than n, and count the number a of atoms of a q-regular semi-primary lattice of type (1^n). In two different ways, we will count triples (u, v, w) of elements of the lattice, where u, v are distinct atoms and $w = u \vee v$. First, pick u in a ways, then an atom w of $[u, \hat{1}]$ in $\langle n-1\rangle$ ways (as the interval $[u, \hat{1}]$ has type (1^{n-1})), and then let v be any of the q atoms of $[\hat{0}, w]$ other than u; altogether, there are $a\langle n-1\rangle q$ such triples. Second, pick u in a ways and v in $a-1$ ways, and then $w = u \vee v$ in one way; so there are $a(a-1)$ triples. Thus, $a(a-1) = a\langle n-1\rangle q$, so $a = \langle n-1\rangle q + 1 = \langle n\rangle$. $\square$

Theorem 4.36. *In a q-regular semi-primary lattice of type λ, the number of coatoms of type $\mu \lessdot \lambda$ (dually, atoms of cotype $\mu \lessdot \lambda$) is $\langle\lambda_c'\rangle - \left\langle\lambda_{c+1}'\right\rangle$, where $c = \operatorname{col}(\lambda/\mu)$.*

Proof. The coatoms x of type μ satisfy $x \geq \hat{1}[c-1]$ but not $x \geq \hat{1}[c]$. Thus, we want the coatoms of the interval $\left[\hat{1}[c-1], \hat{1}\right]$ not in the subinterval $\left[\hat{1}[c], \hat{1}\right]$. The types of these intervals are $\mathcal{C}^{c-1}\lambda$ and $\mathcal{C}^c\lambda$, respectively, and so they contain $\langle\lambda_c'\rangle$ and $\left\langle\lambda_{c+1}'\right\rangle$ coatoms, respectively, by the preceding theorem. $\square$

Definition 4.37. By Theorem 4.11, the types of elements in a flag increase in Young's lattice, so we may define for each $x \leq f_l$ the **flag type** $\operatorname{ftype}_x f$ of a flag $f = (f_l, \ldots, f_h)$ as the chain of partitions

$$\begin{aligned}\operatorname{ftype}_x f &\stackrel{\text{def}}{=} (\operatorname{type}[x, f_l], \ldots, \operatorname{type}[x, f_h]) \\ \operatorname{ftype} f &\stackrel{\text{def}}{=} (\operatorname{type}[f_l, f_l], \ldots, \operatorname{type}[f_l, f_h]).\end{aligned}$$

This is a skew tableau of shape $\operatorname{type}[x, f_h]/\operatorname{type}[x, f_l]$ in which for each $k = l{+}1, \ldots, h$, the squares of $\operatorname{type}[x, f_k]/\operatorname{type}[x, f_{k-1}]$ are filled with k.

Theorem 4.38. *Let $\mathcal{L}$ be a q-regular semi-primary lattice of type λ, and $\boldsymbol{P}$ be a standard skew tableau of outer shape λ with distinct entries. The number of flags with* $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ *is*

$$F^{\boldsymbol{P}}(q) \stackrel{def}{=} \prod_{i \,\in\, \boldsymbol{P}} \Big(\langle \operatorname{row}(\boldsymbol{P}, i) \rangle - \langle \operatorname{row}_1(\boldsymbol{P}, i) \rangle \Big).$$

This is a monic polynomial of degree $n(\operatorname{sh} \boldsymbol{P})$.

Proof. We have $f_h = \hat{1}$. Given $f_h \geq \cdots \geq f_l$ with types given by $\boldsymbol{P}$, we want to choose $f_{i-1} \leq f_i$. If $\lambda^{(i-1)} = \lambda^{(i)}$ (equivalently, i doesn't appear in $\boldsymbol{P}$) then $f_{i-1} = f_i$. So consider $\lambda^{(i-1)} \lessdot \lambda^{(i)}$, with these differing in the cell at the bottom of column c. Then the number of coatoms f_{i-1} of $[\hat{0}, f_i]$ of type $\lambda^{(i-1)}$ is

$$\left\langle \lambda_c^{(i)\prime} \right\rangle - \left\langle \lambda_{c+1}^{(i)\prime} \right\rangle = \langle \operatorname{row}(\boldsymbol{P}, i) \rangle - \langle \operatorname{row}_1(\boldsymbol{P}, i) \rangle.$$

This is a monic polynomial of degree $\operatorname{row}(\boldsymbol{P}, i) - 1$, The number of flags of type $\boldsymbol{P}$ is the product of these polynomials over all entries i of $\boldsymbol{P}$, and is hence monic of degree $n(\operatorname{sh} \boldsymbol{P})$. $\square$

Note. Many theorems concerning flag types will be stated for saturated flags with $f = (f_0 \lessdot \cdots \lessdot f_n)$, so that $\operatorname{ftype}_x f$ contains $1, \ldots, n$; however, they can always be applied to multisaturated flags with different indices $f = (f_l \leq f_{l+1} \leq \cdots \leq f_h)$, where the tableau $\operatorname{ftype}_x f$ has entries $\{\, k : f_{k-1} \lessdot f_k \,\}$. Simply change the indices used, and when some k doesn't appear in the flag type, take f_k to be the same as f_{k-1} or conversely, depending on which is constructed first in the proof. Also, when a theorem is stated concerning $\operatorname{ftype}_{\hat{0}} f$, it can be applied to $\operatorname{ftype}_x f$ with any $x \leq f_l$ by applying the theorem to the semi-primary lattice $[x, \hat{1}]$ instead of $\mathcal{L} = [\hat{0}, \hat{1}]$.

Definition 4.39. The **superstandard skew tableau of shape** λ/μ is the tableau of shape λ/μ in which the entries are filled in consecutively along the first row, then the second, and so on. If the entries aren't specified, count beginning with 1. So the superstandard tableau of shape 4332/221 is

$$\begin{array}{cccc} & & \boxed{1} & \boxed{2} \\ & & \boxed{3} & \\ & \boxed{4} & \boxed{5} & \\ \boxed{6} & \boxed{7} & & \end{array}$$

Definition 4.40. The following quantities frequently appear in enumerating various objects in semi-primary lattices. Let λ be a partition.

$$M_\lambda(q) \stackrel{\text{def}}{=} \prod_{j=1}^{\lambda_1} \left\langle \lambda'_j - \lambda'_{j+1} \right\rangle ! = \prod_{j=1}^{\lambda_1} \langle m_j(\lambda)\rangle !$$

$$n_0(\lambda) \stackrel{\text{def}}{=} \sum_{j=1}^{\lambda_1} \binom{m_j(\lambda)}{2}$$

We will sometimes use $M_\lambda(q^{-1}) = q^{-n_0(\lambda)} M_\lambda(q)$, which has only non-positive powers of q, including a constant term 1. For any polynomial $p(q)$ divisible by $q^{n_0(\lambda)}$, the product $p(q)M_\lambda(q^{-1})$ is a polynomial of the same degree as $p(q)$. A fact we will not use that may be of interest is that $M_\lambda(q)$ is a polynomial whose coefficients form a symmetric, unimodal sequence, i.e., if we write $M_\lambda(q) = \sum_{i=0}^{d} a_i q^i$ where d is the degree of $M_\lambda(q)$, then $a_i = a_{d-i}$ and $0 < a_0 \le \cdots \le a_{\lfloor d/2 \rfloor}$. This is because $M_\lambda(q)$ is the product of terms $\langle n\rangle$, each of which is unimodal and symmetric, and the product of unimodal symmetric polynomials is again unimodal symmetric.

Corollary 4.41. *Let $\boldsymbol{P}$ be the superstandard tableau of shape λ/μ in a q-regular semi-primary lattice of type λ. Let ν be the partition given by $m_j(\nu) = \#\{\, i : \lambda_i = \mu_i = j \,\}$. The number of flags of type $\boldsymbol{P}$ is*

$$q^{n(\lambda/\mu)} M_\lambda(q^{-1})/M_\nu(q^{-1}).$$

Proof. We evaluate the product over $i \in \boldsymbol{P}$ of

$$\langle \mathrm{row}(\boldsymbol{P}, i)\rangle - \langle \mathrm{row}_1(\boldsymbol{P}, i)\rangle = q^{\mathrm{row}_1(\boldsymbol{P},i)} \langle \mathrm{row}(\boldsymbol{P}, i) - \mathrm{row}_1(\boldsymbol{P}, i)\rangle.$$

First consider the differences on the right side. Since $\boldsymbol{P}$ is superstandard, the differences $\mathrm{row}(\boldsymbol{P}, i) - \mathrm{row}_1(\boldsymbol{P}, i)$ equal 1 when i is not at the end of a row of $\boldsymbol{P}$, and since $\langle 1\rangle = 1$, we can ignore these terms. The $m_j(\lambda) - m_j(\nu)$ rows of length j with at least one entry contribute consecutive factors $\langle m_j(\nu)+1\rangle, \langle m_j(\nu)+2\rangle, \ldots, \langle m_j(\lambda)\rangle$ to the product, which we collect into $\langle m_j(\lambda)\rangle!/\langle m_j(\nu)\rangle!$. The product of this over all row lengths j is $M_\lambda(q)/M_\nu(q)$.

Next consider the product over $i \in \boldsymbol{P}$ of the powers of q. The exponent of q is $d = \sum_i \mathrm{row}_1(\boldsymbol{P}, i)$. Consider the contribution of the cell (r, j) of λ/μ to this sum. Each cell of λ/μ not at the end of a row contributes $r-1$ to the sum. An entry is at the end of a row when $\lambda'_{j+1} + \nu'_j < r \le \lambda'_j$, and contributes λ'_{j+1} to the sum. The sum of $r-1$ over all squares of λ/μ is $n(\lambda/\mu)$, while the sum of $r-1-\lambda'_{j+1}$ over all squares at the end of rows that contain entries is $n_0(\lambda) - n_0(\nu)$. The difference is d. The total number of flags is then

$$q^{n(\lambda/\mu)-(n_0(\lambda)-n_0(\nu))} M_\lambda(q)/M_\nu(q) = q^{n(\lambda/\mu)} M_\lambda(q^{-1})/M_\nu(q^{-1}). \quad \square$$

4.5. Lattice automorphisms. We examine lattice automorphisms of $\mathcal{L} = L(V, N)$. As usual, let V be a vector space, and choose a basis of V with elements

$$e_{ij} \text{ with } (i,j) \in \lambda, \text{ such that } Ne_{i1} = 0 \text{ and } Ne_{ij} = e_{i,j-1} \text{ when } j > 1.$$

Let $\vec{x} = (x_1, \ldots, x_{\lambda'_1})$ be given componentwise by $x_i = \mathrm{span}\{\, e_{ij} : 1 \le j \le \lambda_i \,\}$; these are N-invariant subspaces of V.

Let $A = \mathrm{Aut}(\mathcal{L})$. Let E be the group of endomorphisms (invertible linear transformations) on V that commute with N. Any endormorphism in E induces a lattice automorphism of $\mathcal{L}$, and since nonzero scalar multiples of the same endomorphism induce the same lattice automorphism, we may projectively identify E with a subgroup E' of A. However, E' may not be all of A. For example, suppose the field is $\mathbb{C}$, and for any $W \in L(V,N)$, let $\overline{W} = \left\{ \sum_{(i,j)\in\lambda} \overline{a_{ij}} e_{ij} : \sum_{(i,j)\in\lambda} a_{ij} e_{ij} \in W \right\}$; this fixes $\vec{x}$. More generally, for any field, the Galois group induces distinct lattice automorphisms fixing $\vec{x}$, and these are distinct from the automorphisms in E', so that A contains at least the product of E' with the automorphisms induced by the Galois group acting on coordinates with respect to a fixed basis.

We will not determine A, but we will show that the cardinality of the orbit under A of any saturated flag from $\hat{0}$ to $\hat{1}$ is divisible by $M_\lambda(q)$.

Theorem 4.42. *In $\mathcal{L} = L(V,N)$, the orbit of each maximal length saturated flag under the automorphism group has cardinality divisible by $M_\lambda(q)$. The minimal orbit has cardinality exactly $M_\lambda(q)$.*

Note. We may replace $\mathcal{L}$ by any q-regular Arguesian lattices of the form given by Jónsson and Monk (summarized in Theorem 4.30), but this result may not apply to non-Desarguean q-regular semi-primary lattices.

Proof. Let $V_k = \mathrm{span}\{ e_{ik} : \lambda_i = k \}$. Let G_k be the set of endomorphisms on V_k. With $m = m_k(\lambda)$,

$$\#G_k = (q^m - q^0)(q^m - q^1)\cdots(q^m - q^{m-1}) = (q-1)^m q^{\binom{m}{2}} \langle m \rangle !.$$

To any tuple $(g_1, \ldots, g_{\lambda_1}) \in G_1 \times \cdots \times G_{\lambda_1}$, associate an endomorphism g on V, commuting with N, by defining $g(e_{ij}) = N^{\lambda_i - j}(g_{\lambda_i}(e_{ik}))$ and extending linearly to all of V. Let G be the group of all such endomorphisms; then $\#G = (q-1)^l q^{n_0(\lambda)} M_\lambda(q)$, where $l = \lambda'_1$.

Define projections by linearly extending their action on a basis of V:

$$\begin{aligned} \pi_k : V &\longrightarrow V_k \\ e_{ij} &\longmapsto \begin{cases} e_{ij} & \text{if } \lambda_i = j = k; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For $x \in \mathcal{L}$, define $\pi(x) = (\pi_1(x), \ldots, \pi_{\lambda_1}(x))$.

For any saturated flag $0 = f_0 \lessdot f_1 \lessdot \cdots \lessdot f_n = V$, let $G_{\pi(f)}$ be the subgroup of G fixing $\pi(f) = (\pi(f_0), \ldots, \pi(f_n))$. We can explicitly determine $G_{\pi(f)}$ as follows. For each k with $m_k(\lambda) > 0$, consider the sequence $0 = \pi_k(f_0) \leq \cdots \leq \pi_k(f_n)$. Among these are $m = m_k(\lambda)$ distinct spaces forming a saturated chain, $0 = W_0 \lessdot \cdots \lessdot W_m = V_k$. Initially, the action of g_k on $W_0 = 0$ is void. Assume g_k fixes each of $W_0, \ldots, W_{d-1}$. Arbitrarily choose a vector in W_d not in W_{d-1}, and extend g_k to fix W_d as well by mapping this vector to any of the $q^d - q^{d-1}$ vectors in W_d not also in W_{d-1}. Altogether there are $\prod_{d=1}^m (q^d - q^{d-1}) = (q-1)^m q^{\binom{m}{2}}$ possible g_k's fixing $W_0, \ldots, W_d$. Making independent choices of $g_1, \ldots, g_{\lambda_1}$ to form all g fixing $\pi(f)$, there are $(q-1)^l q^{n_0(\lambda)}$ endomorphisms in $G_{\pi(f)}$.

Thus, $\#G / \#G_{\pi(f)} = M_\lambda(q)$ is the size of the orbit of $\pi(f)$ under the action of G.

Let G_f be the subgroup of G fixing f. If f is fixed, so is $\pi(f)$, so G_f is a subgroup of $G_{\pi(f)}$. Thus, $\#G_f$ divides $\#G_{\pi(f)}$, so the size of the orbit of f under G is a multiple of $M_\lambda(q)$, specifically $\#G/\#G_f = (\#G/\#G_{\pi(f)})(\#G_{\pi(f)}/\#G_f) = M_\lambda(q) \cdot (\#G_{\pi(f)}/\#G_f)$.

Although G is not a subgroup of A, it may be projectively identified with some subgroup H of A: each element of G induces an automorphism in A, and only nonzero scalar multiples yield the same automorphism. Consider right cosets Ha with $a \in A$. Apply two cosets Ha_1 and Ha_2 to f. Either they yield the exact same orbit, or they yield disjoint orbits: if $h_1a_1(f) = h_2a_2(f)$ with $h_1, h_2 \in H$, the orbits are the same because for any $h \in H$ we have $ha_1(f) = (hh_1^{-1}h_2)a_2(f)$ and $ha_2(f) = (hh_2^{-1}h_1)a_1(f)$. Each coset yields an orbit $(Ha)(f) = H(a(f))$ with cardinality divisible by $M_\lambda(q)$, and unequal orbits are disjoint, so the orbit of f under A also has cardinality divisible by $M_\lambda(q)$.

The minimum cardinality of an orbit is in fact exactly $M_\lambda(q)$. Let $\boldsymbol{P}$ be the superstandard tableau of shape λ. Since the flag type is an automorphism invariant, the set of flags of type $\boldsymbol{P}$ is the union of orbits of A, and since there are $F^{\boldsymbol{P}}(q) = M_\lambda(q)$ such flags, they form one orbit of size $M_\lambda(q)$. □

4.6. Complemented elements.

Theorem 4.43. *Let $\mathcal{L}$ be a semi-primary lattice of type λ. An element of $\mathcal{L}$ is complemented iff it has type μ and cotype ν with $\mu \cup \nu = \lambda$ (i.e., μ and ν are complementary subsets of the rows of λ).*

Proof. If x and y are complements, a join decomposition of $\hat{1}$ may be formed from any join decompositions of x and y, so we manifestly have $\operatorname{type} \hat{1} = \operatorname{type} x \cup \operatorname{type} y$. But also $[x, \hat{1}] = [x, x \vee y] \cong [x \wedge y, y] = [\hat{0}, y]$, so $\operatorname{type} \hat{1} = \operatorname{type} x \cup \operatorname{cotype} x$.

We prove the converse by induction on the length of $\mathcal{L}$. Suppose x has type μ and cotype ν where $\mu \cup \nu = \lambda$. Either $\mu_1 = \lambda_1$ or $\nu_1 = \lambda_1$ (or both). In the former case, pick a μ_1-cycle c in $[\hat{0}, x]$, and a complement c' to c in $\mathcal{L}$. Then $x \wedge c'$ and c are complementary in $[\hat{0}, x]$ because $(x \wedge c') \vee c = x \wedge (c' \vee c) = x \wedge \hat{1} = x$ and $(x \wedge c') \wedge c = x \wedge (c' \wedge c) = x \wedge \hat{0} = \hat{0}$. Thus, $\operatorname{type}(x \wedge c') \cup \operatorname{type} c = \operatorname{type} x$, so $\operatorname{type}(x \wedge c') = (\mu_2, \mu_3, \ldots)$. Also, $\operatorname{type}[x \wedge c', c'] = \operatorname{type}[x, x \vee c'] = \operatorname{type}[x, \hat{1}] = \nu$, and $\operatorname{type}[\hat{0}, c'] = (\lambda_2, \lambda_3, \ldots)$. Inductively apply the theorem to find a complement x' of $x \wedge c'$ in the subinterval $[\hat{0}, c']$ of $\mathcal{L}$, i.e., $\hat{0} = (x \wedge c') \wedge x'$ and $c' = (x \wedge c') \vee x'$. In fact, x' is a complement to x in $\mathcal{L}$: since $x' \leq c'$, we have $x \wedge x' = x \wedge (x' \wedge c') = \hat{0}$, and since $x = c \vee (x \wedge c')$, we have

$$x \vee x' = (c \vee (x \wedge c')) \vee x' = c \vee ((x \wedge c') \vee x') = c \vee c' = \hat{1}.$$

In case $\nu_1 = \lambda_1$, we dualize the above argument, choosing a cocycle c of maximum rank in $[x, \hat{1}]$, a complement c' to c in $\mathcal{L}$, and then a complement x' to $x \vee c'$ in $[c', \hat{1}]$. □

Theorem 4.44. *The number of complemented cycles of length $r > 0$ in a q-regular semi-primary lattice $\mathcal{L}$ of type λ is*

$$q^{(\lambda'_1 + \cdots + \lambda'_r) - r} \langle m_r(\lambda) \rangle_{q^{-1}}.$$

Hence, the number of maximum length cycles is

$$q^{|\lambda|-\lambda_1}\langle m_{\lambda_1}(\lambda)\rangle_{q^{-1}} = q^{\lambda_2+\lambda_3+\cdots}\langle m_{\lambda_1}(\lambda)\rangle_{q^{-1}}.$$

In the trivial case $r = 0$, the number of complemented cycles is 1.

Proof. We induct on r and λ_1. The only complemented cycle of length $r = 0$ is $\hat{0}$.

When $r = 1$, the complemented atoms are the atoms whose cotype is $(\lambda'_1 - 1, \lambda'_2, \lambda'_3, \ldots)'$. Equivalently, by evaluating the cotype as $(\mathrm{Ctype}[v, \hat{1}])'$, an atom v is complemented iff $v \not\leq C\hat{1}$. The number of atoms in $\mathcal{L}$, less the number in $[\hat{0}, C\hat{1}]$, is

$$\langle \lambda'_1\rangle - \langle \lambda'_2\rangle = q^{\lambda'_1-1}\langle \lambda'_1 - \lambda'_2\rangle_{q^{-1}} = q^{\lambda'_1-1}\langle m_1(\lambda)\rangle_{q^{-1}},$$

as the theorem claims for $r = 1$.

When $\lambda_1 = 0$, there are no cycles of positive length, and indeed we have $m_r(\lambda) = 0$ when $r > 0$.

Now take $\lambda_1 > 0$ and $r > 1$ and assume the theorem is true for smaller values. In two ways, we count pairs (c, d) where c is a complemented r-cycle in the lattice; d is a complemented $(r-1)$-cycle in $[A\hat{0}, \hat{1}]$; and $c \leq d$.

Let c be a complemented r-cycle in $\mathcal{L}$, with a complement c'. We require $d \geq c$ and $d \geq A\hat{0}$, so $d \geq A\hat{0} \vee c$. Now $\mathrm{type}[A\hat{0}, A\hat{0} \vee c] = \mathrm{type}[A\hat{0} \wedge c, c] = (r-1)$, so only $d = A\hat{0} \vee c$ is possible (as a larger d would not be an $(r-1)$-cycle). We will show that this d is complemented in $[A\hat{0}, \hat{1}]$ by showing that $d' = c' \vee A\hat{0}$ is a complement. By Corollary 3.10, $\mathrm{type}[\hat{0}, d']$ is the type of c' with the first column lengthened by 1. We have $d \vee d' = c \vee c' \vee A\hat{0} = \hat{1} \vee A\hat{0} = \hat{1}$. Also, $d \wedge d' \geq A\hat{0}$ and

$$\begin{aligned}
\rho(d \wedge d') &= \rho(d) + \rho(d') - \rho(\hat{1}) \\
&= (\rho(c) + \rho(A\hat{0}) - 1) + (\rho(c') + 1) - \rho(\hat{1}) \\
&= (\rho(c) + \rho(c') - \rho(\hat{1})) + \rho(A\hat{0}) \\
&= \rho(A\hat{0})
\end{aligned}$$

so in fact, $d \wedge d' = A\hat{0}$. So the number of pairs (c, d) is the same as the number of complemented r-cycles c of $\mathcal{L}$.

On the other hand, let d be a complemented $(r-1)$-cycle in $[A\hat{0}, \hat{1}]$, in one of $q^{(\lambda'_2+\cdots+\lambda'_r)-(r-1)}$ ways. This has type $(\lambda'_1, 1^{r-1})'$ in $\mathcal{L}$. Consider the atom $v = C^{r-1}d$ of $\mathcal{L}$. Any r-cycle c of $\mathcal{L}$ with $c \leq d$ must have $c \geq v$, for we must have $d = c \vee A\hat{0}$, and then $C^{r-1}d = C^{r-1}c \vee C^{r-1}A\hat{0} = C^{r-1}c$. Also, $\mathrm{type}[v, d] = (\lambda'_1, 1^{r-2})'$. So c is an $(r-1)$-cycle in $[v, d]$, hence a maximum length cycle, hence complemented. The number of choices of c is, by induction, $q^{(\lambda'_1+(r-2)\cdot 1)-(r-1)} = q^{\lambda'_1+1}$. So the number of pairs (c, d) is $q^{(\lambda'_1+\cdots+\lambda'_r)-r}$. By the first count, this number is also the number of complemented r-cycles. $\square$

Definition 4.45 (see [20, p. 38]). In a semi-primary lattice $\mathcal{L}$, the set of complements of $x \in \mathcal{L}$ is $\Gamma_{\mathcal{L}}(x)$.

An enumerative counterpart to Theorem 4.5 is the following.

Theorem 4.46 (see [20, pp. 38–39]). *If c is a cycle of rank highest rank* $\mathrm{arank}\,\mathcal{L}$ *in a q-regular semi-primary lattice $\mathcal{L}$, then the number of complements to c in $\mathcal{L}$ is $q^{\rho(\hat{1})-\rho(c)}$.*

We now generalize this to enumerate the complements of all complemented elements.

Theorem 4.47. *Let x be a complemented element of type μ and cotype ν in a q-regular semi-primary lattice of type $\lambda = \mu \cup \nu$. The number of complements to x is*

$$\#\Gamma_{\mathcal{L}}(x) = q^{\sum_j \mu'_j \nu'_j}.$$

We present two proofs. The first is analytic, and the second is synthetic, so it applies to any q-regular semi-primary lattice.

Analytic proof. Let V be a vector space with a nilpotent action of type λ. Choose a basis of V with two kinds of elements,

$$\begin{array}{lll} e_{ij} \text{ with } 1 \leq j \leq \mu_i & Ne_{i1} = 0 & Ne_{ij} = e_{i,j-1} \text{ for } j > 1, \text{ and} \\ e'_{ij} \text{ with } 1 \leq j \leq \nu_i & Ne'_{i1} = 0 & Ne'_{ij} = e'_{i,j-1} \text{ for } j > 1, \end{array}$$

such that the e_{ij} span the subspace x of V. A complement to x is some y which can be expressed canonically as follows. It is spanned by the vectors $N^k z_m$ where $(m, k+1) \in \nu$ and z_m is of the form

$$z_m = e'_{m,\nu_m} + \sum_{(i,j)\in\mu:\, j\leq\nu_m} a^{(m)}_{ij} e_{ij}.$$

The restriction $j \leq \nu_m$ is because the space spanned by $\{z_m, Nz_m, \ldots, N^{\nu_m - 1} z_m\}$ must be annihilated by N^{ν_m}.

We can set all the parameters $a^{(m)}_{ij}$ independently to any of the q elements of the field. For given m and $j \leq \nu_m$, the number of i with $(i,j) \in \mu$ is μ'_j. Thus, the total number of parameters $a^{(m)}_{ij}$ is

$$\sum_m \sum_{j\leq\nu_m} \mu'_j = \sum_j \sum_{m:\,\nu_m \geq j} \mu'_j = \sum_j \nu'_j \mu'_j. \quad \square$$

Synthetic proof. We induct on λ_1. When $\lambda_1 = 0$, then $x = \hat{0}$, its only complement is $\hat{0}$, and $\mu = \nu = \lambda = \emptyset$, so $\sum_j \mu'_j \nu'_j = 0$. Now take $\lambda_1 > 0$ and assume lattices with smaller values of λ_1 obey the theorem.

Let $k = \mu'_1$. In two ways, we count flags $(c_0 \lessdot c_1 \lessdot \cdots \lessdot c_k)$ where c_0 is a complement to x in $\mathcal{L}$ and c_k is a complement to $x \vee A\hat{0}$ in $[A\hat{0}, \hat{1}]$. Note that $\operatorname{type} x \vee A\hat{0} = (\lambda'_1, \mu'_2, \mu'_3, \ldots)'$ and $\operatorname{type} c_k = (\lambda'_1, \nu'_2, \nu'_3, \ldots)'$. So in any such sequence, we have $c_k = c_0 \vee A\hat{0}$, and $\operatorname{type} c_i = (\nu'_1 + i, \nu'_2, \nu'_3, \ldots)'$.

Let c_0 be a complement of x in $\mathcal{L}$. Successively choose $c_1, c_2, \ldots, c_k$ so that c_i is an atom of $[c_{i-1}, c_0 \vee A\hat{0}]$. This interval is elementary of rank $k - i + 1$, because it is contained in the elementary interval $[c_0, c_0 \vee A\hat{0}]$, so it has type (1^{k-i+1}), and the number of choices of c_i is $\langle k - i + 1 \rangle$. The final term is $c_k = c_0 \vee A\hat{0}$. So there are

$$\#\Gamma_{\mathcal{L}}(x) \cdot \prod_{i=1}^{k} \langle k - i + 1 \rangle = \#\Gamma_{\mathcal{L}}(x) \cdot \langle k \rangle!$$

sequences.

On the other hand, choose a complement c_k to $x \vee A\hat{0}$ in $[A\hat{0}, \hat{1}]$. First we show that $\operatorname{type} c_k \wedge x = (1^k)$. We have $c_k \wedge (x \vee A\hat{0}) = A\hat{0}$ because the elements on the left are complementary. By modularity, $(c_k \wedge x) \vee A\hat{0} = A\hat{0}$, so $c_k \wedge x \leq A\hat{0}$, so $c_k \wedge x = c_k \wedge A\hat{0} \wedge x = A\hat{0} \wedge x$, which has type (1^k).

Next choose c_{i-1} to be a coatom in $[\hat{0}, c_i]$ with $c_{i-1} \vee x = c_i \vee x = \hat{1}$ and $c_{i-1} \wedge x \lessdot c_i \wedge x$; thus, c_{i-1} is any coatom of $[\hat{0}, c_i]$ not also in $[c_i \wedge x, c_i]$. The type of $[\hat{0}, c_i]$ has first column $\nu_1' + i$, so this interval has $\langle \nu_1' + i \rangle$ coatoms. The interval $[c_i \wedge x, c_i] \cong [x, x \vee c_i] = [x, \hat{1}]$ has type ν and hence $\langle \nu_1' \rangle$ coatoms. The type of $[\hat{0}, x \wedge c_i]$ is (1^i), so c_0 is a complement to x.

The number of sequences counted in this second fashion is

$$\begin{aligned}
\#\Gamma_{[A\hat{0},\hat{1}]}(x \vee A\hat{0}) \cdot \prod_{i=1}^{k} (\langle \nu_1' + i \rangle - \langle \nu_1' \rangle) &= q^{\sum_{j \geq 2} \mu_j' \nu_j'} \cdot \prod_{i=1}^{k} q^{\nu_1'} \langle \nu_1' + i - \nu_1' \rangle \\
&= q^{\sum_{j \geq 2} \mu_j' \nu_j'} \cdot q^{\mu_1' \nu_1'} \cdot \langle k \rangle!.
\end{aligned}$$

Equate this with the number of sequences counted in the first fashion to obtain the theorem. $\square$

4.7. The algebra of independent elements. For a tuple $\vec{x} = (x_1, \ldots, x_k)$ of elements of a semi-primary lattice, and a weak composition $\mu = (\mu_1, \ldots, \mu_k)$, let

$$\begin{aligned}
\vec{x}[[\mu]] &\stackrel{\text{def}}{=} x_1[\mu_1] \vee \cdots \vee x_k[\mu_k] \\
\vec{x}((\mu)) &\stackrel{\text{def}}{=} x_1(\mu_1) \wedge \cdots \wedge x_k(\mu_k).
\end{aligned}$$

Ordinarily, we will apply the former (latter) to a tuple $\vec{x}$ of independent join (meet) irreducibles.

A **join decomposition** of x is a tuple $\vec{x}$ of independent join-irreducibles whose join is x, such that $\rho(x_1) \geq \cdots \geq \rho(x_k)$. A **meet decomposition** is defined dually.

Theorem 4.48. *The number of join decompositions of $\hat{1}$ in a q-regular semi-primary lattice of type λ is*

$$q^{2n(\lambda)} M_\lambda(q^{-1}).$$

Proof. Let $\lambda^{(k)} = (\lambda_k, \lambda_{k+1}, \ldots)$, and t_k be the multiplicity of λ_k in $\lambda^{(k)}$. Let $n_k = \left|\lambda^{(k)}\right| = \lambda_k + \lambda_{k+1} + \cdots$.

Choose a maximum length cycle x_1 in $\hat{1}$ in one of $q^{n_1 - \lambda_1} \langle t_1 \rangle_{q^{-1}}$ ways, and then a complement y_1 to x_1 in one of $q^{n_1 - \lambda_1} = q^{n_2}$ ways.

Suppose we have alternately chosen $y_0, x_1, y_1, \ldots, x_{k-1}, y_{k-1}$ (where $y_0 = \hat{1}$) so that x_i is a maximum length cycle, and y_i is a complement to it, in y_{i-1}, for $i = 1, \ldots, k-1$. Then type $y_i = \lambda^{(i)}$. The number of ways to pick a maximum length cycle x_k in y_{k-1} is $q^{n_{k+1}} \langle t_k \rangle_{q^{-1}}$. The number of complements y_k to x_k in y_{k-1} is $q^{n_{k+1}}$.

The number of sequences $y_0, x_1, y_1, \ldots, x_k, y_k$ of the above form with $y_k = \hat{0}$, is the number of join decompositions of $\hat{1}$, because the y's are recoverable from the x's: $y_i = x_i \vee x_{i+1} \vee \cdots$. This number is

$$\prod_k q^{2n_{k+1}} \langle t_k \rangle_{q^{-1}} = q^{2n(\lambda)} M_\lambda(q^{-1}).$$

The exponent follows from $\sum_k n_{k+1} = \sum_k (\lambda_{k+1} + \lambda_{k+2} + \cdots)$, noting that each λ_i appears $i-1$ times. For each $j > 0$, there is a string of consecutive t_k's of the form $m_j, m_j - 1, \ldots, 1$, so that the product of all $\langle t_k \rangle_{q^{-1}}$ is $M_\lambda(q^{-1})$. $\square$

If x, y are independent elements in a modular lattice, the interval $[\hat{0}, x \vee y]$ is not in general isomorphic to the direct product of lattices $[\hat{0}, x]$ and $[\hat{0}, y]$. However, we have the following result.

Lemma 4.49. *Let x, y be independent elements in a modular lattice. When $x_1, x_2 \leq x$ and $y_1, y_2 \leq y$, we have*

$$\begin{aligned}(x_1 \vee y_1) \vee (x_2 \vee y_2) &= (x_1 \vee x_2) \vee (y_1 \vee y_2) \\ (x_1 \vee y_1) \wedge (x_2 \vee y_2) &= (x_1 \wedge x_2) \vee (y_1 \wedge y_2)\end{aligned}$$

Proof. The first part is clear from associativity and commutativity. For the second part, $(x_1 \vee y_1) \wedge (x_2 \vee y_2) \geq (x_1 \wedge x_2) \vee (y_1 \wedge y_2)$ because each parenthesized expression on the left is weakly greater than each parenthesized expression on the right. Because x and y are independent, any element less than or equal to x and any less than or equal to y are independent, so the rank of the left-hand side is

$$\begin{aligned}&\rho(x_1 \vee y_1) + \rho(x_2 \vee y_2) - \rho(x_1 \vee y_1 \vee x_2 \vee y_2) \\ &\qquad = \rho(x_1) + \rho(y_1) + \rho(x_2) + \rho(y_2) - \rho(x_1 \vee x_2) - \rho(y_1 \vee y_2) \\ &\qquad\qquad = \rho(x_1 \wedge x_2) + \rho(y_1 \wedge y_2),\end{aligned}$$

which is the rank of the right-hand side, so we have equality. □

Corollary 4.50. *Let $\vec{x}$ be a join decomposition of $\hat{1}$ of type λ. Then for all compositions $\mu, \nu \leq \lambda$, we have $\vec{x}[[\mu]] \vee \vec{x}[[\nu]] = \vec{x}[[\mu \vee \nu]]$ and $\vec{x}[[\mu]] \wedge \vec{x}[[\nu]] = \vec{x}[[\mu \wedge \nu]]$. Dual equalities apply to meet decompositions of $\hat{0}$.*

Theorem 4.51. *Let $\vec{x} = (x_1, \ldots, x_k)$ be a join decomposition of $\hat{1}$ and $\vec{y} = (y_1, \ldots, y_k)$ be defined by $y_i = \vee_{j \neq i} x_j$. Then $\vec{y}$ is a meet decomposition of $\hat{0}$. Further, for any weak compositions μ, ν with $\mu_i + \nu_i = \lambda_i$, we have $\vec{x}[[\mu]] = \vec{y}((\nu))$. Further, $C\vec{x}[[\mu]] = \vec{x}[[\mu^-]]$, $A\vec{y}((\nu)) = \vec{y}((\nu^-))$, $A\vec{x}[[\mu]] = \vec{y}((\mu^+))$, and $C\vec{y}((\nu)) = \vec{y}((\nu^+))$, where $\mu_i^- = \max\{\mu_i - 1, 0\}$, $\mu_i^+ = \min\{\mu_i + 1, \lambda_i\}$, and similarly for ν.*

Proof. The intervals $[\hat{0}, x_i] = [x_i \wedge y_1, x_i]$ and $[y_i, \hat{1}] = [y_i, x_i \vee y_i]$ are perspective, so the y_i are cocycles with $\bar{\rho}(y_i) = \rho(x_i) = \lambda_i$. The element μ_i from the bottom of either interval is $\lambda_i - \mu_i = \nu_i$ from the top, so $[y_i, \hat{1}](\nu_i) = y_i \vee x_i[\mu_i] = \vec{x}[[\lambda^{(i)}]]$, where $\tilde{\lambda}$ is the weak composition $\lambda_j^{(i)} = \lambda_j$ if $j \neq i$, and $\lambda_i^{(i)} = \mu_i$. Finally,

$$\vec{y}((\nu)) = \bigwedge_i [y_i, \hat{1}](\nu_i) = \bigwedge_i \vec{x}[[\lambda^{(i)}]] = \vec{x}[[\bigwedge_i \lambda^{(i)}]] = \vec{x}[[\mu]],$$

where we applied Corollary 4.50 to obtain the third equality.

Next,

$$C\vec{y}((\nu)) = C\vec{x}[[\mu]] = C\left(\bigvee_i x_i[\mu_i]\right) = \bigvee_i C\left(x_i[\mu_i]\right) = \bigvee_i x_i[\mu_i^-] = \vec{x}[[\mu^-]] = \vec{y}((\nu^+)),$$

and similarly for the other pair of equalities. □

Definition 4.52. Given weak compositions $\mu \leq \lambda$, define weak compositions $\mathcal{A}_\lambda \mu$ and $\mathcal{C}_\mu \lambda$ in the interval $[\mu, \lambda]$ by specifying their parts:

$$\begin{aligned}(\mathcal{A}_\lambda \mu)_i &= \min\{\mu_i + 1, \lambda_i\} \\ (\mathcal{C}_\mu \lambda)_i &= \max\{\lambda_i - 1, \mu_i\}.\end{aligned}$$

The weak composition $\mathcal{A}_\lambda\mu$ adds a cell to the end of each row of μ unless it would exceed the length of the corresponding row of λ, while the weak composition $\mathcal{C}_\mu\lambda$ deletes a cell from the end of each row of λ unless it would be shorter than the same row of μ.

Corollary 4.53. *Let $\vec{x}, \vec{y}, \lambda, \mu, \nu$ be as in Theorem 4.51. Then*

$$\begin{aligned} C_{\vec{x}[[\mu]]}\big(\vec{x}[[\lambda]]\big) &= \vec{x}[[\mathcal{C}_\mu\lambda]] \\ A_{\vec{x}[[\mu]]}\big(\vec{x}[[\lambda]]\big) &= \vec{x}[[\mathcal{A}_\lambda\mu]] \\ C_{\vec{y}((\nu))}\big(\vec{y}((\lambda))\big) &= \vec{y}((\mathcal{A}_\nu\lambda)) \\ A_{\vec{y}((\nu))}\big(\vec{y}((\lambda))\big) &= \vec{y}((\mathcal{C}_\lambda\nu)) \end{aligned}$$

Definition 4.54. Given partitions μ, λ, we say that μ is a **subcolumn partition** of λ, written $\mu \sqsubseteq \lambda$, if μ can be obtained from λ by deleting columns, or equivalently, $m_i(\mu') \le m_i(\lambda')$ for all possible column lengths i.

Lemma 4.55. *If $\mu \sqsubseteq \lambda$ then $\mu \sqsubseteq \mathcal{A}_\lambda\mu \sqsubseteq \lambda$ and $\mu \sqsubseteq \mathcal{C}_\mu\lambda \sqsubseteq \lambda$.*

Proof. We obtain $\mathcal{A}_\lambda\mu$ from μ by inserting the largest column of λ omitted from μ, while we obtain $\mathcal{C}_\mu\lambda$ from λ by deleting the largest omitted column. □

Lemma 4.56. *If $\mu, \nu \sqsubseteq \lambda$ then $\mu \vee \nu, \mu \wedge \nu \sqsubseteq \lambda$.*

Proof. Let $\rho = \mu \wedge \nu$, so $\rho'_i = \min\{\mu'_i, \nu'_i\}$. Each ρ'_i is the length of some column of μ or ν and hence of λ. We must make sure that the multiplicity of each length is no greater than in λ. Suppose i is minimal with $\rho'_i = k$, and that, without loss of generality, $\mu'_i = k$. The $\big(i + m_k(\mu')\big)$th column of μ, and hence of ρ, has length strictly less than k, so $m_k(\rho') \le m_k(\mu')$. Also, $m_k(\mu') \le m_k(\lambda')$, so $m_k(\rho') \le m_k(\lambda')$. This holds for all k, so $\rho \sqsubseteq \lambda$.

The argument for joins is similar, but instead of choosing the leftmost column of length k and examining the one $m_k(\mu')$ to its right, we choose the rightmost one of length k and examine the column (if any) $m_k(\mu')$ to its left. □

Theorem 4.57. *Let S be the minimal set containing $\emptyset, \lambda$ and closed under $\wedge, \vee, \mathcal{A}, \mathcal{C}$. Then S is precisely the set of all subcolumn partitions of λ.*

Proof. If $\mu \sqsubseteq \nu \sqsubseteq \lambda$ then $\mathcal{A}_\nu\mu$ adds a column to μ of length equal to the largest i with $m_i(\nu') > m_i(\mu')$, while $\mathcal{C}_\nu\mu$ subtracts a column of this length from ν. So all elements of S are subcolumn partitions of λ.

Conversely, let $\mu \sqsubseteq \lambda$ be obtained from λ by deleting columns $c_1 > c_2 > \cdots > c_k > 0$. Abbreviate $\mathcal{A}x = \mathcal{A}_\lambda x$ and $\mathcal{C}x = \mathcal{C}_\emptyset x$. Then $\mu = \mathcal{A}^{c_k-1}\mathcal{C}^{c_k}\cdots\mathcal{A}^{c_1-1}\mathcal{C}^{c_1}\lambda$, because $\mathcal{A}^{c_i-1}\mathcal{C}^{c_i}$ deletes the first c_i columns, which happen to coincide with the first c_i columns of λ, and then inserts the $c_i - 1$ largest columns of λ that are missing, which in fact are the $c_i - 1$ largest columns of λ. □

Theorem 4.58. *Let $\mu \sqsubseteq \lambda = \operatorname{type}\hat{1}$. Let $\vec{x}$ and $\vec{y}$ be two join decompositions of $\hat{1}$. Then $\vec{x}[[\mu]] = \vec{y}[[\mu]]$.*

Proof. By Theorem 4.57, we can express μ as a polynomial in $\wedge, \vee, \mathcal{A}, \mathcal{C}, \emptyset, \lambda$. Replace $\emptyset$ with $\hat{0}$; λ with $\hat{1}$; $\mathcal{A}$ with A; $\mathcal{C}$ with C; and interpret $\wedge$ and $\vee$ in $\mathcal{L}$ instead of Young's lattice. This gives a decomposition-independent expression of an element equal to either of $\vec{x}[[\mu]]$ and $\vec{y}[[\mu]]$, by recursive applications of Corollaries 4.50 and 4.53. □

Definition 4.59. This last theorem allows us, for all $\mu \sqsubseteq \lambda$, to define a special element $\hat{1}[[\mu]]$. Its type is μ and its cotype is $(\lambda_1 - \mu_1, \lambda_2 - \mu_2, \ldots)$, or alternately, the columns of λ missing from μ. Further, for any interval $[x, y]$, we may define $[x, y][[\mu]]$ as $\vec{z}[[\mu]]$, where $\mu \sqsubseteq \operatorname{type}[x, y]$ and $\vec{z}$ is any join decomposition of y in the interval $[x, y]$. We abbreviate $[\hat{0}, y][[\mu]]$ by $y[[\mu]]$.

Theorem 4.60. *Let λ, μ, ν be partitions with $\mu + \nu = \lambda$. The unique element of type μ and cotype ν in a semi-primary lattice of type λ is $\hat{1}[[\mu]]$.*

Proof. The previous theorem establishes existence. To prove uniqueness, we induct on the length of the lattice. For $\lambda = \mu = \nu = \emptyset$, there is only one element in the lattice, so take a larger case. Suppose x has type μ and cotype ν. Either $\mu'_1 = \lambda'_1$, or $\nu'_1 = \lambda'_1$ (or both). In the former case, $A_x\hat{0} = A\hat{0}$, so that x has type $\mathcal{C}\mu$ and cotype ν in the interval $[A\hat{0}, \hat{1}]$ of type $\mathcal{C}\lambda$, and since this theorem inductively applies to this smaller interval, x is unique. In the latter case, $C_x\hat{1} = C\hat{1}$, so that x has type μ and cotype $\mathcal{C}\nu$ in $[\hat{0}, C\hat{1}]$ (of type $\mathcal{C}\lambda$), and again x is unique. □

4.8. Hereditary Decompositions.

Definition 4.61. A **hereditary (join) decomposition** of a (nonsaturated) flag $f = (f_0 \leq \cdots \leq f_n)$ is a join decomposition $\vec{x}$ of f_n such that $f_i = \vec{x}[[\operatorname{type} f_i]]$ for all i.

Theorem 4.62. *Let $\mathcal{L}$ be a semi-primary lattice of type λ. Let $f = (f_0, \ldots, f_n)$ have a hereditary join decomposition in $\mathcal{L}$. Let $p(v_0, \ldots, v_n)$ be a lattice polynomial constructed from the variables $v_0, \ldots, v_n$; constant $\hat{0}$; and binary operators $\wedge, \vee, C, A$. Then* type $p(f_0, \ldots, f_n)$ *is computed by substituting into $p(v_0, \ldots, v_n)$ the following: v_i becomes* type f_i*; C becomes $\mathcal{C}$; A becomes $\mathcal{A}$; $\hat{0}$ becomes $\emptyset$; and $\wedge$ and $\vee$ are interpreted in Young's lattice instead of $\mathcal{L}$.*

Proof. Consider any hereditary decomposition $\vec{x}$ of f. Then

$$p(f_0, \ldots, f_n) = p(\vec{x}[[\operatorname{type} f_0]], \ldots, \vec{x}[[\operatorname{type} f_n]]).$$

Also $\hat{0}$ can be replaced by $\vec{x}[[\emptyset]]$. Recursively apply Corollaries 4.50 and 4.53 to this form of the polynomial to obtain the result. □

Now we'll examine some conditions under which flags have hereditary join decompositions.

Theorem 4.63. *Let $x < y$ in a semi-primary lattice $\mathcal{L}$. There is a hereditary decomposition of (x, y) in which a single part is reduced in length by 1. Consequently,* type $x <$ type y *in Young's lattice. If the types differ in the first component, then all join decompositions of y restrict to join decompositions of x.*

Proof. If $x = \hat{0}$ then y is an atom, and the decomposition is simply y.

If $\operatorname{arank} x = \operatorname{arank} y$ then pick a cycle $c \leq x$ of rank $\operatorname{arank} x$. It will be one element of the decompositions of both x and y. Take a complement c' of c in $[\hat{0}, y]$. Then $y \vee c' \geq x \vee c' \geq c \vee c' = \hat{1}$ so $y \vee c' = x \vee c' = \hat{1}$, and then by Lemma 2.3, $c' = y \wedge c' \gtrdot x \wedge c'$. Since $\rho(c') < \rho(y)$, inductively there is a hereditary decomposition of $(x \wedge c', c')$ satisfying the conditions of the theorem, and to this decomposition we prepend a cycle c to obtain a decomposition of (x, y).

If $\operatorname{arank} x < \operatorname{arank} y$ then pick any cycle $c \leq y$ of rank $\operatorname{arank} y$. Let $d = c \wedge x \lessdot c \wedge y = c$. Let c' be a complement of d in $[\hat{0}, x]$. Then $c' \wedge c \geq c' \wedge d = \hat{0}$, but the only atom in the chain $[\hat{0}, c]$ is also in $[\hat{0}, d]$, so $c' \wedge c = \hat{0}$. Thus, $c' \vee c \gtrdot c' \vee d = x$; in fact, $c' \vee c = y$, since $c' \vee c \leq y$ and $y \gtrdot x$. Thus c' is also a complement of c in y, so we add c to any decomposition of c' to obtain a hereditary decomposition of (x, y). ☐

As a corollary, we obtain a different proof of Theorem 4.11.

Theorem 4.11. *Let* $[a', b'] \subseteq [a, b]$. *Then* $\operatorname{type}[a', b'] \leq \operatorname{type}[a, b]$.

Proof. By repeated applications of the preceding theorem on a saturated chain from b down to b', we have $\operatorname{type}[a, b'] \leq \operatorname{type}[a, b]$. By a dual argument, we obtain $\operatorname{type}[a', b'] \leq \operatorname{type}[a, b']$. ☐

We obtain an enumerative version of Theorem 4.63.

Corollary 4.64. *Let x be a coatom of type μ in a q-regular semi-primary lattice of type λ. Suppose λ and μ differ in their ith part.*

(1) *The number of complemented cycles c of length $r = \lambda_i$ with $c \not\leq x$ is*
$$q^{(\lambda'_1 + \cdots + \lambda'_r) - r}.$$
(2) *The number of hereditary decompositions of $(x, \hat{1})$ is*
$$\frac{q^{2n(\lambda) - i + 1} M_\lambda(q^{-1})}{\langle m_r(\lambda) \rangle_{q^{-1}}}.$$

Proof.

(1) Let $r = \lambda_i$. The number of complemented cycles of length r in $[\hat{0}, \hat{1}]$, less the number of complemented cycles of length r in $[\hat{0}, x]$, is
$$\begin{aligned} &q^{(\lambda'_1 + \cdots + \lambda'_r) - r} \langle m_r(\lambda) \rangle_{q^{-1}} - q^{(\mu'_1 + \cdots + \mu'_r) - r} \langle m_r(\mu) \rangle_{q^{-1}} \\ &\qquad = q^{(\lambda'_1 + \cdots + \lambda'_r) - r} \langle m_r(\lambda) \rangle_{q^{-1}} - q^{(\lambda'_1 + \cdots + \lambda'_r - 1) - r} \langle m_r(\lambda) - 1 \rangle_{q^{-1}} \\ &\qquad\qquad = q^{(\lambda'_1 + \cdots + \lambda'_r) - r}. \end{aligned}$$
by Theorem 4.47. The parts of λ' and μ' are the same except $\mu'_r = \lambda'_r - 1$, so $m_r(\mu) = m_r(\lambda) - 1$. The two terms subtracted in the second expression are each sums of consecutive powers of q, differing only in the single power given in the third expression.

(2) Let $d = (\lambda'_1 + \cdots + \lambda'_r) - r$. Consider any hereditary decomposition $\vec{x}$ of $(x, \hat{1})$. We have $x_i \leq \hat{1}$; $x_i \not\leq x$; and $C x_i \leq x$. Let $c' = \vee_{j \neq i} x_j$. Then c' is a complement to x_i in $[\hat{0}, \hat{1}]$, and to $C x_i$ in $[\hat{0}, x]$. Note that in fact any complement y to $C x_i$ in $[\hat{0}, x]$ is a complement to x in $[\hat{0}, \hat{1}]$ (though the

converse is not true): we have $x_i \vee y \geq Cx_i \vee y = x$ but $x_i \not\leq x$, so $x_i \vee y = \hat{1}$, and so $x_i \wedge y = Cx_i \wedge y = \hat{0}$.

To enumerate the hereditary decompositions, first choose a complemented cycle $x_i \leq \hat{1}$ but $x_i \not\leq x$ in one of $q^d \langle m_r(\lambda)\rangle_{q^{-1}}$ ways. This has type $(1^r)'$ and cotype

$$\nu = (\lambda'_1 - 1, \ldots, \lambda'_r - 1, \lambda'_{r+1}, \lambda'_{r+2}, \ldots)'.$$

Then we pick a complement c' to Cx_i in x, with type ν and cotype (1^{r-1}) in $[\hat{0}, x]$, in one of

$$q^{\lambda'_1 + \cdots + \lambda'_{r-1} - (r-1)} = q^{d - \lambda'_r + 1} = q^{d-i+1}$$

ways by Theorem 4.47. Obtain a decomposition of c' in one of

$$q^{2n(\nu)} M_\nu(q^{-1}) = q^{2(n(\lambda)-d)} \frac{M_\lambda(q^{-1})}{\langle m_r(\lambda)\rangle_{q^{-1}}}$$

ways, and insert x_i into this decomposition as the ith term to obtain a hereditary decomposition of $(x \lessdot \hat{1})$. Multiply the number of choices together to obtain the value stated in the theorem. $\square$

The next several theorems are different generalizations of Theorem 4.63 to hereditary decompositions of flags.

Definition 4.65. An interval is **elementary** if its type is of the form (1^n).

Definition 4.66. A **vertical strip** is a skew partition λ/μ with at most one cell in each row; equivalently, $\lambda_i \geq \mu_i$ for all $i > 0$. The notation $\lambda \geq_{\vee} \mu$ means $\lambda \geq \mu$ and λ/μ is a vertical strip.

Dually, a **horizontal strip** is a skew partition λ/μ with at most one cell in each column; equivalently, $\lambda'_i \geq \mu'_i$ for all $i > 0$.

Proposition 4.67. *Let $[x, y]$ be an elementary interval in a semi-primary lattice. Then* $\operatorname{type} y / \operatorname{type} x$ *is a vertical strip.*

Proof. Since $[x, y]$ is elementary, it is contained in $[Cy, y]$, whence $\operatorname{type} Cy \leq \operatorname{type} x \leq \operatorname{type} y$. Since $\operatorname{type} y / \operatorname{type} Cy$ has one cell on each nonempty row of $\operatorname{type} y$, and no other cells, it is a vertical strip. $\square$

Definition 4.68. A **row-end** of a tableau $\boldsymbol{P}$ of shape λ/μ is any cell (i, λ_i) with $1 \leq i \leq \lambda'_1$.

Theorem 4.69. *Let $f = (f_0, \ldots, f_n)$ be a flag in which the interval $[f_0, f_n]$ is elementary.*

(1) *f has a hereditary decomposition.*
(2) *Let $\boldsymbol{P} = (\lambda^{(0)} \lessdot \cdots \lessdot \lambda^{(n)})$ be a standard skew tableau whose entries form a vertical strip, $\lambda^{(n)}/\lambda^{(0)}$. In a q-regular semi-primary lattice of shape $\lambda^{(n)}$, the number of these flags with* $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ *is*

$$\prod_{k \,\in\, \boldsymbol{P}} \left(\langle l_1\rangle - \langle l_2\rangle\right)$$

where

$$\begin{aligned} l_1 &= \#\ \textit{row-ends of } \boldsymbol{P} \textit{ weakly right of } k \textit{ containing any of } \bullet, 1, \ldots, k \\ l_2 &= \#\ \textit{row-ends of } \boldsymbol{P} \textit{ strictly right of } k \textit{ containing any of } \bullet, 1, \ldots, k \end{aligned}$$

Proof.

(1) We may assume f is saturated, because if it's not, we can always remove duplicate elements and add additional intermediate elements to saturate it; doing so does not change the type of the interval from the first to the last element, and still provides decompositions of all the original elements.

Any hereditary decomposition of $f_{n-1} \lessdot f_n$ yields a cycle c and an independent element c' with $c \vee c' = f_n$ and $Cc \vee c' = f_{n-1}$.

For $0 \leq i \leq n$, the interval $[c \wedge f_i, c \wedge f_n]$ is contained in $[\hat{0}, c]$, so its type, α, is a one row partition. Also,

$$[f_i \wedge c, f_n \wedge c] \subseteq [f_i \wedge c, c] \cong [f_i, f_i \vee c] \subseteq [f_i, f_n],$$

so α is a one column partition. Thus α is $\emptyset$ or (1). Further, $c \wedge f_{n-1} \lessdot c \wedge f_n$, so $\alpha = (1)$ for $i = 0, \ldots, n-1$. Thus $c \wedge f_i = Cc$ for $i = 0, \ldots, n-1$.

Let $g = (g_0 \leq \cdots \leq g_{n-1})$ be the flag given by $g_i = f_i \wedge c'$. The elements Cc and g_i are independent, and

$$Cc \vee g_i = Cc \vee (c' \wedge f_i) = (Cc \vee c') \wedge f_i = f_{n-1} \wedge f_i = f_i.$$

The interval $[g_0, g_{n-1}]$ is elementary because

$$Ag_0 = A(f_0 \wedge c') = Af_0 \wedge Ac' \geq f_n \wedge Ac' \geq f_{n-1} \wedge c' = g_{n-1}.$$

Since the flag g is saturated and shorter than f, by induction we find a hereditary decomposition of g, and then add c to it to obtain a hereditary decomposition of f.

(2) Set $f_n := \hat{1}$. After choosing $f_n, \ldots, f_k$, choose f_{k-1} to be a coatom of $[\hat{0}, f_k]$ of type $\lambda^{(k-1)}$ so that $[f_{k-1}, \hat{1}]$ is elementary, as follows. Let $r = \mathrm{col}(\lambda^{(k)}/\lambda^{(k-1)})$. The condition that f_{k-1} is a coatom of $[\hat{0}, f_k]$ of type $\lambda^{(k-1)}$ is equivalent to

$$\begin{aligned} f_{k-1} &\lessdot f_k \\ f_{k-1} &\geq f_k[r-1] \\ f_{k-1} &\not\geq f_k[r] \end{aligned}$$

while the elementary condition is

$$f_{k-1} \leq C\hat{1}.$$

So f_{k-1} is any coatom of $I_1 = \left[C\hat{1} \vee f_k[r-1], f_k\right]$ not also in $I_2 = \left[C\hat{1} \vee f_k[r], f_k\right]$. These are both elementary intervals since they are subintervals of $[C\hat{1}, \hat{1}]$. By (1) there is a hereditary decomposition of $(f_k, \hat{1})$, so by Theorem 4.62, $\mathrm{type}(C\hat{1} \vee f_k[r-1]) = \mathcal{C}\lambda^{(n)} \vee \lambda^{(k)}[r-1]$. Compare this rowwise to $\lambda^{(k)}$: when $r - 1 < \lambda_i^{(k)} = \lambda_i^{(n)}$ the ith row is $\lambda_i^{(k)} - 1$, and otherwise, the ith row is $\lambda_i^{(k)}$. We have $\lambda_i^{(k)} \lessdot \lambda_n^{(k)}$ when the entry at the end of row i of $\boldsymbol{P}$ is larger than k. Thus, $\rho(I_1)$ is the number of entries weakly smaller than k at the end of a row in a column weakly right of k. Similarly, $\rho(I_2)$ is the number of entries weakly smaller than k at the end of a row in a column strictly right of k. $\square$

Theorem 4.70. *Let $f = (f_0 \lessdot \cdots \lessdot f_n)$ be a flag in a semi-primary lattice $\mathcal{L}$ with* $\operatorname{ftype}_{\hat{0}} f$ *equal to the superstandard skew tableau of shape λ/μ.*

(1) *f has a hereditary decomposition.*
(2) *If $\mathcal{L}$ is q-regular, there are*

$$q^{n(\lambda)+n(\mu)} M_\nu(q^{-1})$$

such decompositions, where ν is the partition given by

$$m_i(\nu) = \#\{\, j \,:\, \lambda_j = \mu_j = i \,\} \text{ for } i \geq 1.$$

Proof. In both parts we induct on the number of rows in λ. When there are no rows, the flag is $(\hat{0})$, which has the empty sequence as its sole decomposition.

(1) Let the first row of $\operatorname{ftype}_{\hat{0}} f$ have entries $1, \ldots, k$, with $k = \lambda_1 - \mu_1$. Choose a cycle c of rank λ_1 with $c \leq f_k$, and take a complement c' of c in f_n. Since $\operatorname{arank} f_k = \cdots = \operatorname{arank} f_n = \lambda_1$, we see that c is a maximal cycle in each of $f_k, \ldots, f_n$, and so the type of each is the type of any relative complement of c, with a part λ_1 prepended. The element $c' \wedge f_j$ is a complement to c in f_j (with $k \leq j \leq n$) because $c \vee (c' \wedge f_j) = (c \vee c') \wedge f_j = f_n \wedge f_j = f_j$ and $c \wedge (c' \wedge f_j) \leq c \wedge c' = \hat{0}$. So the flag $g = (f_k \wedge c', \ldots, f_n \wedge c')$ of complements to c has type $\operatorname{ftype}_{\hat{0}} f$ with the first row deleted; that is, it is the superstandard skew tableau of shape $\tilde{\lambda}/\tilde{\mu}$ where $\tilde{\lambda} = (\lambda_2, \lambda_3, \ldots)$ and $\tilde{\mu} = (\mu_2, \mu_3, \ldots)$ and entries $k+1, \ldots, n$. Inductively applying the theorem to g, we obtain a hereditary decomposition $\vec{x}$ of g, to which we prepend the cycle c to obtain a hereditary decomposition $\vec{y}$ of $(f_k, \ldots, f_n)$. Finally, because the types of $f_0, \ldots, f_k$ differ in only their first component, all join decompositions of f_k are hereditary decompositions of $f_0, \ldots, f_k$, so $\vec{y}$ is a hereditary decomposition of f.

(2) We count the choices made in (1). The type of f_k is $\gamma = (\lambda_1, \mu_2, \mu_3, \ldots)$, so the number of choices of c is

$$q^{\mu_2+\mu_3+\cdots} \langle m_{\lambda_1}(\gamma) \rangle_{q^{-1}} = \begin{cases} q^{|\mu|-\mu_1} & \text{if } \lambda_1 > \mu_1; \\ q^{|\mu|-\mu_1} \langle m_{\mu_1}(\mu) \rangle_{q^{-1}} & \text{if } \lambda_1 = \mu_1. \end{cases}$$

by Theorem 4.44 (note that if $\lambda_1 > \mu_1$ then $\lambda_1 > \mu_2$, so $m_{\lambda_1}(\gamma) = 1$, and $\langle 1 \rangle_{q^{-1}} = 1$). The number of choices of c' is

$$q^{|\lambda|-\lambda_1}$$

by Theorem 4.47. Let $\tilde{\nu}$ be the partition given by

$$m_i(\tilde{\nu}) = \#\left\{\, j \,:\, \tilde{\lambda}_j = \tilde{\mu}_j = i \,\right\}.$$

Then inductively there are

$$q^{n(\tilde{\lambda})+n(\tilde{\mu})} M_{\tilde{\nu}}(q^{-1})$$

decompositions of g.

If we fill in each cell of λ with its row number minus 1 and add up these entries, we obtain $n(\lambda)$; subtract an additional 1 from each entry not on the first row, and add them up to obtain $n(\tilde{\lambda})$. A similar argument computes $n(\tilde{\mu})$. Thus,

$$\begin{aligned} n(\tilde{\lambda}) &= n(\lambda) - |\lambda| + \lambda_1 \\ n(\tilde{\mu}) &= n(\mu) - |\mu| + \mu_1 \end{aligned}$$

Now we multiply all the numbers of choices together to obtain the number of decompositions:

$$\begin{array}{ll} q^{n(\lambda)+n(\mu)} M_{\tilde{\nu}}(q^{-1}) & \text{if } \lambda_1 > \mu_1; \\ q^{n(\lambda)+n(\mu)} \langle m_{\mu_1}(\mu) \rangle_{q^{-1}} M_{\tilde{\nu}}(q^{-1}) & \text{if } \lambda_1 = \mu_1. \end{array}$$

If $\lambda_1 > \mu_1$ then $\nu = \tilde{\nu}$, so that we have the total stated in the theorem.

If $\lambda_1 = \mu_1$ then $\nu = (\mu_1, \tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_3, \ldots)$. Since the parts of λ weakly decrease, but are bounded below by the parts of μ, the first $m_{\mu_1}(\mu)$ parts of λ are also equal to μ_1, so $m_{\mu_1}(\nu) = m_{\mu_1}(\mu)$ and $M_\nu(q^{-1}) = \langle m_{\nu_1}(\nu) \rangle_{q^{-1}} M_{\tilde{\nu}}(q^{-1})$. So again, the total is as stated in the theorem. $\square$

Example 4.71. Consider

$$\boldsymbol{P} = \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & 1 \\ \bullet & \bullet & \bullet & 2 & 3 \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & & \\ \bullet & 4 & 5 & & \\ \bullet & & & & \\ 6 & & & & \end{array}$$

This has shape $\lambda/\mu = (5,5,5,5,4,4,4,4,1,1)/(5,5,4,3,3,3,3,1,1)$. The rows with no entries form the partition $\nu = (5,5,3,3,3,1)$. A given flag with type $\boldsymbol{P}$ has

$$q^{n(\lambda)+n(\mu)} M_\nu(q^{-1}) = q^{173+82} \langle 2 \rangle_{q^{-1}} \langle 3 \rangle_{q^{-1}} \langle 1 \rangle_{q^{-1}} = q^{255} \langle 2 \rangle_{q^{-1}} \langle 3 \rangle_{q^{-1}} \langle 1 \rangle_{q^{-1}}$$

hereditary decompositions.

Corollary 4.72. *Let f be a saturated flag and* $\mathrm{low}(f) < k < \mathrm{high}(f)$.

(1) *If* $\mathrm{ftype}_{\hat{0}} f$ *has $k+1$ on a lower row than k, then* $\mathrm{type}[f_{k-1}, f_{k+1}] = \begin{smallmatrix}\square\\\square\end{smallmatrix}$*;*
(2) *if k and $k+1$ are on the same row, then* $\mathrm{type}[f_{k-1}, f_{k+1}] = \square\square$*;*
(3) *if $k+1$ is on an earlier row, either type is possible (although generically we expect $\square\square$).*

Proof. In the first two cases, we have $\mathrm{ftype}_{\hat{0}}(f_{k-1} < f_k < f_{k+1})$ is a superstandard skew tableau, so that these three consecutive elements have a hereditary decomposition. In (1), two join-irreducibles in the decomposition of f_{k+1} are each reduced in length by 1 to obtain f_{k-1}, while in (2), one join-irreducible is reduced in length by 2. So we obtain the types indicated.

The smallest example of (3) is

$$\mathrm{ftype}_{\hat{0}} f = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

We see that $\operatorname{type} Cf_3 = (1)$. So the only value of f_1 with $\operatorname{type}[f_1, f_3] = \substack{\square\\\square}$ is Cf_1, and all other atoms f_1 of $[f_0, f_2]$ yield $\operatorname{type}[f_1, f_3] = \square\square$.

More generally in (3), let $\boldsymbol{P} = (\lambda^{(k-1)} \lessdot \lambda^{(k)} \lessdot \lambda^{(k+1)})$ be a skew tableau of shape λ/μ with just 2 entries, k in cell (r_0, c_0) and $k+1$ in cell (r_1, c_1), with $c_0 < c_1$ and $r_0 > r_1$. Choose any f_k, f_{k+1} with $\operatorname{ftype}_{\hat{0}}(f_k, f_{k+1}) = (\lambda^{(k)}, \lambda^{(k+1)})$. We will show that of those $f_{k-1} \lessdot f_k$ with type $\lambda^{(k-1)}$, fewer have $\operatorname{type}[f_{k-1}, f_{k+1}] = \substack{\square\\\square}$ than $\square\square$.

We have $f_{k-1} \lessdot f_k$ and $\operatorname{type} f_{k-1} = \lambda^{(k-1)}$ iff f_{k-1} is a coatom of the interval $I_1 = \big[f_k[c_0-1] \vee Cf_k, f_k\big]$ not also in $I_2 = \big[f_k[c_0] \vee Cf_k, f_k\big]$.

If we further want $\operatorname{type}[f_{k-1}, f_{k+1}] = \substack{\square\\\square}$, then since $f_{k-1} \geq Cf_{k+1}$, we obtain that f_{k-1} is any coatom of $I_3 = \big[f_k[c_0-1] \vee Cf_{k+1}, f_k\big]$ not also in $I_4 = \big[f_k[c_0] \vee Cf_{k+1}, f_k\big]$.

Let $x_1 = f_k[c_0-1] \vee Cf_k$, and similarly, let $x_i = \hat{0}_{I_i}$ for $i = 1, 2, 3, 4$. Since $f_k \lessdot f_{k+1}$, these have a hereditary decomposition, so the types of $x_1, \ldots, x_4$ can be computed by Theorem 4.62 by replacing f_k and f_{k+1} with their types, and C with $\mathcal{C}$.

The type of x_1 is $\lambda^{(k)}$ with each row at least c_0 long reduced in length by 1, while in $\operatorname{type} x_2$, each row of $\lambda^{(k)}$ at least c_0+1 long is reduced in length by 1. The types of x_3 and x_4 are almost the same as x_1 and x_2 except that an additional cell is added at (r_1, c_1-1). Since $c_1 > c_0$, this cell is not present in $\operatorname{type} x_1$, and when $c_1 > c_0+1$, it is not present in $\operatorname{type} x_2$. Note that in any case, we do not have $x_3 = x_4$ because the cell (r_0, c_0) is in $\operatorname{type} x_4$ but not $\operatorname{type} x_3$. Thus, $I_3 - I_4$ is nonempty, and any coatom yields f_{k-1} with $\operatorname{type}[f_{k-1}, f_{k+1}] = \substack{\square\\\square}$. Any coatom of $I_1 - I_2$ not also in $I_3 - I_4$ yields $\square\square$. Since I_1 is longer than I_3 by 1, there are roughly q times as many possibilities yielding $\square\square$ as $\substack{\square\\\square}$ in a q-regular semi-primary lattice. $\square$

Example 4.73. Suppose

$$\operatorname{ftype} f = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 8 \\ \hline 2 & 5 & 6 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}$$

Then intervals $[f_0, f_2]$, $[f_3, f_5]$, $[f_5, f_7]$, and $[f_7, f_9]$ have type $\substack{\square\\\square}$. Intervals $[f_2, f_4]$ and $[f_4, f_6]$ have type $\square\square$. The types of the intervals $[f_1, f_3]$ and $[f_6, f_8]$ cannot be determined from $\operatorname{ftype} f$.

Definition 4.74. Let $\boldsymbol{A}$ be a skew tableau and $\boldsymbol{B}$ be an ordinary tableau. The tableau $\boldsymbol{A} + \boldsymbol{B}$ is obtained by appending the ith row of $\boldsymbol{B}$ to the ith row of $\boldsymbol{A}$, for all i. Its shape is $\operatorname{osh} \boldsymbol{A} + \operatorname{sh} \boldsymbol{B} = \big((\operatorname{osh} \boldsymbol{A})' \cup (\operatorname{sh} \boldsymbol{B})'\big)'$, so $\operatorname{osh} \boldsymbol{A}$ and $\operatorname{sh} \boldsymbol{B}$ are both subcolumn partitions of $\operatorname{osh} \boldsymbol{A} + \boldsymbol{B}$. For example,

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 8 & 10 \\ \hline 9 & 11 \\ \hline 12 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 7 & 8 & 10 \\ \hline 2 & 5 & 9 & 11 \\ \cline{1-4} 4 & 12 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$$

This is also defined for composition tableaux, but the relation for transposed shapes doesn't apply. The definition in terms of the chains $\boldsymbol{A} = (\lambda^{(l)}, \lambda^{(l+1)}, \ldots, \lambda^{(k)})$ and $\boldsymbol{B} = (\gamma^{(k)}, \gamma^{(k+1)}, \ldots, \gamma^{(h)})$ is

$$\boldsymbol{A} + \boldsymbol{B} = (\lambda^{(l)}, \lambda^{(l+1)}, \ldots, \lambda^{(k)} = \gamma^{(k)} + \lambda^{(k)}, \gamma^{(k+1)} + \lambda^{(k)}, \ldots, \gamma^{(h)} + \lambda^{(k)}).$$

Definition 4.75. For any tableau $\boldsymbol{A} = (\lambda^{(l)}, \ldots, \lambda^{(h)})$, let $d\boldsymbol{A} = (\lambda^{(l)}, \ldots, \lambda^{(h-1)})$, so $\operatorname{low}(d\boldsymbol{A}) = \operatorname{low}(\boldsymbol{A})$ and $\operatorname{high}(d\boldsymbol{A}) < \operatorname{high}(\boldsymbol{A})$. This can be thought of as deleting the largest entry of $\boldsymbol{A}$, though when $\operatorname{high}(\boldsymbol{A})$ doesn't occur in $\boldsymbol{A}$, this isn't strictly correct.

Theorem 4.76. *Let f be a saturated flag. Suppose $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{A} + \boldsymbol{B}$, where $\boldsymbol{A}$ is a standard skew tableau and $\boldsymbol{B}$ is an ordinary standard tableau. Let $h = \operatorname{high}(\boldsymbol{B})$ and $k = \operatorname{high}(\boldsymbol{A}) = \operatorname{low}(\boldsymbol{B})$. Let $\mu = \operatorname{sh} \boldsymbol{A}$. Then $f_k = f_m[[\mu]]$ for $m = k, \ldots, h$, and $\operatorname{ftype}(f_k, \ldots, f_h) = \boldsymbol{B}$.*

Proof. We induct on h. When $h = k$ it is true, so take $h > k$. Then $\operatorname{ftype} df = \boldsymbol{A} + d\boldsymbol{B}$, and inductively we assume the theorem applies to df. Let $\vec{x}$ be a hereditary decomposition of $(f_{h-1} < f_h)$.

Since $\mu \sqsubseteq \operatorname{sh} \boldsymbol{A} + \boldsymbol{B} = \operatorname{type} f_h$, by Theorem 4.58, the shape μ restriction of all join decompositions of f_h are the same. Thus $f_h[[\mu]] = \vec{x}[[\mu]] = f_{h-1}[[\mu]] = f_k$. Because $f_k = \vec{x}[[\mu]]$, we have $\operatorname{type}[f_k, f_h]$ is obtained from the composition $\operatorname{type} f_h - \operatorname{type} f_k = \operatorname{sh} \boldsymbol{B}$ by sorting the parts into decreasing order to form a partition, and they already are in order. By applying the theorem to df and $\operatorname{ftype} df = d(\boldsymbol{A} + \boldsymbol{B}) = \boldsymbol{A} + d\boldsymbol{B}$, we have $\operatorname{ftype}(f_k, \ldots, f_{h-1}) = d\boldsymbol{B}$, and by adding a cell with h to achieve the shape $\operatorname{sh} \boldsymbol{B}$ of $\operatorname{type}[f_k, f_h]$, we obtain $\operatorname{ftype}(f_k, \ldots, f_h) = \boldsymbol{B}$. ◻

Example 4.77. For all flags $f = (f_0 < \cdots < f_{11})$,

$$\text{if} \quad \operatorname{ftype} f = \begin{ytableau} 1 & 2 & 4 & 7 & 8 \\ 3 & 5 & 9 & 11 \\ 6 & 10 \end{ytableau} = \begin{ytableau} 1 & 2 & 4 \\ 3 & 5 \\ 6 \end{ytableau} + \begin{ytableau} 7 & 8 \\ 9 & 11 \\ 10 \end{ytableau} \quad \text{then} \quad \operatorname{ftype} \partial^6 f = \begin{ytableau} 7 & 8 \\ 9 & 11 \\ 10 \end{ytableau}.$$

See Theorem 4.93 for a generalization of this to $\boldsymbol{B}$ being a row and column strict composition tableau.

4.9. Hall Polynomials.

The first two definitions are standard.

Definition 4.78. The **reverse row word** of a skew tableau $\boldsymbol{P}$ is the sequence of digits obtained by reading the entries of $\boldsymbol{P}$ from right to left on the first row, then the second row, and so on to the bottom. For example, the reverse row word of

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & 1 & 1 & 1 \\ \bullet & \bullet & 1 & 2 & 2 & \\ \bullet & 1 & 3 & & & \\ 4 & & & & & \end{array}$$

is 111221314.

Definition 4.79. A **ballot sequence** is a sequence of integers such that in any initial segment of the sequence, there are at least as many i's as $(i+1)$'s for all $i \in \mathbb{P}$. So 111221314 is a ballot sequence, but 112221314 is not because in the initial segment 11222, there are more 2's than 1's.

Definition 4.80 (see [16, p. 90]). A tableau $\boldsymbol{P}$ representing the chain of partitions $(\lambda^{(0)} \leq \lambda^{(1)} \leq \ldots \leq \lambda^{(r)})$ is called an **LR-sequence of type** $(\mu, \nu; \lambda)$ if

(1) $\lambda^{(0)} = \mu$ and $\lambda^{(r)} = \lambda$;
(2) $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip of length ν_i for $1 \leq i \leq r$;

(3) the reverse row word of $\boldsymbol{P}$ is a ballot sequence.

The number of LR-sequences of type $(\mu, \nu; \lambda)$ is the **Littlewood-Richardson coefficient** $c^{\lambda}_{\mu\nu}$. Their properties include $c^{\lambda}_{\mu\nu} = c^{\lambda}_{\nu\mu} = c^{\lambda'}_{\mu',\nu'}$, proofs of which can also be found in [**16**].

Theorem 4.81. *Let μ, ν, λ be partitions. There are polynomials $g^{\lambda}_{\mu\nu}(q)$ with integer coefficients, such that in every q-regular semi-primary lattice of type λ, the number of elements of type μ and cotype ν is $g^{\lambda}_{\mu\nu}(q)$. When $|\mu| + |\nu| \neq |\lambda|$, the polynomial is 0. Otherwise, the leading coefficient is $c^{\lambda}_{\mu\nu}$ and the degree is $n(\lambda) - n(\mu) - n(\nu)$.*

These polynomials are called **Hall polynomials**.

Hall proved the existence of polynomials $g^{\lambda}_{\mu\nu}(p)$ such that in the lattice of subgroups of an abelian p-group of type λ, the number of subgroups of type μ and cotype ν is $g^{\lambda}_{\mu\nu}(p)$; a summary of his theory is in [**9**]. He determined the degree and leading coefficient of the polynomials. Klein [**14**] was the first to explicitly determine these polynomials for all parameters, and did so in the wider context of p-modules. Macdonald [**16**, §II.4] later determined a different (but of course equal) formula, in the context of modules over discrete valuation rings, and replaced the prime p by the cardinality q of the residue field of the ring.

Macdonald's construction of these polynomials goes through to the wider context of q-regular semi-primary lattices by making minor modifications to his proofs. There are syntactic changes, since we are considering elements of a lattice instead of submodules of a module: the module M is the maximal element $\hat{1}$ of $\mathcal{L}$; the sum of modules is replaced by the join, and the intersection by the meet; and quotients of modules M/N are replaced by intervals $[N, M]$. The socle of M is replaced by $M[1] = A\hat{0}$, while $\mathfrak{p}^i N$ is replaced by $C^i N$ (where $\mathfrak{p}$ is the maximal ideal of the ring). There are further semantic changes required because submodules can be viewed as collections of vectors, while elements of a semi-primary lattice cannot, because not all semi-primary lattices have a submodule lattice representation. Nonetheless, semi-primary lattices have sufficient structure to carry out the vector dependent portions of the proof. This problem arises in his Proposition (3.4) which we redo below as Theorem 4.83 in the context of semi-primary lattices, and in his Proposition (4.4), which requires selection of equal length sequences of linearly independent vectors in two elementary quotients in M. We cannot do this, because in our wider context, the elements of an arbitrary semi-primary lattice do not further decompose into collections of vectors. He counts the number of ways of choosing each sequence, and takes the ratio. We circumvent this by choosing sequences of independent atoms in these elementary intervals, which diminishes the number of choices of each term in each sequence by a factor $q - 1$. Since the number of terms in each sequence is the same, these diminishing factors cancel when the ratio is taken.

Maley [**18**] showed that $g^{\lambda}_{\mu\nu}(p)$ is a polynomial in $(p-1)$ with nonnegative integer coefficients. His formula, though still long, is much simpler than those of Klein and Macdonald. In his proof, the number of subgroups satisfying certain conditions is determined by a canonical construction of matrices whose row spans are the subgroups in question. This dependence on the coordinatization of groups is critical to the relative simplicity of his formula and proof, but would require extensive changes to be applied directly to semi-primary lattices. However, since Maley and Macdonald's polynomials are equal, and Macdonald's proof goes through to semi-primary lattices, Maley's formula may also be used for q-regular semi-primary lattices. Refer to Klein, Macdonald, or Maley for explicit formulas for Hall polynomials.

Definition 4.82. Let y be an element of a semi-primary lattice. Define

$$\begin{aligned}\mathrm{ALR}(y) &\stackrel{\text{def}}{=} ((\operatorname{type} A^i y)')_{i\geq 0}\\ \mathrm{CLR}(y) &\stackrel{\text{def}}{=} ((\operatorname{cotype} C^i y)')_{i\geq 0}\end{aligned}$$

More generally, if $[x,z]$ is an interval and y is in this interval, define

$$\begin{aligned}\mathrm{ALR}_{[x,z]}(y) &\stackrel{\text{def}}{=} ((\operatorname{type}[x, A_z^i y])')_{i\geq 0}\\ \mathrm{CLR}_{[x,z]}(y) &\stackrel{\text{def}}{=} ((\operatorname{type}[C_x^i y, z])')_{i\geq 0}\end{aligned}$$

Green [**8**] introduced the LR-sequence of an element (our $\mathrm{CLR}(y)$) as a means of determining $g^{\lambda}_{\mu\nu}(p)$ in the context of p-modules. He shows that for any LR-sequence S of type $(\mu',\nu';\lambda')$, the number of elements y with $\mathrm{CLR}(y) = S$ is a monic polynomial in q with degree $n(\lambda)-n(\mu)-n(\nu)$. Summing over all such LR-sequences, we obtain that $g^{\lambda}_{\mu\nu}(q)$ is a polynomial of degree $n(\lambda)-n(\mu)-n(\nu)$ with leading coefficient $c^{\lambda'}_{\mu',\nu'} = c^{\lambda}_{\mu\nu}$.

The following theorem was originally done for our $\mathrm{CLR}(y)$ by Green in the context of p-modules, and later Macdonald [**16**, p. 91] in the context of modules over discrete valuation rings. It illustrates how to recast proofs in semi-primary lattice notation, and has additional minor differences that are discussed afterwards.

Theorem 4.83. *Let $\mathcal{L}$ be a semi-primary lattice of type λ, and y be an element of type μ and cotype ν. Then $\mathrm{ALR}(y)$ is an LR-sequence of type $(\mu',\nu';\lambda')$, and $\mathrm{CLR}(y)$ is an LR-sequence of type $(\nu',\mu';\lambda')$.*

Let $x \leq y \leq z$. Let $\lambda = \operatorname{type}[x,z]$, $\mu = \operatorname{type}[x,y]$, and $\nu = \operatorname{type}[y,z]$. Then $\mathrm{ALR}_{[x,z]}(y)$ is an LR-sequence of type $(\mu',\nu';\lambda')$ and $\mathrm{CLR}_{[x,z]}(y)$ is an LR-sequence of type $(\nu',\mu';\lambda')$.

Proof. It suffices to prove this for $\mathrm{ALR}(y)$; it then holds for $\mathrm{CLR}(y)$ by applying the theorem to the dual lattice, and for the interval $[x,z]$ by applying the theorem to this sublattice. We verify that the three conditions in the definition of an LR-sequence hold. Let $\lambda^{(i)} = (\operatorname{type} A^i y)'$.

(1) $(\operatorname{type} A^0 y)' = (\operatorname{type} y)' = \mu'$, and when r is sufficiently large, $A^r y = \hat{1}$ so $(\operatorname{type} A^r y)' = \lambda'$.
(2) The number of cells in $\lambda^{(i)}/\lambda^{(i-1)}$ is $\rho(A^i y) - \rho(A^{i-1}y) = \nu'_i$. The interval $[A^{i-1}y, A^i y]$ is elementary, so $\operatorname{type} A^i y / \operatorname{type} A^{i-1} y$ is a vertical strip by Proposition 4.67, whence its transpose $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip.
(3) The reverse row word of a tableau $\boldsymbol{P}$ is a ballot sequence iff the number of i's in the first k rows is at least the number of $(i+1)$'s in the first $k+1$ rows, for all $i,k \in \mathbb{P}$. This is because if there is any initial segment of the reverse row word of $\boldsymbol{P}$ that has more $(i+1)$'s than i's for some i, we can adjust the initial sequence to be cut off after all the $(i+1)$'s in the last row partially contained in the initial sequence, and since this weakly increases the number of $(i+1)$'s and weakly decreases the number of i's, we still have a violation.

For $\boldsymbol{P} = \mathrm{ALR}(y)$, the number of entries at most i in the first k rows is

$$\left|(\operatorname{type} A^i y)[k]\right| = \rho((A^i y)[k]) = \rho(A^i y \wedge A^k \hat{0}).$$

Thus, the number of i's in the first k rows is the length of the interval $[A^{i-1}y \wedge A^k\hat{0}, A^i y \wedge A^k\hat{0}]$. Similarly, the number of $(i+1)$'s in the first $k+1$ rows is the length of the interval $[A^i y \wedge A^{k+1}\hat{0}, A^{i+1}y \wedge A^{k+1}\hat{0}]$. The latter interval is obtained from the former one by applying A, so the latter one has smaller length by Theorem 4.15(7). $\square$

The means used in the last sentence of this proof differs from Macdonald; he gives a surjective homomorphism from the quotient module corresponding to the former interval, to the one for the latter interval, thus showing the latter interval has smaller length. We use less structure to obtain this in the wider context of semi-primary lattices. Note also that this applies to all semi-primary lattices, not just regular ones.

Macdonald's other results [**16**, §II.1] about the lattice of submodules of a module over a discrete valuation ring do not all go through to arbitrary q-regular semi-primary lattices. The construction of the dual of a module M may not apply to a semi-primary lattice not derived from a submodule lattice, and in fact, semi-primary lattices do not have to be self-dual. He computes the number of automorphisms of a module through an analytic argument, which doesn't apply if the lattice is not a submodule lattice.

4.10. The Littlewood-Richardson sequences of adjacent elements. We will compare the tableau $\mathrm{ALR}_{[x,z]}(y)$ (or dually, $\mathrm{CLR}_{[x,z]}(y)$) to the corresponding tableau obtained by increasing one of x, y, or z by a single cover relation. The tableaux will almost be the same, except along an insertion/deletion path.

Given a cell $c \in \mathbb{P} \times \mathbb{P}$, and a skew tableau $\boldsymbol{A}$ of shape λ/μ, the entry in cell c of $\boldsymbol{A}$ is denoted $\mathrm{entry}(\boldsymbol{A}, c)$. An entry $\bullet$ has numerical value $\mathrm{low}(\boldsymbol{A})$ and denotes cells in μ, while o denotes a cell outside λ, and is considered to be a greater value than all numbers in the tableau.

Definition 4.84. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be skew tableaux, and let $c_1, \ldots, c_N$ be cells in $\mathbb{P} \times \mathbb{P}$ such that the entries in $\boldsymbol{A}$ and $\boldsymbol{B}$ agree in all cells except the c_k's. The tableau $\boldsymbol{B}$ is obtained by **inserting** e into $\boldsymbol{A}$ on the **insertion path** $c_1, \ldots, c_N$, and that $\boldsymbol{A}$ is obtained by **deleting** e from $\boldsymbol{B}$ on the **deletion path** $c_N, \ldots, c_1$, when the entries in $c_1, \ldots, c_N$ are $e_1, \ldots, e_{N-1}, o$ in $\boldsymbol{A}$ and $e, e_1, \ldots, e_{N-1}$ in $\boldsymbol{B}$, where e_k are some entries. The insertion path and deletion path have **length** N.

The tableau $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by inserting e and deleting d on the path $c_1, \ldots, c_N$ if the consecutive entries in these cells are $e_1, \ldots, e_{N-1}, d$ in $\boldsymbol{A}$, and are $e, e_1, \ldots, e_{N-1}$ in $\boldsymbol{B}$. Equivalently, $\boldsymbol{A}$ is obtained from $\boldsymbol{B}$ by inserting d and deleting e on the path $c_N, \ldots, c_1$. The insertion/deletion path has **length** $N+1$.

These paths may be represented by diagrams such as the following ones. The two tableaux shown are obtained from each other by means of the indicated paths. A squiggly arrow $\rightsquigarrow$ indicates that an entry moves into or out of the tableau, and a solid arrow indicates that an entry moves into a cell, **bumping** the entry there out to another cell or out of the tableau (according to further arrows). The number of arrows, both solid and squiggly, is the length of the path. For example, these two