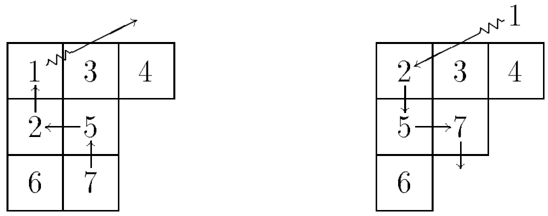
tableaux are obtained from each other by the indicated paths of length 4.



deletion path insertion path

Many of the insertion/deletion paths we encounter will have a similar form.

**Proposition 4.85.** *Let $\boldsymbol{A} = (\gamma^{(i)}, \ldots, \gamma^{(j)})$ and $\boldsymbol{B} = (\mu^{(i)}, \ldots, \mu^{(j)})$ be skew tableaux with $\gamma^{(k)} < \mu^{(k)}$ for $e \leq k < d$ (for some $d \leq j$) and $\gamma^{(k)} = \mu^{(k)}$ otherwise. Let $c_k = \mu^{(k)}/\gamma^{(k)}$ be a single cell for $k = e, \ldots, d-1$. Some consecutive $c_k$'s may be equal; select the subsequence of $e, \ldots, j$ consisting of the first $k$ in each run of equal $c_k$'s, by defining the sequence $e = k_0 < k_1 < \ldots < k_N$ so that $c_{k_m} = c_{k_m+1} = \cdots = c_{k_{m+1}-1} \neq c_{k_{m+1}}$ and $c_{k_N} = c_{d-1}$. Then $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by inserting $e$ and deleting $d$ on the path $c_{k_0}, \ldots, c_{k_N}$.*

*If we set $d = j+1$ above, so that $e \leq i \leq j$ and $k = e, \ldots, j$, then $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by inserting $e$ along the insertion path $c_{k_0}, \ldots, c_{k_N}$, and $\boldsymbol{A}$ is obtained from $\boldsymbol{B}$ by deleting $e$ from the deletion path $c_{k_N}, \ldots, c_{k_0}$.*

*In both cases, if $c_{k-1}$ and $c_k$ are distinct cells on the same row, then all cells between them contain $k$ in both $\boldsymbol{A}$ and $\boldsymbol{B}$.*

*Proof.* The entries in $\boldsymbol{A}$ and $\boldsymbol{B}$ are the same in all cells except the $c_k$'s, because the $k$'s in $\boldsymbol{A}$ are in cells $\gamma^{(k)}/\gamma^{(k-1)}$, and are in $\boldsymbol{B}$ in cells $\mu^{(k)}/\mu^{(k-1)}$, and these differ only in $c_k$ and $c_{k-1}$. Consecutive $c_k$'s may be equal, but if $c_{k-1} \neq c_k$ then $c_{k-1}$ is in both $\boldsymbol{A}$ and $\boldsymbol{B}$, so no later cell equals $c_{k-1}$.

The value of entry$(\boldsymbol{B}, c)$ is the smallest $k$ for which $c$ is in $\mu^{(k)}$, and similarly for $\boldsymbol{A}$, so entry$(\boldsymbol{B}, c_{k_m}) = k_m$ for $m = 0, \ldots, r$, and entry$(\boldsymbol{A}, c_{k_m}) = k_{m+1}$ for $m = 0, \ldots, r-1$. Since $c_N$ isn't in $\gamma^{(d-1)}$, it follows that entry$(\boldsymbol{A}, c_N)$ is $d$ when $d \leq j$ and $o$ when $d = j+1$.

If $c_{k-1}$ and $c_k$ are distinct cells on the same row, the cells between them all appear in $\gamma^{(k)}$ and $\mu^{(k)}$ and none appear in $\mu^{(k-1)}$ or $\gamma^{(k-1)}$, so all contain $k$ in both $\boldsymbol{A}$ and $\boldsymbol{B}$. $\square$

We will examine how $\mathrm{ALR}_{[x,z]}(y)$ changes in three situations with one of $x, y, z$ varying to an element covering it, and the other two staying constant:

(x) $x < \tilde{x} \leq y \leq z$;
(y) $x \leq y < \tilde{y} \leq z$;
(z) $x \leq y \leq z < \tilde{z}$.

These three cases and the following definitions apply to the following lemma and theorem. Let

$$\begin{aligned} w_k = w_k(x,y,z) &= A^k y \wedge z \\ w_{kr} = w_{kr}(x,y,z) &= A^r x \wedge A^k y \wedge z \end{aligned}$$

(so $(\mathrm{type}[x, A^k_z y])'_r = \rho(w_{kr}) - \rho(w_{k,r-1})$)

$$\begin{aligned} \boldsymbol{P} &= \mathrm{ALR}_{[x,z]}(y) \\ \mu &= \mathrm{type}[x,y] \\ \nu &= \mathrm{type}[y,z] \\ \lambda &= \mathrm{type}[x,z] \end{aligned}$$

Define $\tilde{w}_k$, $\tilde{w}_{kr}$, $\widetilde{\boldsymbol{P}}$, $\tilde{\mu}$, $\tilde{\nu}$, and $\tilde{\lambda}$ similarly by replacing $x$ with $\tilde{x}$ in case (x); $y$ with $\tilde{y}$ in case (y); and $z$ with $\tilde{z}$ in case (z).

We always have $w_{kr} \leq \tilde{w}_{kr}$ because $A$ and $\wedge$ both preserve $\leq$, so we are interested in determining for which $k, r$ these are equal and for which they are not. Let $S = \{(k,r) : w_{kr} = \tilde{w}_{kr}\}$.

**Lemma 4.86.** *If $(k,r) \in S$, then $(k',r') \in S$, where $k' \geq 0$ and $r' \geq 0$ depend on the case:*

(x) *$r' \geq r$ and $k' - k \leq r' - r$; so $(k+1, r+1) \in S$ and $(k', r) \in S$ for $k' \leq k$.*
(y) *$k' \geq k$ and $r' - r \leq k' - k$; so $(k+1, r+1) \in S$ and $(k, r') \in S$ for $r' \leq r$.*
(z) *$k' \leq k$ and $r' \leq r$.*

*Proof.*

(x) Given $(k,r) \in S$, we have $w_{kr} = \tilde{w}_{kr}$, or
$$A^r x \wedge A^k y \wedge z = A^r \tilde{x} \wedge A^k y \wedge z.$$
For $r' \geq r$, applying $A_z^{r'-r}$ to both sides of the equality preserves the equality and adds $r' - r$ to both $k$ and $r$, yielding $w_{k+r'-r,r'} = \tilde{w}_{k+r'-r,r'}$. Next, for $k' \leq k + r' - r$, taking the meet of each side with $A^{k'} y$ preserves the equality and changes $k + r' - r$ to $k'$, so $w_{k',r'} = \tilde{w}_{k',r'}$.
(y) This is similar to (x), but with various variables swapped.
(z) Given $(k,r) \in S$, if $0 \leq k' \leq k$ and $0 \leq r' \leq r$ then $A^{k'} y \leq A^k y$ and $A^{r'} x \leq A^r x$, so
$$w_{kr} \wedge A^{k'} y \wedge A^{r'} x = (A^r x \wedge A^{r'} x) \wedge (A^k y \wedge A^{k'} y) \wedge z = A^{r'} x \wedge A^{k'} y \wedge z = w_{k',r'},$$
and similarly, $\tilde{w}_{kr} \wedge A^{k'} y \wedge A^{r'} x = \tilde{w}_{k',r'}$. $\square$

**Theorem 4.87.** *We obtain $\widetilde{\boldsymbol{P}}$ from $\boldsymbol{P}$ by insertion or deletion paths as follows; see Figure 2.*

(x) *Delete $\bullet$ from $\boldsymbol{P}$ on a path $C_N, \ldots, C_0$, where*
  (1) *$\bullet = \mathrm{entry}(\boldsymbol{P}, C_0) < \cdots < \mathrm{entry}(\boldsymbol{P}, C_N)$;*
  (2) *$\mathrm{row}(C_0) \leq \cdots \leq \mathrm{row}(C_N)$;*
  (3) *If $\mathrm{row}(C_i) < \mathrm{row}(C_{i+1})$ then $\mathrm{col}(C_i) \geq \mathrm{col}(C_{i+1})$, while if $\mathrm{row}(C_i) = \mathrm{row}(C_{i+1})$ then $\mathrm{col}(C_i) < \mathrm{col}(C_{i+1})$.*
(y) *Insert $\bullet$ and delete $e = \mathrm{col}(\nu/\tilde{\nu})$ from a vertical strip $C_0, C_1, \ldots, C_{e-1}$ in $\boldsymbol{P}$, where*
  (1) *$\mathrm{entry}(\boldsymbol{P}, C_i) = i + 1$;*
  (2) *$\mathrm{row}(C_0) < \cdots < \mathrm{row}(C_{e-1})$;*
  (3) *$\mathrm{col}(C_0) \geq \cdots \geq \mathrm{col}(C_{e-1})$.*
(z) *Insert $e = \mathrm{col}(\tilde{\nu}/\nu)$ into a "horizontal strip" $C_0, \ldots, C_N$ of $\boldsymbol{P}$, where*
  (1) *$e < \mathrm{entry}(\boldsymbol{P}, C_0) < \cdots < \mathrm{entry}(\boldsymbol{P}, C_N) = o$;*
  (2) *$\mathrm{col}(C_0) < \cdots < \mathrm{col}(C_e)$.*

(3) $\mathrm{row}(C_0) \geq \cdots \geq \mathrm{row}(C_e)$;

*Proof.*

(x) We have $\tilde\mu \lessdot \mu$, $\tilde\nu = \nu$, $\tilde\lambda \lessdot \lambda$, and $w_k = \tilde w_k$ for all $k$. Thus $\mathrm{type}[x, w_k] \gtrdot \mathrm{type}[\tilde x, \tilde w_k]$, so $c_k = \mathrm{type}[x, w_k]'/\,\mathrm{type}[\tilde x, w_k]'$ is a single cell of $\lambda'$. By Proposition 4.85, $\widetilde{\boldsymbol{P}}$ is obtained from $\boldsymbol{P}$ by deleting $\bullet$ from a path consisting of the distinct cells among $\dots, c_1, c_0$.

Let $r_k = \mathrm{row}(c_k) = \min\{\, r \geq 0 \,:\, w_{kr} = \tilde w_{kr} \,\}$, so $(k, r) \in S$ iff $r \geq r_k$. First we show that $r_{k+1} \geq r_k$, or equivalently, $r_k + 1 \geq r_{k+1} \geq r_k$. By Lemma 4.86, $(k+1, r_k+1) \in S$ because $(k, r_k) \in S$. Thus $r_k + 1 \geq r_{k+1}$. Conversely, by Lemma 4.86, $(k, r_{k+1}) \in S$ because $(k+1, r_{k+1}) \in S$. Thus $r_{k+1} \geq r_k$.

If $c_k$ and $c_{k+1}$ are on the same row, $c_{k+1}$ is weakly right of $c_k$. Suppose they are on consecutive rows $r, r+1$. Apply $A_z$ to the interval $[w_{r-1,k}, w_{r,k}]$ to obtain the interval $[w_{r,k+1}, w_{r+1,k+1}]$ of weakly smaller length. The lengths are $\mathrm{col}(c_k) \geq \mathrm{col}(c_{k+1})$.

(y) We have $\tilde\mu \gtrdot \mu$, $\tilde\nu \lessdot \tilde\nu$, and $\tilde\lambda = \lambda$. For each $0 \leq k < e$, we have $w_k \lessdot \tilde w_k$, while for $k \geq e$, we have $w_k = \tilde w_k$. So $c_k = \mathrm{type}[x, \tilde w_k]'/\,\mathrm{type}[x, w_k]'$ is a single cell for $k = 0, \dots, e-1$. Since $w_k \lessdot \tilde w_k$ for $k < e$, we have $w_{k+1} = A_z w_k \geq \tilde w_k$, so the cells $c_k$ are distinct. By Proposition 4.85, $\widetilde{\boldsymbol{P}}$ is obtained from $\boldsymbol{P}$ by inserting $\bullet$ and deleting $e$ on the path $c_0, \dots, c_{e-1}$. Cell $c_k$ contains $k+1$ in $\boldsymbol{P}$ and $k$ in $\widetilde{\boldsymbol{P}}$.

Let $r_k = \mathrm{row}(c_k) = \min\{\, r \geq 0 \,:\, w_{kr} \lessdot \tilde w_{kr} \,\} > 0$. Since $(k, r_k - 1) \in S$, we have $(k+1, r_k) \in S$ by Lemma 4.86, so that $r_{k+1} > r_k$. Thus the rows of the successive cells strictly increase. Since $c_k$ is an outer corner of $\mathrm{type}[x, \tilde w_k]'$, and $\mathrm{type}[x, \tilde w_{k+1}]'/\,\mathrm{type}[x, \tilde w_k]'$ is a horizontal strip, all cells on higher numbered rows of this strip are in columns weakly smaller than the column of $c_k$.

(z) We have $\tilde\mu = \mu$, $\tilde\nu \lessdot \tilde\nu$, and $\tilde\lambda \lessdot \lambda$. For each $k < e$, we have $w_k = \tilde w_k$, while for $k \geq e$, we have $w_k \lessdot \tilde w_k$. Let $c_k = \mathrm{type}[x, \tilde w_k]'/\,\mathrm{type}[x, w_k]'$ for $k \geq e$. The cells $c_k$ need not be distinct. By Proposition 4.85, $\widetilde{\boldsymbol{P}}$ is obtained from $\boldsymbol{P}$ by inserting $e$ along a path consisting of the distinct cells among the $c_k$'s.

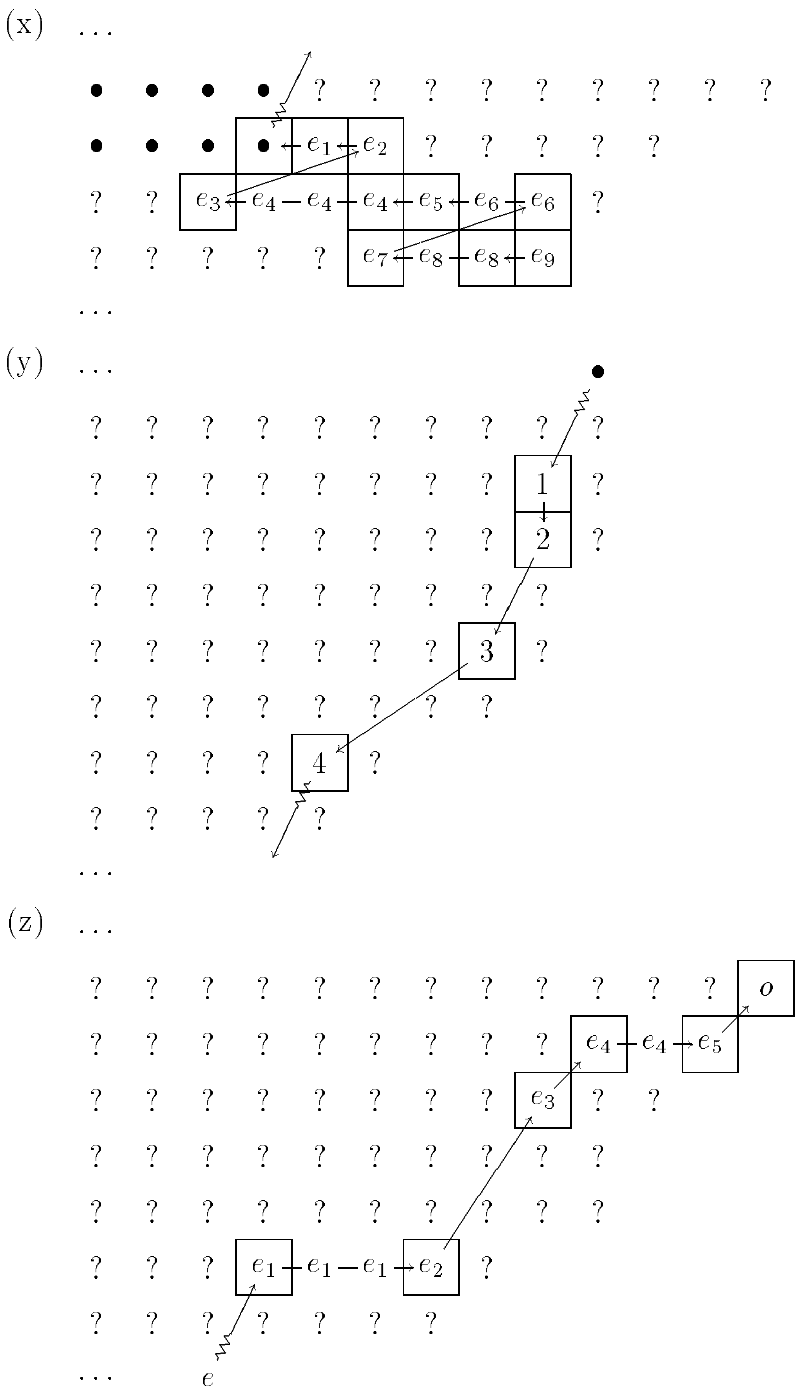
Let $r_k = \mathrm{row}(c_k) = \min\{\, r \geq 0 \,:\, w_{kr} \lessdot \tilde w_{kr} \,\} > 0$. So $(k+1, r_{k+1} - 1) \in S$, whence $(k, r_{k+1} - 1) \in S$ by Lemma 4.86. Thus $r_k > r_{k+1} - 1$, or equivalently, $r_k \geq r_{k+1}$, so the row numbers weakly decrease. □

### 4.11. Applications of Littlewood-Richardson sequences.

**Definition 4.88.** The **straightening** of a weak composition $\lambda$ is the partition $S(\lambda)$ obtained from it by sorting the parts into decreasing order. The **straightening** of a tableau of weak composition shape whose rows and columns have weakly increasing entries (ignoring the holes) is the Young tableau whose columns have the same entries in the same order as in the corresponding columns of $\boldsymbol{P}$, by pushing the entries up into the holes left by short rows. In terms of the chain of weak compositions $\boldsymbol{P} = (\lambda^{(l)} \leq \lambda^{(l+1)} \leq \cdots \leq \lambda^{(h)})$, this is

$$S(\boldsymbol{P}) \stackrel{\text{def}}{=} \left( S(\lambda^{(l)}), S(\lambda^{(l+1)}), \dots, S(\lambda^{(h)}) \right).$$

**Definition 4.89.** Let $\mu \leq \lambda$ be partitions. We define two LR-sequences in tableaux of shape $\lambda'/\mu'$:




Apply the insertion/deletion path to $\boldsymbol{P}$ (shown) to obtain $\widetilde{\boldsymbol{P}}$. The cells marked ? have entries that do not change. The boxed cells $C_k$ have entries $e_k$ in (x) and (z), and $k$ in (y). When an entry moves horizontally, all the cells it passes across must contain that entry, since the entry weakly bounds them above and below.

FIGURE 2. Insertion/deletion paths for LR-sequences of adjacent elements.

(1)
$$\mathrm{LR}_*(\mu';\lambda') \overset{\text{def}}{=} (\mu', (\mathcal{A}_\lambda \mu)', (\mathcal{A}_\lambda^2 \mu)', \ldots)$$
is the LR-sequence of type $(\mu', S(\lambda-\mu)'; \lambda')$ obtained by filling column $j$ of $\lambda'/\mu'$ with entries $1, 2, \ldots, \lambda_j - \mu_j$ from top to bottom, for each $j$.

(2) $\mathrm{LR}^*(\mu';\lambda')$ is the LR-sequence of type $(\mu', S(\lambda'-\mu'); \lambda')$ obtained as follows. Start with the shape $\lambda'/\mu'$. Place 1 in the upper rightmost empty cell, 2 in the upper rightmost empty cell strictly below that, 3 in the upper rightmost empty cell strictly below that, and so on, until there are no lower empty cells. This fills the strip $\lambda'/\mathcal{C}_{\mu'}\lambda'$ with entries $1, 2, \ldots$ from top to bottom. In the cells remaining, do this again: place 1 in the upper rightmost empty cell, 2 in the upper rightmost empty cell strictly below that, and so on, thus filling the strip $\mathcal{C}_{\mu'}\lambda'/\mathcal{C}^2_{\mu'}\lambda'$ with entries $1, 2, \ldots$ from top to bottom. Continue in this fashion until the whole tableau is filled.

These are both LR-sequences because they are semistandard tableau with an injection from entries $k > 1$ to entries $k-1$ on higher rows.

**Example 4.90.** Let $\mu = (5, 4, 4, 2)$ and $\lambda = (8, 8, 6, 5, 3)$. Then

$$\mathrm{LR}_*(\mu';\lambda') = \begin{array}{lllll} \bullet & \bullet & \bullet & \bullet & 1 \\ \bullet & \bullet & \bullet & \bullet & 2 \\ \bullet & \bullet & \bullet & 1 & 3 \\ \bullet & \bullet & \bullet & 2 & \\ \bullet & 1 & 1 & 3 & \\ 1 & 2 & 2 & & \\ 2 & 3 & & & \\ 3 & 4 & & & \end{array} \qquad \mathrm{LR}^*(\mu';\lambda') = \begin{array}{lllll} \bullet & \bullet & \bullet & \bullet & 1 \\ \bullet & \bullet & \bullet & \bullet & 2 \\ \bullet & \bullet & \bullet & 1 & 3 \\ \bullet & \bullet & \bullet & 4 & \\ \bullet & 1 & 2 & 5 & \\ 2 & 3 & 6 & & \\ 4 & 7 & & & \\ 5 & 8 & & & \end{array}$$

have respective contents $S(\lambda-\mu)' = S(3, 4, 2, 3, 3)' = (4, 3, 3, 3, 2)' = (5, 5, 4, 1)$ and $S(\lambda'-\mu') = S(1, 1, 2, 1, 1, 3, 3, 2, 2) = (3, 3, 2, 2, 2, 1, 1, 1, 1)$.

**Definition 4.91.** The **content** of a tableau is the weak composition $\nu$ such that there are $\nu_i$ $i$'s in the tableau for each $i$.

**Theorem 4.92.** *Let $\mu \le \lambda$ be partitions. In $q$-regular semi-primary lattices, for sufficiently large $q$, the least frequently occuring value of $\mathrm{ALR}_{[\hat{0},y]}(x)$ for pairs in the set*
$$\{\, (x, y) : x \le y \text{ and } \operatorname{type} x = \mu \text{ and } \operatorname{type} y = \lambda \,\}$$
*is $\mathrm{LR}_*(\mu';\lambda')$, and the most frequently occurring value is $\mathrm{LR}^*(\mu';\lambda')$.*

*Proof.* Fix $y$ of type $\lambda$. The number of $x \le y$ of type $\mu$ and cotype $\nu$ is $g^\lambda_{\mu\nu}(q)$, a polynomial in $q$ of degree $n(\lambda) - n(\mu) - n(\nu)$ with leading coefficient $c^\lambda_{\mu\nu}$. To determine the least frequently occurring LR-sequence, we seek $\nu$ with $c^\lambda_{\mu\nu} \ne 0$ that minimizes $n(\lambda) - n(\mu) - n(\nu)$, or equivalently, maximizes $n(\nu)$; the most frequently occurring LR-sequence is similarly obtained by minimizing $n(\nu)$.

Let $\boldsymbol{Q}$ be an LR-sequence of shape $\lambda'/\mu'$ and content $\nu'$. Clearly, the entry in each cell is weakly bounded below by that in $\mathrm{LR}_*(\mu';\lambda')$ and above by that in $\mathrm{LR}^*(\mu';\lambda')$. Since $\mathrm{LR}_*(\mu';\lambda')$ is obtained from $\boldsymbol{Q}$ by weakly lowering the entry in each cell, the content is obtained from the partition $\nu'$ by moving squares up to higher rows, which increases $n(\nu)$. Similarly, $\mathrm{LR}^*(\mu';\lambda')$ is obtained from $\boldsymbol{Q}$ by weakly raising the entry in each cell, thus altering the content $\nu'$ by moving squares down to lower rows, decreasing $n(\nu)$. The extremes are uniquely obtained by $\mathrm{LR}_*(\mu';\lambda')$ and $\mathrm{LR}^*(\mu';\lambda')$. $\square$

A useful application of one of these LR-sequences is the following generalization of part of Theorem 4.76.

**Theorem 4.93.** *Let $f = (f_l, \dots, f_h)$ be a saturated flag. Suppose* $\mathrm{ftype}_{\hat{0}} f = \boldsymbol{A} + \boldsymbol{B}$, *where $\boldsymbol{A}$ is a standard tableau on entries $l+1, \dots k$ and $\boldsymbol{B}$ is a row and column strict composition tableau on distinct entries $k+1, \dots h$. Then* $\mathrm{ALR}_{[\hat{0}, f_m]}(f_k) = \mathrm{LR}_*((\mathrm{type}\, f_k)'; (\mathrm{type}\, f_m)')$ *for each $m = k, \dots, h$. Thus,* $\mathrm{ftype}\, \partial^{k-l} f = S(\boldsymbol{B})$.

*Proof.* The ALR sequence is as specified for $m = k$.

Let $\boldsymbol{P} = \mathrm{ALR}_{[\hat{0}, f_{m-1}]}(f_k)$, and assume this equals $\mathrm{LR}_*(\mu';\lambda')$ where $(\mu';\lambda') = ((\mathrm{type}\, f_k)'; (\mathrm{type}\, f_{m-1})')$. We want to find $\widetilde{\boldsymbol{P}} = \mathrm{ALR}_{[\hat{0}, f_m]}(f_k)$, of inner shape $\mu'$ and outer shape $\tilde{\lambda}' = (\mathrm{type}\, f_m)'$. The outer shapes of $\widetilde{\boldsymbol{P}}$ and $\boldsymbol{P}$ differ in some cell $c$ in column $k$, where $k = \mathrm{row}(\boldsymbol{A} + \boldsymbol{B}, m) = \mathrm{row}(\boldsymbol{B}, m)$. By Theorem 4.87(z), we obtain $\widetilde{\boldsymbol{P}}$ from $\boldsymbol{P}$ by inserting a number on a horizontal strip terminating in $c$. If there are cells besides $c$ in the insertion path, their entries in $\boldsymbol{P}$ are replaced by smaller entries in $\widetilde{\boldsymbol{P}}$. Since the columns of $\boldsymbol{P}$ consist of consecutive numbers, an entry cannot be replaced by a smaller one without violating column strictness. Thus, only $c$ is on the insertion path, so $\widetilde{\boldsymbol{P}}$ is obtained from $\boldsymbol{P}$ by placing an entry at $c$ and leaving all other cells intact.

Now $m$ occurs in $\boldsymbol{B}$ in cell $(k, e)$ where $e = \lambda_k - \mu_k + 1$. All rows of $\boldsymbol{B}$ below this cell have fewer than $e$ entries less than $m$, because $\boldsymbol{B}$ is column strict. The entry in the cell above $c$ in $\boldsymbol{P}$ (and thus $\widetilde{\boldsymbol{P}}$) is $e-1$ (or $c$ is at the top of the column if $\lambda_k = \mu_k = 0$), so the minimum entry possible in $c$ in $\widetilde{\boldsymbol{P}}$ is $e$, and any occurrences of $e$ in $\boldsymbol{P}$ and $\widetilde{\boldsymbol{P}}$ on rows above $c$ must be in columns right of $c$. The number of occurrences of $e$ in $\boldsymbol{P}$ in columns right of $c$ is $\#\{\, r > k : \lambda_r - \mu_r \geq e \,\}$. This equals the number of rows of $\boldsymbol{B}$ below the cell with $m$ with at least $e$ entries less than $m$; it has already been shown that there are no such rows. Thus, we cannot place a number larger than $e$ at $c$ in $\widetilde{\boldsymbol{P}}$ because the reverse row word of $\widetilde{\boldsymbol{P}}$ would not be a ballot sequence. Since $e$ is both a lower and upper bound for $\mathrm{entry}(\widetilde{\boldsymbol{P}}, c)$, this entry equals $e$.

Since $\widetilde{\boldsymbol{P}}$ is obtained from $\boldsymbol{P}$ by placing $e$ in cell $c$, it is $\mathrm{LR}_*(\mu';\tilde{\lambda}')$.

We have $\mathrm{type}[f_k, f_m] = S(\mathrm{type}\, f_m - \mathrm{type}\, f_k)$ for $m = k, \dots, h$, because $\mathrm{ALR}_{[\hat{0}, f_m]}(f_k)$ is an LR-sequence of type $((\mathrm{type}\, f_k)', S(\mathrm{type}\, f_m - \mathrm{type}\, f_k)'; (\mathrm{type}\, f_m)')$. The partition $S(\mathrm{type}\, f_m - \mathrm{type}\, f_k)$ is computed by taking the shape of the subtableau of $\boldsymbol{B}$ of entries at most $m$, and sorting the rows into decreasing order. Thus,

$$\begin{aligned} \mathrm{ftype}\, \partial^{k-l} f &= \mathrm{ftype}(f_k, f_{k+1}, \dots, f_h) = \Big(\mathrm{type}[f_k, f_m]\Big)_{m=k}^{h} \\ &= \Big(S(\mathrm{type}\, f_m - \mathrm{type}\, f_k)\Big)_{m=k}^{h} = S(\boldsymbol{B}). \quad \square \end{aligned}$$

**Example 4.94.** Suppose

$$\operatorname{ftype}_{\hat{0}} f = \begin{array}{|c|c|c|c|c|}\cline{3-5} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 & 5 & 7 \\ \cline{2-5} \multicolumn{1}{c|}{} & 1 & 6 & 9 & 10 \\ \cline{1-5} 2 & 4 & 8 & \multicolumn{2}{c}{} \\ \cline{1-3} 11 & 12 & 13 & \multicolumn{2}{c}{} \\ \cline{1-3}\end{array} = \begin{array}{|c|c|c|}\cline{3-3} \multicolumn{2}{c|}{} & 3 \\ \cline{2-3} \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\cline{1-2} 5 & 7 & \multicolumn{1}{c}{} \\ \hline 6 & 9 & 10 \\ \hline 8 & \multicolumn{2}{c}{} \\ \hline 11 & 12 & 13 \\ \hline\end{array} = \boldsymbol{A} + \boldsymbol{B}.$$

For each entry of $\operatorname{ftype}_{\hat{0}} f$ that appears in $\boldsymbol{B}$, put a subscript on it indicating its column number in $\boldsymbol{B}$, and then transpose the tableau.

$$\begin{array}{|c|c|c|c|}\cline{3-4} \multicolumn{2}{c|}{} & 2 & 11_1 \\ \cline{2-4} \multicolumn{1}{c|}{} & 1 & 4 & 12_2 \\ \hline 3 & 6_1 & 8_1 & 13_3 \\ \hline 5_1 & 9_2 & \multicolumn{2}{c}{} \\ \cline{1-2} 7_2 & 10_3 & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array}$$

To find $\mathrm{ALR}_{[f_0,f_m]}(f_4)$ for $4 \le m \le 13$, take the subtableau of entries at most $m$, and replace each entry by its subscript if it has one, or $\bullet$ if it doesn't. The successive values of $\mathrm{ALR}_{[f_0,f_m]}(f_4)$ from $m = 4$ through 13 are

$$\begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & & \end{array}, \quad \begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & & \\ 1 & & \end{array}, \quad \begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & 1 & \\ 1 & & \end{array}, \quad \begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & 1 & \\ 1 & & \\ 2 & & \end{array}, \quad \begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & 1 & 1 \\ 1 & & \\ 2 & & \end{array}$$

$$\begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & 1 & 1 \\ 1 & 2 & \\ 2 & & \end{array}, \quad \begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & 1 & 1 \\ 1 & 2 & \\ 2 & 3 & \end{array}, \quad \begin{array}{llll} \bullet & \bullet & \bullet & 1 \\ \bullet & \bullet & \bullet & \\ \bullet & 1 & 1 & \\ 1 & 2 & & \\ 2 & 3 & & \end{array}, \quad \begin{array}{llll} \bullet & \bullet & \bullet & 1 \\ \bullet & \bullet & \bullet & 2 \\ \bullet & 1 & 1 & \\ 1 & 2 & & \\ 2 & 3 & & \end{array}, \quad \begin{array}{llll} \bullet & \bullet & \bullet & 1 \\ \bullet & \bullet & \bullet & 2 \\ \bullet & 1 & 1 & 3 \\ 1 & 2 & & \\ 2 & 3 & & \end{array}$$

and $\operatorname{ftype} \partial^4 f = \operatorname{ftype}(f_4, \ldots, f_{13}) = S(B) = \begin{array}{|c|c|c|}\hline 5 & 7 & 10 \\ \hline 6 & 9 & 13 \\ \hline 8 & 12 & \multicolumn{1}{c}{} \\ \cline{1-2} 11 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$. Similarly,

$$\begin{aligned}
\operatorname{ftype} \partial^2 d^8 f &= \operatorname{ftype}(f_2, \ldots, f_5) &= \begin{array}{|c|c|}\hline 3 & 5 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \\
\operatorname{ftype} \partial^6 d^5 f &= \operatorname{ftype}(f_6, f_7, f_8) &= \begin{array}{|c|}\hline 7 \\ \hline 8 \\ \hline\end{array} \\
\operatorname{ftype} \partial^9 f &= \operatorname{ftype}(f_9, \ldots, f_{13}) &= \begin{array}{|c|c|c|}\hline 9 & 10 & 13 \\ \hline 11 & 12 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \\
\operatorname{ftype} \partial^{10} f &= \operatorname{ftype}(f_{10}, \ldots, f_{13}) &= \begin{array}{|c|c|c|}\hline 10 & 12 & 13 \\ \hline 11 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \\
\operatorname{ftype} \partial^{11} f &= \operatorname{ftype}(f_{11}, f_{12}, f_{13}) &= \begin{array}{|c|c|c|}\hline 11 & 12 & 13 \\ \hline\end{array}
\end{aligned}$$

# Part III. Tableau games and flag configurations

## 5. Interval type tables in semi-primary lattices

**5.1. Schützenberger's evacuation algorithm.** Schützenberger [**25**] developed an involution ev on standard tableaux of any given shape, which Fomin [**4**, §8] has described in terms of a triangular array of partitions. We present both versions below. Also see Sagan [**22**, §3.11] for an exposition. Hesselink [**10**, §5] and van Leeuwen [**31**, Theorem 2.3.2] showed that in an invariant subspace lattice over an infinite field, in the Zariski topology, the cotype of a generic flag of type $\boldsymbol{P}$ is $\operatorname{ev}\boldsymbol{P}$. Hesselink also examines degenerate behavior. We will show an analogous genericity result for $q$-regular semi-primary lattices, and will also analyze the degenerate behavior.

**Definition 5.1.** Let $\boldsymbol{P}$ be a skew tableau whose entries are distinct numbers, and symbols $\star$, representing holes. We **slide** an entry $e$ into a hole by placing $e$ at that hole, and replacing the former cell of $e$ with $\star$.

Let $\boldsymbol{P}$ be a standard tableau of shape $\lambda \vdash n$. Define an operator $\Delta\boldsymbol{P}$ as follows.

(1) Replace the top left corner of $\boldsymbol{P}$ (which has the minimum element of $\boldsymbol{P}$) with $\star$.
(2) If $\star$ is not at an inner corner, slide the smaller or only of its right and lower neighbors into $\star$, and then repeat this step.
(3) Finally, $\star$ is at an inner corner. Delete the square with $\star$, obtaining a smaller tableau, $\Delta\boldsymbol{P}$.

We illustrate this with

$$\boldsymbol{P} = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2}\end{array}$$

First replace 1 with $\star$:

$$\begin{array}{|c|c|c|}\hline \star & \boxed{3} & 4 \\ \hline \boxed{2} & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2}\end{array}$$

The right and lower neighbors of $\star$ (shown with bold frames) contain 2 and 3, the smaller of which is 2, so we next obtain

$$\begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline \star & \boxed{5} \\ \cline{1-2} \boxed{6} & 7 \\ \cline{1-2}\end{array}$$

The right and lower neighbors of $\star$ contain 5 and 6, the smaller of which is 5, so we next obtain

$$\begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 5 & \star \\ \cline{1-2} 6 & \boxed{7} \\ \cline{1-2}\end{array}$$

Now there is no right neighbor, but there is a lower neighbor, 7.

$$\begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 5 & 7 \\ \cline{1-2} 6 & \star \\ \cline{1-2}\end{array}$$

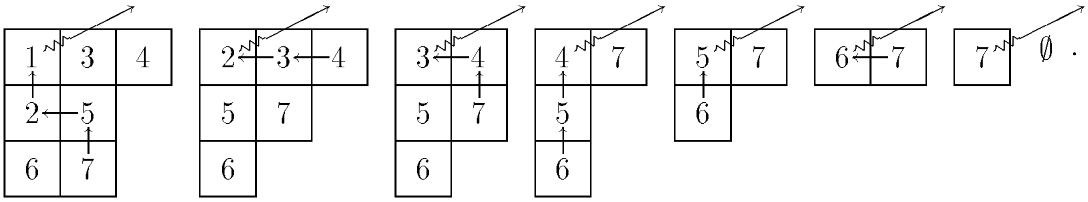
Finally there are no right or lower neighbors, so delete $\star$ to obtain

$$\Delta \boldsymbol{P} = \begin{array}{|c|c|c|} \hline 2 & 3 & 4 \\ \hline 5 & 7 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$$

Apply $\Delta$ to $\Delta \boldsymbol{P}$ to obtain

$$\Delta^2 \boldsymbol{P} = \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 5 & 7 \\ \hline 6 \\ \cline{1-1} \end{array}$$

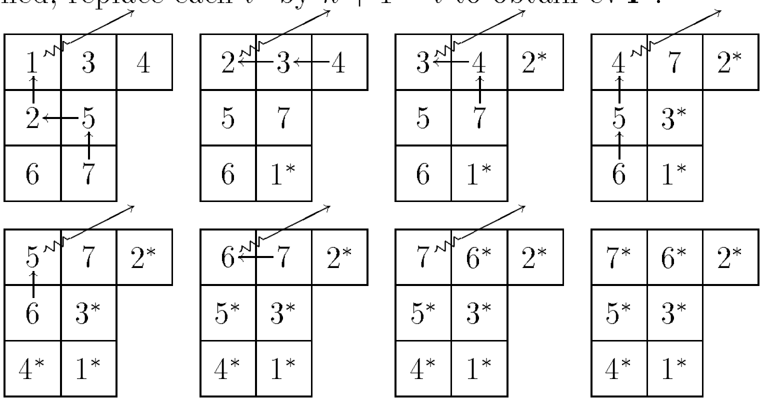
and compute $\Delta^0 \boldsymbol{P}, \Delta^1 \boldsymbol{P}, \ldots, \Delta^7 \boldsymbol{P}$:



The chain $\operatorname{sh} \Delta^n \boldsymbol{P} \lessdot \operatorname{sh} \Delta^{n-1} \boldsymbol{P} \lessdot \cdots \lessdot \operatorname{sh} \Delta^0 \boldsymbol{P}$ in Young's lattice yields a tableau $\operatorname{ev} \boldsymbol{P}$, which in this example is

$$\operatorname{ev} \boldsymbol{P} = \begin{array}{|c|c|c|} \hline 1 & 2 & 6 \\ \hline 3 & 5 \\ \cline{1-2} 4 & 7 \\ \cline{1-2} \end{array} .$$

A convenient way to compute $\operatorname{ev} \boldsymbol{P}$ is to embed all $\Delta^i \boldsymbol{P}$ in a tableau of shape $\lambda$, by placing an entry $i^*$ in the cell that's deleted when $\Delta$ deletes $i$ from the tableau. The starred entries are not part of the tableau $\Delta^i \boldsymbol{P}$, but are simply a bookkeeping device to record the history of the successive shapes. When a tableau with only starred entries is obtained, replace each $i^*$ by $n+1-i$ to obtain $\operatorname{ev} \boldsymbol{P}$.



**Theorem 5.2.** *The operator* $\operatorname{ev}$ *is an involution, that is,* $\operatorname{ev} \operatorname{ev} \boldsymbol{P} = \boldsymbol{P}$ *for any standard tableau* $\boldsymbol{P}$.

An alternate method of computing $\operatorname{ev} \boldsymbol{P}$ due to Fomin [**4**, §8] yields an easier proof of this theorem than the classical proof. We illustrate his method with the previous example. Form an upper triangular grid $\Lambda$ of partitions $\lambda^{(ij)}$, where for $0 \le i \le j \le n$, the entry $\lambda^{(ij)}$ in row $i$, column $j$ is the subshape of $\Delta^i \boldsymbol{P}$ consisting of entries $i+1, \ldots, j$; see Figure 3.

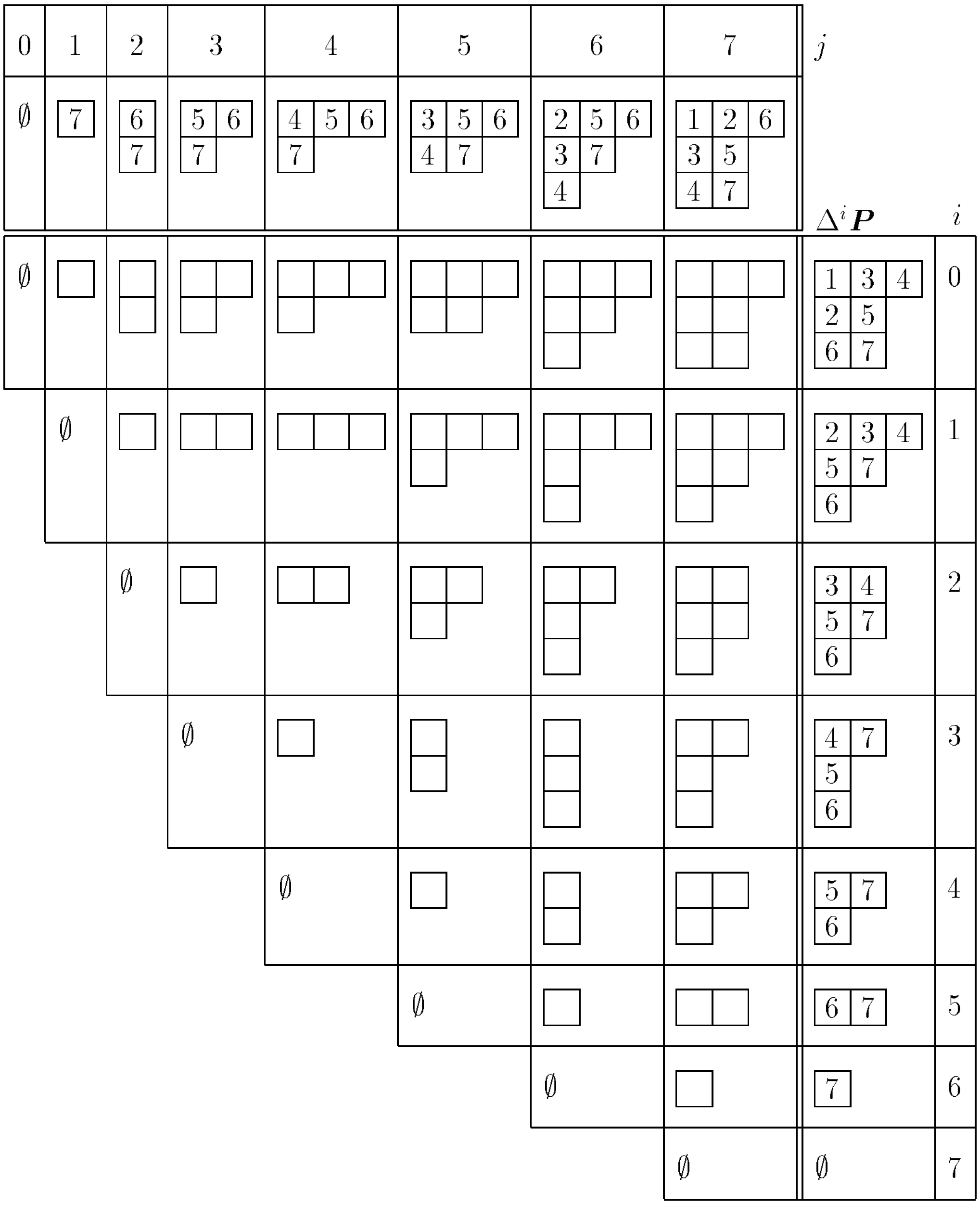



FIGURE 3. Fomin's computation of evacuation.

Not only do the rows of $\Lambda$ form saturated chains in Young's lattice, but the columns do as well; comparing the contents of the cells of $\Delta^i \boldsymbol{P}$ and $\Delta^{i+1}\boldsymbol{P}$, we see that each cell of the former stays the same, increases, or is emptied in the latter. If the contents had been at most $j$ and are emptied or increased above $j$, the cell is in $\lambda^{(ij)}$ but not $\lambda^{(i+1,j)}$, and otherwise is in both partitions. Thus, $\lambda^{(ij)} \geq \lambda^{(i+1,j)}$ in Young's lattice. Further, the number of cells in $\lambda^{(ij)}$ is $j-i$, so the columns of $\Lambda$ also form saturated chains. Notice in the figure that the chain represented by the last column is $\operatorname{ev} \boldsymbol{P}$.

Now we show how to compute this table without first computing $\Delta^i \boldsymbol{P}$ and expanding it into a chain. We are given $\boldsymbol{P}$ of shape $\lambda \vdash n$. Form an upper triangular grid $\Lambda$ with entries $\lambda^{(ij)}$, where $0 \leq i \leq j \leq n$. Set $(\lambda^{(00)}, \ldots, \lambda^{(0n)})$ to $\boldsymbol{P}$ and $\lambda^{(jj)} = \emptyset$ for $0 \leq j \leq n$. We use repeated applications of a local rule to compute the values of the remaining entries of $\Lambda$. Consider a 2 by 2 section of $\Lambda$:

| $\lambda^{(ab)}$ | $b = j-1$ | $b = j$ |
|---|---|---|
| $a = i$ | $\gamma$ | $\delta$ |
| $a = i+1$ | $\alpha$ | $\beta$ |

If we are given $\alpha, \gamma, \delta$, we want to compute $\beta$. Since the rows of the grid increase left to right, and the columns increase top to bottom, $\beta$ is between $\alpha$ and $\delta$ in Young's lattice. These differ by just two squares, in positions denoted $x$ and $y$, where $x, y \in \mathbb{P} \times \mathbb{P}$.

**Definition 5.3.** A **brick** is a skew partition consisting of two adjacent squares. A **horizontal brick** is a brick with two horizontally adjacent squares, that is, $\lambda/\mu$ where for some $j$, we have $\lambda_j = \mu_j + 2$ and $\lambda_i = \mu_i$ for $i \neq j$. A **vertical brick** is a brick with two vertically adjacent squares.

If $\delta/\alpha$ is a brick, there is only one partition between $\alpha$ and $\delta$, and both $\gamma$ and $\beta$ are equal to it. If $\delta/\alpha$ is not a brick, there are two intermediate partitions, one of which is $\gamma$; for evacuation, we choose $\beta$ to be the other one. These two situations appear as follows; the notation $\alpha \lessdot_x \beta$ means $\alpha \lessdot \beta$ and $\beta \ominus \alpha = x$.

(a) $x, y$ adjacent

$$\begin{array}{ccc} \gamma & \lessdot_y & \delta \\ \curlyvee_x & & \curlyvee_y \\ \alpha & \lessdot_x & \beta \end{array}$$

(b) $x, y$ not adjacent

$$\begin{array}{ccc} \gamma & \lessdot_y & \delta \\ \curlyvee_x & & \curlyvee_x \\ \alpha & \lessdot_y & \beta \end{array}$$

Now we compare Fomin's algorithm with the classical evacuation algorithm. Given a tableau $\boldsymbol{P}$ as the top row, produce the table $\Lambda$ by the local rules just given. Define the tableau $\boldsymbol{P}^{(ij)}$ (with $0 \leq i \leq j \leq n$) as the following chain of partitions, with the entry $e$ to fill the square added between two partitions shown between them as $\boxed{e}$:

$$\boldsymbol{P}^{(ij)} = \begin{array}{lll} & & \lambda^{(i-1,j)} \lessdot_{\boxed{j+1}} \cdots \lessdot_{\boxed{n}} \lambda^{(i-1,n)} \\ & & \curlyvee_{\boxed{\star}} \\ \lambda^{(ii)} \lessdot_{\boxed{i+1}} \lambda^{(i,i+1)} \lessdot_{\boxed{i+2}} \cdots \lessdot_{\boxed{j}} & & \lambda^{(ij)} \end{array}$$

Note that $\boldsymbol{P} = \Delta^0 \boldsymbol{P}$ is the top row of $\Lambda$. Given that $\Delta^{i-1}\boldsymbol{P}$ is the $(i-1)$th row of $\Lambda$, we want to show that $\Delta^i \boldsymbol{P}$ is the $i$th row of $\Lambda$. The tableau $\boldsymbol{P}^{(ii)}$ is $\Delta^{i-1}\boldsymbol{P}$ with the minimal entry replaced by $\star$, just as in the first step of the classical algorithm for computing $\Delta(\Delta^{i-1}\boldsymbol{P})$. The only place the chains $\boldsymbol{P}^{(i,j-1)}$ and $\boldsymbol{P}^{(ij)}$ may differ is in the 2 by 2 section of $\Lambda$,

$$\begin{array}{cc} \lambda^{(i-1,j-1)} & \lambda^{(i-1,j)} \\ \lambda^{(i,j-1)} & \lambda^{(ij)} \end{array}$$

So all entries except possibly for $j$ and $\star$ are in the same position in both $\boldsymbol{P}^{(i,j-1)}$ and $\boldsymbol{P}^{(ij)}$. If this section is in configuration (a) then $\star$ and $j$ are swapped, while if it is in configuration (b), they are in the same position in both tableaux. We are in configuration (a) iff the cell $y$ of $\boldsymbol{P}^{(i,j-1)}$ containing $j$ is adjacent, right or below, the cell $x$ with $\star$, and is smaller than the other cell adjacent to $x$ if the other cell is nonempty. So the rule for swapping $\star$ with an adjacent entry is the same as in the classical computation. Finally, $\star$ is at an inner corner of $\boldsymbol{P}^{(in)}$, and removing it yields the $i$th row of $\Lambda$ as $\Delta^i\boldsymbol{P}$.

Since $\lambda^{(in)} = \operatorname{sh}\Delta^i P$, the chain represented by the last column of $\Lambda$ is $\operatorname{ev}\boldsymbol{P}$.

Now every 2 by 2 segment of this table obeys the rule, given partitions $\alpha$ and $\delta$, either a unique partition is between them, and both $\beta$ and $\gamma$ equal it, or there are two partitions between them, and $\beta$ is one and $\gamma$ the other. So given initial values of the final column instead of the first row, we could build the exact same table. Since the rule is symmetric, $\operatorname{ev}\operatorname{ev}\boldsymbol{P} = \boldsymbol{P}$, so ev is an involution.

**5.2. The interval type table of a flag.** The interval type table of a saturated flag $f$ in a semi-primary lattice is the triangular array $\operatorname{itype} f = (\operatorname{type}[f_i, f_j])_{0\le i\le j\le n}$.

We first develop necessary (but not sufficient) conditions for a triangular array $(\lambda^{(ij)})_{0\le i\le j\le n}$ of partitions to be realizable as the interval type table of some flag. Then we show that if $\mathcal{L}$ is $q$-regular of type $\lambda$, and $\operatorname{sh}\boldsymbol{P} = \lambda$, most flags whose type is $\boldsymbol{P}$ have the same interval type table as as the evacuation table of $\boldsymbol{P}$.

For any saturated flag $f$, $\operatorname{type}[f_i, f_j]$ is increasing in Young's lattice as $i$ decreases and $j$ increases, and has rank $j-i$. Thus, the triangular array $\operatorname{itype} f$ is quite similar to the array described above for Fomin's version of evacuation.

$$\begin{array}{r|ccc} \operatorname{type}[f_a, f_b] & b=j-1 & & b=j \\ \hline a=i & \gamma & \lessdot & \delta \\ & \lessdot & & \lessdot \\ a=i+1 & \alpha & \lessdot & \beta \end{array}$$

Given $\alpha < \delta$ of ranks differing by 2, there are still either one or two elements between them in Young's lattice, and if there is only one, then both $\gamma$ and $\beta$ must equal it. If there are two, while the evacuation algorithm specifies that $\beta$ must be one and $\gamma$ the other, this needn't be the case for the interval types of flags. We will show that in a quantifiable sense, they usually are different, but also determine occasions when they may be equal.

Let $\boldsymbol{P}_i = \operatorname{ftype}\partial^i f$ denote the tableau associated to the partitions on the $i$th row of $\operatorname{itype} f$. We generalize the tableau operator $\Delta$ to a nondeterministic game $\tilde{\Delta}$, such that $\boldsymbol{P}_{i+1}$ has form $\tilde{\Delta}\boldsymbol{P}_i$ for $i = 0, \ldots, n-1$. The notation $\boldsymbol{P}_{i+1} = \tilde{\Delta}\boldsymbol{P}_i$ means that $\boldsymbol{P}_{i+1}$ is one of the possible values of $\tilde{\Delta}\boldsymbol{P}_i$.

Let $\boldsymbol{P}$ be a standard tableau.

(1) Replace the top left corner of $\boldsymbol{P}$ with $\star$.

Consider the cell just below $\star$ and the cells in the column right of $\star$. If a cell amongst these has a larger entry than was initially at $\star$, and this entry can be slid into $\star$ to yield a tableau that is strictly increasing in rows and columns (ignoring $\star$), the cell is **eligible**. Unless $\star$ is an inner corner, there is at least one eligible cell because the smaller or only of the right and lower neighbors of $\star$ is eligible.

(2) If there are any eligible cells, choose one, slide it into $\star$, and then repeat this step.

(3) Finally, $\star$ is at an inner corner. Delete the square with $\star$, obtaining a smaller tableau, $\widetilde{\Delta}\boldsymbol{P}$.

We illustrate this with

$$\boldsymbol{P} = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} \end{array}$$

First replace 1 with $\star$; the eligible cells after the replacement are shown with bold frames:

$$\begin{array}{|c|c|c|}\hline \star & 3 & 4 \\ \hline \boxed{2} & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} \end{array}$$

Only 2 is eligible, because if any number from the second column is slid into $\star$, the first column will no longer be strictly increasing, since these numbers are larger than 2. So slide 2 into $\star$:

$$\begin{array}{|c|c|c|}\hline 2 & \boxed{3} & 4 \\ \hline \star & \boxed{5} \\ \cline{1-2} 6 & 7 \\ \cline{1-2} \end{array}$$

The evacuation algorithm would specify that we slide 5 into $\star$, but in this algorithm, we may choose to slide 3 into $\star$.

$$\begin{array}{|c|c|c|}\hline 2 & \star & \boxed{4} \\ \hline 3 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} \end{array}$$

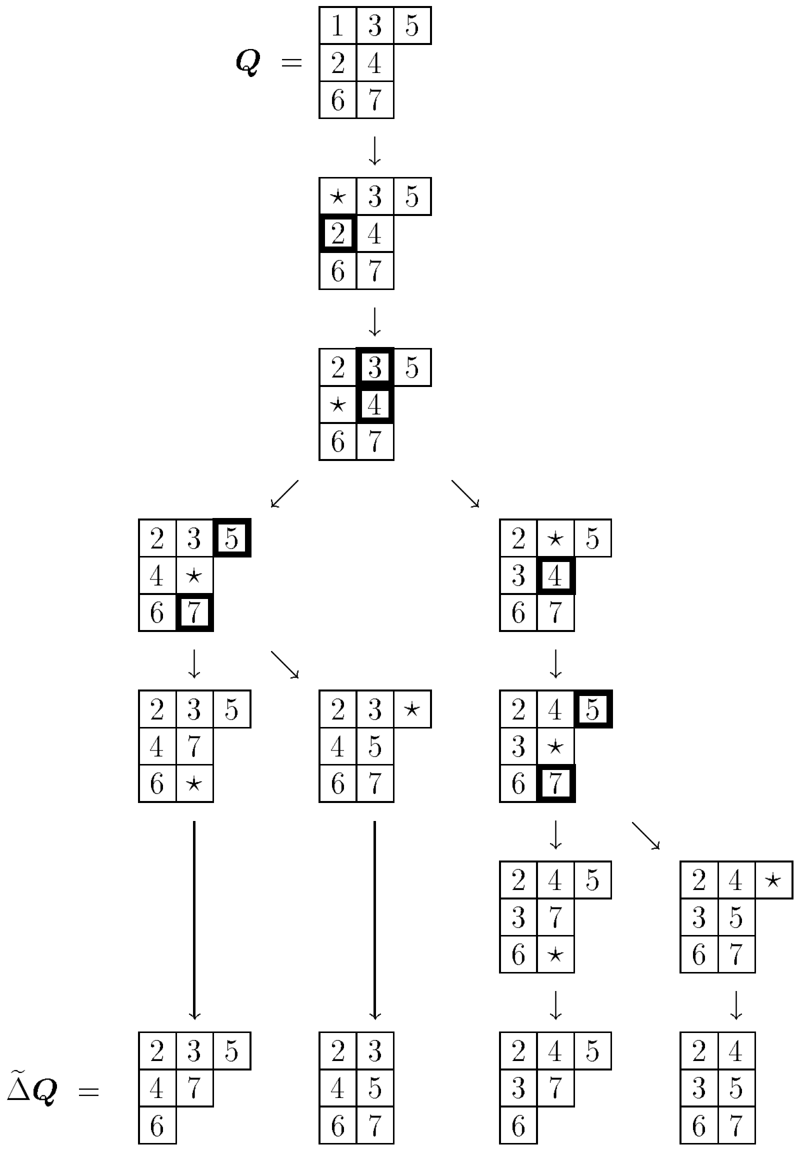
Only 4 is eligible, so slide it into $\star$. Then $\star$ is an inner corner with no further eligible cells, so $\star$ is removed to obtain as one possible value of $\widetilde{\Delta}\boldsymbol{P}$,

$$\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 & 5 \\ \hline 6 & 7 \\ \hline \end{array}$$

This and $\Delta\boldsymbol{P}$ are the only possible values of $\widetilde{\Delta}\boldsymbol{P}$. Swapping the 4 and 5 in $\boldsymbol{P}$ leads to more possibilities, which are shown in Figure 4.

We will show that all flags $f$ of type $\boldsymbol{P}$ have $\operatorname{ftype}\partial f = \widetilde{\Delta}\boldsymbol{P}$, and show that in a $q$-regular semi-primary lattice, the number of ways this happens for each possible $\widetilde{\Delta}\boldsymbol{P}$ is a nonzero monic polynomial in $q$.

*Notation.* For partitions $\alpha \lessdot \beta$, write $\alpha \lessdot_{|r|} \beta$ to mean that these partitions differ in a square in column $r$, that is, $\alpha'_i = \beta'_i$ for $i \neq r$, and $\alpha'_r \lessdot \beta'_r$. Write $\alpha \lessdot_x \beta$ to mean that $x \in \mathbb{P} \times \mathbb{P}$ is the cell in which the diagrams of $\alpha$ and $\beta$ differ.



Begin with $\boldsymbol{Q}$. Replace its top left corner with $\star$. At each step, the cells marked $\boxed{k}$ are eligible to be slid to $\star$. When $\star$ reaches an inner corner, delete that cell to obtain a possible value of $\widetilde{\Delta}\boldsymbol{Q}$. The ordinary $\Delta\boldsymbol{Q}$ is the left-hand path.

FIGURE 4. All possible degenerate elementary evacuation steps.

**Theorem 5.4.** *Let $\mathcal{L}$ be a semi-primary lattice. Let $j - i > 1$, and consider a flag $(f_i \lessdot f_{i+1} \leq f_{j-1} \lessdot f_j)$ in $\mathcal{L}$. The interval types between them have one of the following forms, where the positions $x$ or $y$ of the square two partitions differ by is noted between them.*

$$
\begin{array}{c|ccc}
\text{(a) type}[f_a, f_b] & b = j-1 & & b = j \\
\hline
a = i & \gamma & \lessdot_y & \delta \\
 & \mathrel{\rotatebox[origin=c]{90}{$\lessdot$}}_x & & \mathrel{\rotatebox[origin=c]{90}{$\lessdot$}}_y \\
a = i+1 & \alpha & \lessdot_x & \beta
\end{array}
\qquad
\begin{array}{c|ccc}
\text{(b)} & j-1 & & j \\
\hline
i & \gamma & \lessdot_y & \delta \\
 & \mathrel{\rotatebox[origin=c]{90}{$\lessdot$}}_x & & \mathrel{\rotatebox[origin=c]{90}{$\lessdot$}}_x \\
i+1 & \alpha & \lessdot_y & \beta
\end{array}
$$

*If $\delta/\alpha$ is a brick, we have form (a); if $\operatorname{col}(y) \gtrdot \operatorname{col}(x)$ but $\operatorname{row}(y) \neq \operatorname{row}(x)$, we may have either form; and otherwise, $|\operatorname{col}(y) - \operatorname{col}(x)| > 1$ and we have form (b).*

*Proof.* The type of an interval increases as the lower boundary is lowered or the upper boundary is raised, so we have one of forms (a) and (b). Now suppose we are given the partial flag $(f_i \lessdot f_{i+1} < f_j)$ with type$[f_i, f_j] = \delta$ and type$[f_{i+1}, f_j] = \beta$, and we want to choose a coatom $f_{j-1}$ of $[f_{i+1}, f_j]$ with type$[f_i, f_{j-1}] = \gamma$ and type$[f_{i+1}, f_{j-1}] = \alpha$.

Define $z^k_{ab} = A^k_{f_b} f_a = A^k f_a \wedge f_b$.

Let the columns the partitions differ in be given by $\gamma \lessdot_{|s|} \delta$ and $\alpha \lessdot_{|r|} \beta$. The condition type$[f_i, f_{j-1}] = \gamma$ is satisfied precisely when $f_{j-1}$ is a coatom of $I_1 = [z^{s-1}_{ij}, f_j]$ not also in $I_2 = [z^s_{ij}, f_j]$, while the condition type$[f_{i+1}, f_{j-1}] = \alpha$ is satisfied precisely when $f_{j-1}$ is a coatom of $I_3 = [z^{r-1}_{i+1,j}, f_j]$ not also in $I_4 = [z^r_{i+1,j}, f_j]$. So we count the coatoms of $I_1 \cap I_3$, minus those of $I_1 \cap I_4$, minus those of $I_2 \cap I_3$, plus the doubly subtracted ones in $I_2 \cap I_4$. We tabulate all these intersections below in all cases. Each intersection has the form $[t, f_j] \cap [u, f_j] = [t \vee u, f_j]$. Note for all $k \geq 0$ that $z^{k+1}_{ij} \geq z^k_{i+1,j} \geq z^k_{ij}$: the second inequality expands to $A^k f_{i+1} \wedge f_j \geq A^k f_i \wedge f_j$, which is true because $A$ and $\wedge$ both preserve weak cover relations, while $z^{k+1}_{ij} = A_{f_j} z^k_{ij}$ is the join of all weak upper covers of $z^k_{ij}$ below $f_j$, and $z^k_{i+1,j}$ is such a weak upper cover, so the first inequality holds.

| | | | $x$ in each case | | | |
|---|---|---|---|---|---|---|
| sign | $[x, f_j]$ | $x$ equation | $r < s-1$ | $r = s-1$ | $r = s$ | $r > s$ |
| $+$ | $I_1 \cap I_3$ | $z^{s-1}_{ij} \vee z^{r-1}_{i+1,j}$ | $z^{s-1}_{ij}$ | $z^{s-1}_{ij}$ | $z^{r-1}_{i+1,j}$ | $z^{r-1}_{i+1,j}$ |
| $-$ | $I_1 \cap I_4$ | $z^{s-1}_{ij} \vee z^r_{i+1,j}$ | $z^{s-1}_{ij}$ | $z^r_{i+1,j}$ | $z^r_{i+1,j}$ | $z^r_{i+1,j}$ |
| $-$ | $I_2 \cap I_3$ | $z^s_{ij} \vee z^{r-1}_{i+1,j}$ | $z^s_{ij}$ | $z^s_{ij}$ | $z^s_{ij}$ | $z^{r-1}_{i+1,j}$ |
| $+$ | $I_2 \cap I_4$ | $z^s_{ij} \vee z^r_{i+1,j}$ | $z^s_{ij}$ | $z^s_{ij}$ | $z^r_{i+1,j}$ | $z^r_{i+1,j}$ |
| | | net interval | $\emptyset$ | $I_1 - I_4$ | $I_3 - I_2$ | $\emptyset$ |
| | | # coatoms if $q$-regular | | $\langle \delta'_s \rangle - \left\langle \beta'_{r+1} \right\rangle$ | $\langle \beta'_r \rangle - \left\langle \delta'_{s+1} \right\rangle$ | |

There is no way to choose $f_{j-1}$ unless $r = s-1$ or $s$. If $\delta/\alpha$ is a brick, we are in form (a). If it is not a brick, we are in form (b), except that if $\operatorname{col}(x) \lessdot \operatorname{col}(y)$, we may be in either (a) or (b). □

**Corollary 5.5.** *Let $(f_i \lessdot f_{i+1} < f_j)$ or $(f_i < f_{j-1} \lessdot f_j)$ be given. This may be completed to $(f_i \lessdot f_{i+1} \leq f_{j-1} \lessdot f_j)$ with specified interval types, by choosing $f_{j-1}$ (respectively, $f_{i+1}$) as follows, and if the lattice is $q$-regular, the number of such choices is as indicated.*

$$(1)\quad \begin{array}{c|ccc}
\mathrm{type}[f_a, f_b] & b = j-1 & & b = j \\
\hline
a = i & \gamma & \lessdot_{|r+1|} & \delta \\
 & \vee_{|r|} & & \vee_{|r+1|} \\
a = i+1 & \alpha & \lessdot_{|r|} & \beta
\end{array}$$

*$f_{j-1}$ is a coatom of $[A^r f_i \wedge f_j, f_j]$ not in $[A^r f_{i+1} \wedge f_j, f_j]$;*
*$\left\langle \delta'_{r+1} \right\rangle - \left\langle \beta'_{r+1} \right\rangle$ choices.*

*$f_{i+1}$ is an atom of $[f_i, C^r f_j \vee f_i]$ not in $[f_i, C^r f_{j-1} \vee f_i]$;*
*$\left\langle \delta'_{r+1} \right\rangle - \left\langle \gamma'_{r+1} \right\rangle$ choices.*

$$(2)\quad \begin{array}{c|ccc}
\mathrm{type}[f_a, f_b] & b = j-1 & & b = j \\
\hline
a = i & \gamma & \lessdot_{|r|} & \delta \\
 & \vee_{|t|} & & \vee_{|t|} \\
a = i+1 & \alpha & \lessdot_{|r|} & \beta
\end{array}$$

*$f_{j-1}$ is a coatom of $[A^{r-1} f_{i+1} \wedge f_j, f_j]$ not in $[A^r f_i \wedge f_j, f_j]$;*
*$\langle \beta'_r \rangle - \left\langle \delta'_{r+1} \right\rangle$ choices.*

*$f_{i+1}$ is an atom of $[f_i, C^{t-1} f_{j-1} \vee f_i]$ not in $[f_i, C^t f_{j-1} \vee f_i]$;*
*$\langle \gamma'_t \rangle - \left\langle \delta'_{t+1} \right\rangle$ choices.*

*Note.* Configuration (1) of the corollary always has form (a) of the theorem; configuration (2) has form (a) for $r = t$ and (b) for $r \neq t$.

**Proposition 5.6.** *In a semi-primary lattice of type $\lambda$, if $f$ is a flag of type $\boldsymbol{P}$, then* $\mathrm{ftype}\, d^k f = d^k \boldsymbol{P}$.

**Proposition 5.7.** *If $f$ is a flag with* $\mathrm{ftype}\, f = \boldsymbol{P}$ *in a semi-primary lattice, then* $\mathrm{ftype}\, \partial f = \widetilde{\Delta} \boldsymbol{P}$. *Conversely, consider a $q$-regular semi-primary lattice of type $\lambda \vdash n$. The number of atoms $f_1$ of cotype $\mu \lessdot_{|r|} \lambda$ is $\left\langle \lambda'_{r-1} \right\rangle - \langle \lambda'_r \rangle$. Pick any such atom. Let $\boldsymbol{P}$ be a standard tableau of shape $\lambda$ and $\boldsymbol{Q} = \widetilde{\Delta} \boldsymbol{P}_i$ have shape $\mu$. The number of ways to complete $(\hat{0}, f_1, \hat{1})$ to a flag $f = (\hat{0}, f_1, f_2, \ldots, f_{n-1}, \hat{1})$ with* $\mathrm{ftype}\, f = \boldsymbol{P}$ *and* $\mathrm{ftype}\, \partial f = \boldsymbol{Q}$ *is a monic polynomial in $q$, given as the product over each entry $j$ of $\boldsymbol{Q}$ of terms depending on how $j$ moves between $\boldsymbol{P}$ and $\boldsymbol{Q}$:*

$$\prod_{j \,\in\, \boldsymbol{Q}} \begin{cases} \langle \mathrm{row}(\boldsymbol{Q}, j) \rangle - \langle \mathrm{row}_1(\boldsymbol{P}, j) \rangle & \textit{if } \mathrm{col}(\boldsymbol{P}, j) = \mathrm{col}(\boldsymbol{Q}, j); \\ \langle \mathrm{row}(\boldsymbol{P}, j) \rangle - \langle \mathrm{row}_1(\boldsymbol{Q}, j) \rangle & \textit{otherwise.} \end{cases}$$

*This polynomial has degree*

$$n(\lambda) - \#\ \textit{of entries that move up one cell from } \boldsymbol{P} \textit{ to } \boldsymbol{Q}.$$

*As $q \to \infty$, all but a fraction $O(q^{-1})$ of the flags $f$ with* $\mathrm{ftype}\, f = \boldsymbol{P}$ *have* $\mathrm{ftype}\, \partial f = \Delta \boldsymbol{P}$, *and all but a fraction $O(q^{-1})$ of the flags $(f_1, \ldots, f_n)$ with $f_1$ an atom and* $\mathrm{ftype}(f_1, \ldots, f_n) = \Delta \boldsymbol{P}$ *have* $\mathrm{ftype}(\hat{0}, f_1, \ldots, f_n) = \boldsymbol{P}$.

*Proof.* Let $f = (f_0, \ldots, f_n)$ be a flag of type $\boldsymbol{P}$. The first part of the theorem is obtained from Theorem 5.4 by expressing it in terms of tableaux. We begin with the tableau $\boldsymbol{P}$, and then essentially follow the proof of Section 5.1 that Fomin's computation of $\Delta \boldsymbol{P}$ agrees with the classical one. Place $\star$ in the upper left corner of $\boldsymbol{P}$, and then for each $j = 1, \ldots, n$ successively, if we are in form (a), $j$ is in the column just right of $\star$, and we swap $\star$ and $j$; while if we are in form (b), we leave the tableau intact. At the end, remove $\star$. This is exactly the algorithm for $\widetilde{\Delta}\boldsymbol{P}$.

For the converse, in a $q$-regular semi-primary lattice of type $\lambda$, let $f_0 = \hat{0}$, $f_n = \hat{1}$, and choose an atom $f_1$ of cotype $\mu$ in the indicated number of ways. If $f_n, f_{n-1}, \ldots, f_j$ have been chosen so that $\operatorname{ftype}_{f_0}(f_j, \ldots, f_n)$ and $\operatorname{ftype}_{f_1}(f_j, \ldots, f_n)$ respectively agree with $\boldsymbol{P}$ and $\boldsymbol{Q}$ in entries $j+1, \ldots, n$, then the number of ways to choose $f_{j-1}$ to extend this to entry $j$ is as stated in the theorem, by plugging into Corollary 5.5: when $j$ is in the same column of $\boldsymbol{P}$ and $\boldsymbol{Q}$, use form (2), and when it is in different columns, use form (1). Denote the product in the theorem by $a(q)$.

If $\operatorname{col}(\boldsymbol{P}, j) = \operatorname{col}(\boldsymbol{Q}, j)$, the degree of the $j$th factor is $\operatorname{row}(\boldsymbol{Q}, j) - 1$, which equals $\operatorname{row}(\boldsymbol{P}, j) - 1$ if $j$ is in the same cell of both $\boldsymbol{P}$ and $\boldsymbol{Q}$, and equals $\operatorname{row}(\boldsymbol{P}, j) - 2$ if it moves up one cell. Let $u$ be the number of $j$'s that move up one cell. The degree of $a(q)$ is the sum of $\operatorname{row}(\boldsymbol{P}, j) - 1$ over all $j \in \boldsymbol{Q}$, minus $u$. Since 1 is the only entry in $\boldsymbol{P}$ not in $\boldsymbol{Q}$, and $\operatorname{row}(\boldsymbol{P}, 1) - 1 = 0$, this sum is the sum over all cells of $\boldsymbol{P}$ of their row number minus 1, totaling $n(\lambda)$.

Multiply $a(q)$ by the initial number of choices of $f_1$, to obtain the total number $b(q)$ of flags $f$ with $\operatorname{ftype} f = \boldsymbol{P}$ and $\operatorname{ftype} \partial f = \boldsymbol{Q}$. It is a monic polynomial of degree $n(\lambda) + (\operatorname{row}(\lambda/\mu) - 1) - u$. In the deletion path for the game $\Delta$, we have $u = \operatorname{row}(\lambda/\mu) - 1$ because the deletion path from $\lambda/\mu$ to the upper left corner only moves left one cell or up one cell at each step. In all other games $\widetilde{\Delta}$, there are entries that move to lower rows, so that $u > \operatorname{row}(\lambda/\mu) - 1$. Thus, the maximum degree $n(\lambda)$ of $b(q)$ is uniquely obtained by the game $\boldsymbol{Q} = \Delta\boldsymbol{P}$. The total number of flags $f$ with $\operatorname{ftype} f = \boldsymbol{P}$ is $F^{\boldsymbol{P}}(q)$, which also is a monic polynomial of degree $n(\lambda)$, so the quotient $b(q)/F^{\boldsymbol{P}}(q)$ is $1 - O(q^{-1})$ as $q \to \infty$, so nearly all flags with $\operatorname{ftype} f = \boldsymbol{P}$ satisfy $\operatorname{ftype} \partial f = \boldsymbol{Q}$. The degree of $a(q)$ in the game $\Delta\boldsymbol{P}$ is $n(\lambda) - u = n(\mu)$, so also $a(q)/F^{\Delta\boldsymbol{P}}(q) = 1 - O(q^{-1})$. Thus, nearly all flags $g = (f_1, \ldots, f_n)$, where $f_1$ is an atom and $\operatorname{ftype} g = \Delta\boldsymbol{Q}$, satisfy $\operatorname{ftype}(\hat{0}, f_1, \ldots, f_n) = \boldsymbol{P}$. $\square$

We will extend this to show that $k = 0, \ldots, n$, all but a fraction $O(q^{-1})$ of the flags $f$ with $\operatorname{ftype} f = \boldsymbol{P}$ satisfy $\operatorname{ftype} \partial^k f = \Delta^k \boldsymbol{P}$. First we develop notions of asymptotic enumeration and composition of operators on flags.

**5.3. Generic behavior in regular semi-primary lattices.** For some problems, we can do exact enumerations, and for others, we can do asymptotic enumerations showing what "usually" happens. Given $\operatorname{ftype} f$ but not $f$, we will compute the "usual" flag type ftype or $\operatorname{ftype}_{\hat{0}}$ of $\partial f$, $df$, $(\phi(f_l), \ldots, \phi(f_h))$ for certain latticial growths $\phi$; various other flag operators; the flag cotype of $f$; and compositions of these.

For logical statement $S(s)$ with parameters $s = (x_1, \ldots, x_k)$, we write $s \in S$ to mean that $S(s)$ is true.

**Definition 5.8 (topological).** Let $S(s)$ and $T(s)$ be logical statements about indeterminates $s = (x_1, \ldots, x_k)$ in an invariant subspace lattice over an infinite field. Then $T(s)$ **is generically true of** $S(s)$ if

$$\overline{\{\, s : S(s) \text{ and } T(s) \text{ are true} \,\}} \supseteq \{\, s : S(s) \text{ is true} \,\},$$

where the closure is taken in the Zariski topology.

We obtain an analogue of this notion for $q$-regular semi-primary lattices by replacing closure in a specific lattice with a limit over all finite lattices. For a logical statement $S(s)$ and a finite semi-primary lattice $\mathcal{L}$, let $N_S = N_S(\mathcal{L})$ denote the number of $s$ for which $S(s)$ is true.

**Definition 5.9 ($q$-regular).** Let $S(s)$ and $T(s)$ be logical statements about indeterminates $s$ in semi-primary lattices. Then $T$ **is generically true of** $S$ if

$$\lim_{q\to\infty} \frac{\#\{\, s \in S \,:\, T(s)\,\}}{N_S} = 1$$

where the limit is taken over all $q$-regular semi-primary lattices as $q \to \infty$. Note that the counts in the numerator and denominator may be different for different $q$-regular semi-primary lattices that have the same value of $q$.

Let $s$ and $t$ be indeterminates; $S(s)$ be a logical statement about indeterminates $s$; and $T(t)$ be a logical statement about indeterminates $t$. Let $\phi$ be a map from the space in which $s$ lies to the space in which $t$ lies.

**Definition 5.10.** We say that $\phi$ **generically maps** $S$ **to** $T$ when $T(\phi(s))$ is generically true of $S(s)$. Equivalently, there is a function $\gamma_\phi(q)$ (of $q$, not $\mathcal{L}$) taking on real values from 0 to 1, vanishing as $q \to \infty$, such that

$$\#\{\, s \in S \,:\, T(\phi(s))\,\} \geq (1 - \gamma_\phi(q))N_S$$

in all $q$-regular semi-primary lattices.

**Definition 5.11.** $\phi$ **uniformly maps** $S$ **to** $T$ when

(1) $\phi$ generically maps $S$ to $T$;
(2) There is a function $\epsilon(q)$ that vanishes as $q \to \infty$, such that

$$\lim_{q\to\infty} \frac{\#\left\{\, t \in T \,:\, \left|\frac{\#\{\, s\in S\,:\,\phi(s)=t\,\}}{N_S/N_T} - 1\right| < \epsilon(q)\,\right\}}{N_T} = 1.$$

Equivalently, there are functions $\delta_\phi(q)$ and $\epsilon_\phi(q)$ taking on real values from 0 to 1 and vanishing as $q \to \infty$, such that

$$\#\left\{\, t \in T \,:\, \left|\frac{\#\{\, s \in S \,:\, \phi(s) = t\,\}}{N_S/N_T} - 1\right| < \epsilon_\phi(q)\right\} \geq (1 - \delta_\phi(q))N_T$$

for all $q$-regular semi-primary lattices.

In other words, (1) most variables $s \in S$ also satisfy $\phi(s) \in T$, and (2) conversely, most variables $t \in T$ have approximately the same number $N_S/N_T$ of inverses under $\phi$ that satisfy $S$.

**Theorem 5.12.** *If $\phi$ uniformly maps $S$ to $T$, and $\theta$ generically maps $T$ to $U$, the composition $\psi = \theta \circ \phi$ generically maps $S$ to $U$. If $\theta$ is uniform then so is $\psi$.*

*Proof.* Let $\phi$ be generic and $\theta$ be uniform. Choose $\gamma_\phi$, $\delta_\phi$, $\epsilon_\phi$, and $\gamma_\theta$ as in the definitions of genericity and uniformity above. All $\gamma$, $\delta$, and $\epsilon$'s in this proof, with any subscripts, are functions of $q$, while all set cardinalities are functions of $\mathcal{L}$ (and hence not solely functions of $q$).

For genericity, we must prove

$$\lim_{q\to\infty} \frac{\#\{\, s \in S \,:\, U(\psi(s)) \,\}}{N_S} = 1.$$

Consider any $q$-regular semi-primary lattice. We have

$$\begin{aligned} \#\{\, t \in T \,:\, U(\theta(t)) \,\} &\geq (1-\gamma_\theta)N_T \\ \#\left\{ t \in T \,:\, \left| \frac{\#\{\, s \in S \,:\, \phi(s) = t \,\}}{N_S/N_T} - 1 \right| < \epsilon_\phi \right\} &\geq (1-\delta_\phi)N_T \end{aligned}$$

so the cardinality of the intersection $I$ of the two sets on the left is at least $(1-\gamma_\theta-\delta_\phi)N_T$. Thus,

$$\begin{aligned} N_S \geq \#\{\, s \in S \,:\, U(\psi(s)) \,\} &\geq \sum_{t\in T} \#\{\, s \in S \,:\, \phi(s) = t \text{ and } U(\theta(t)) \,\} \\ &\geq \sum_{t\in I} \#\{\, s \in S \,:\, \phi(s) = t \text{ and } U(\theta(t)) \,\} \\ &= \sum_{t\in I} \#\{\, s \in S \,:\, \phi(s) = t \,\} \\ &\geq \#I \cdot \frac{N_S}{N_T}(1-\epsilon_\theta) \\ &\geq (1-\gamma_\theta-\delta_\phi)(1-\epsilon_\theta)N_S \end{aligned}$$

Divide through by $N_S$ to obtain

$$1 \geq \frac{\#\{\, s \in S \,:\, U(\psi(s)) \,\}}{N_S} \geq (1-\gamma_\theta-\delta_\phi)\,(1-\epsilon_\phi).$$

The left and right expressions depend only on $q$, not on $\mathcal{L}$, and are 1 in the limit $q \to \infty$; thus the middle expression, which depends on $\mathcal{L}$, also has limit 1 as $q \to \infty$. Thus $\psi$ generically distributes $S$ over $U$.

For uniformity, we will make a number of approximations to $S$, $T$, and $U$.

Let

$$T_1 = \left\{ t \in T \,:\, \left| \frac{\#\{\, s \in S \,:\, \phi(s) = t \,\}}{N_S/N_T} - 1 \right| < \epsilon_\phi \right\}.$$

Note $\#T_1 \geq (1-\delta_\phi)N_T$.

For $u \in U$, let $B_u = \{\, t \in T \,:\, \theta(t) = u \,\}$. At least $(1-\delta_\theta)N_U$ different $u$'s satisfy $\left|\frac{\#B_u}{N_T/N_U} - 1\right| < \epsilon_\theta$. Let $C_u = \{\, t \in T_1 \,:\, \theta(t) = u \,\}$. Then

$$\delta_\phi N_T \geq \#T - \#T_1 = \sum_{u\in U} (\#B_u - \#C_u)\,.$$

Each term of the sum is nonnegative since $B_u \supseteq C_u$. At most $\sqrt{\delta_\phi}N_U$ values of $u$ have $\#B_u - \#C_u > \sqrt{\delta_\phi}\frac{N_T}{N_U}$ because if more did, the rightmost expression would exceed the leftmost. So

$$U_1 = \left\{ u \in U \,:\, \#B_u - \#C_u \leq \sqrt{\delta_\phi}\frac{N_T}{N_U} \right\}$$

has cardinality at least $(1-\sqrt{\delta_\phi})N_U$.

Now let

$$\begin{aligned}
\delta_2 &= \delta_\theta + \sqrt{\delta_\phi} \\
\epsilon_2 &= \epsilon_\theta + \sqrt{\delta_\phi} \\
U_2 &= \left\{ u \in U \,:\, \left| \frac{\#\{\, t \in T_1 \,:\, \theta(t) = u \,\}}{N_T/N_U} - 1 \right| < \epsilon_2 \right\}.
\end{aligned}$$

Then $U_2$ contains the intersection of $\theta(T)$ and $U_1$, whose respective sizes are at least $(1-\delta_\theta)N_U$ and $(1-\sqrt{\delta_\phi})N_U$. Thus their intersection has size at least $(1-\delta_\theta-\sqrt{\delta_\phi})N_U$, so $\#U_2 \geq (1-\delta_2)N_U$.

For each $u \in U$, let

$$\begin{aligned}
D_u &= \{\, s \in S \,:\, \phi(s) \in T_1 \text{ and } \psi(s) = u \,\} \\
E_u &= \{\, s \in S \,:\, \psi(s) = u \,\}.
\end{aligned}$$

For at least $(1-\delta_2)N_U$ different $u$'s, there are at least $(1-\epsilon_2)N_T/N_U$ inverses of $u$ in $T$ under $\theta$ that in turn have at least $(1-\epsilon_\phi)N_S/N_T$ inverses each in $S$ under $\phi$. So at least $(1-\delta_2)N_U$ different $u$'s have $\#D_u \geq (1-\epsilon_\phi)(1-\epsilon_2)N_S/N_U$. Thus, the union of all $D_u$'s has cardinality at least $(1-\delta_2)(1-\epsilon_\phi)(1-\epsilon_2)N_S$. Let

$$\delta_3 = 1 - (1-\delta_2)(1-\epsilon_\phi)(1-\epsilon_2).$$

We have $D_u \subseteq E_u$ so $\#D_u \leq \#E_u$. At most $\sqrt{\delta_3}N_U$ different $u$'s can have

$$\#E_u \geq \frac{N_S}{N_U}\left((1-\epsilon_\phi)(1-\epsilon_2) + \sqrt{\delta_3}\right)$$

because if more do, we have

$$\sum_{u\in U} \#E_u > \frac{N_S}{N_U} \cdot \left((1-\delta_2)(1-\epsilon_\phi)(1-\epsilon_2) + \sqrt{\delta_3}\cdot\sqrt{\delta_3}\right) N_U = N_S,$$

but the union of the disjoint sets $E_u$'s is a subset of $S$, so the sum is at most $N_S$.

So at least $(1-\delta_2-\sqrt{\delta_3})N_U$ different $u$'s have

$$(1-\epsilon_\phi)(1-\epsilon_2) + \sqrt{\delta_3} \geq \frac{\#E_u}{N_S/N_U} \geq (1-\epsilon_\phi)(1-\epsilon_2).$$

So

$$\begin{aligned}
\delta_\psi &= 1 - \delta_2 - \sqrt{\delta_3} \\
\epsilon_\psi &= \max\left\{1-(1-\epsilon_\phi)(1-\epsilon_2), \sqrt{\delta_3} + (1-\epsilon_\phi)(1-\epsilon_2) - 1\right\}
\end{aligned}$$

satisfy the definition of $\psi$ being uniform. □

**Definition 5.13.** The **dual** of a flag $f = (f_0, \ldots, f_n)$ is $f^* = (g_0, \ldots, g_n)$ where $g_i = f^*_{n-i}$ combines inclusion of $\mathcal{L}$ into the dual lattice $\mathcal{L}^*$ with relabeling the indices.

We now present an asymptotic analogue for finite $q$-regular semi-primary lattices of the result of Hesselink [**10**, §5] and van Leeuwen [**31**, Theorem 2.3.2] that the generic cotype of a flag of type $\boldsymbol{P}$ in an invariant subspace lattice over an infinite field is ev $\boldsymbol{P}$.

**Theorem 5.14.** *Let $\boldsymbol{P}$ be a standard tableau of shape $\lambda \vdash n$. In a $q$-regular semi-primary lattice of type $\lambda$, all but a fraction $O(q^{-1})$ of the flags $f$ with* $\operatorname{ftype} f = \boldsymbol{P}$ *satisfy* $\operatorname{ftype} \partial^k = \Delta^k \boldsymbol{P}$ *for $k = 0, \ldots, n$. Consequently, all but a fraction $O(q^{-1})$ of flags $f$ with* $\operatorname{ftype} f = \boldsymbol{P}$ *satisfy* $\operatorname{ftype} f^* = \operatorname{ev} \boldsymbol{P}$.

*Proof.* By Proposition 5.7, $\partial$ uniformly maps flags $f$ with $\operatorname{ftype} f = \boldsymbol{P}$ to flags $g$ from $f_1$ to $\hat{1}$ with $\operatorname{ftype} g = \Delta \boldsymbol{P}$. By Theorem 5.12, the composition $\partial^k$ uniformly maps flags $f$ with $\operatorname{ftype} f = \boldsymbol{P}$ to flags $g = (g_k, \ldots, g_n)$ such that $\operatorname{ftype} g = \Delta^k \boldsymbol{P}$ and there is a sequence $\hat{0} \lessdot g_1 \lessdot g_2 \lessdot \cdots \lessdot g_k$ with $\operatorname{cotype} g_i = \operatorname{sh} \Delta^i \boldsymbol{P}$ for $i = 0, \ldots, k$.

Thus, all but a fraction $O(q^{-1})$ of the flags $f$ with $\operatorname{ftype} f = \boldsymbol{P}$ have $\operatorname{ftype} \partial^k f = \Delta^k \boldsymbol{P}$ for $k = 0, \ldots, n$. Since $\operatorname{cotype} f_k = \operatorname{sh}(\operatorname{ftype} \partial^k f)$ for all $k$, the generic cotype of $f_k$ is $\operatorname{sh} \Delta^k \boldsymbol{P}$. Thus, the tableau $\operatorname{ftype} f^*$ encoding the cotypes is generically $(\operatorname{sh} \Delta^{n-0} \boldsymbol{P}, \operatorname{sh} \Delta^{n-1} \boldsymbol{P}, \ldots) = \operatorname{ev} \boldsymbol{P}$.

Similarly, flags $g$ in the dual lattice with $\operatorname{ftype} g = \operatorname{ev} \boldsymbol{P}$ generically have $\operatorname{ftype} g^* = \operatorname{ev} \operatorname{ev} \boldsymbol{P} = \boldsymbol{P}$. So the uniform cotype of flags of type $\boldsymbol{P}$ is $\operatorname{ev} \boldsymbol{P}$. $\square$

**5.4. Counting the number of flags achieving a partial interval type table.** Consider semi-primary lattices of type $\lambda \vdash n$. An **interval type table** is a specification of partitions $\Lambda = (\lambda^{(ij)})_{0 \le i \le j \le n}$, with $\left|\lambda^{(ij)}\right| = j - i$, and $\lambda^{(ij)}$ increasing in Young's lattice as $j$ increases or $i$ decreases. A **partial interval type table** has only some of the values of $\lambda^{(ij)}$ specified, but their rank and order is still the same, and the entry $\lambda^{(0n)}$ must be specified. The **shape** of an interval type table is the partition $\lambda^{(0n)}$.

When for some $i$, all of $\lambda^{(ii)}, \ldots, \lambda^{(in)}$ are defined, let $\boldsymbol{P}_i$ be the tableau (with entries $i+1, \ldots, n$) corresponding to this chain. If $\lambda^{(ij)}, \lambda^{(i,j+1)}, \ldots, \lambda^{(in)}$ are defined, we let $\boldsymbol{P}_i$ be the skew tableau (with entries $j+1, \ldots, n$) corresponding to these. Let $k^* = n - k$, and if $\lambda^{(i^*,j)}, \lambda^{((i+1)^*,j)}, \ldots, \lambda^{(n^*,j)}$ are defined, let $\boldsymbol{P}^*_j$ be the (skew) tableau (with entries $i+1, \ldots, n$) corresponding to them.

A (partial) interval type table is **realizable** if there is a semi-primary lattice $\mathcal{L}$ with a flag $f$ for which $\operatorname{type}[f_i, f_j] = \lambda^{(ij)}$ whenever $\lambda^{(ij)}$ is defined. We are interested in what interval type tables are realizable, and in $q$-regular semi-primary lattices, determining the number of flags achieving particular (partial) interval type tables. Let $r_{ij} = \operatorname{col}(\boldsymbol{P}_i, j)$ be the column in which $\lambda^{(i,j-1)}$ and $\lambda^{(ij)}$ differ, and $r^*_{ij} = \operatorname{col}(\boldsymbol{P}^*_j, i^*)$ be the column in which $\lambda^{(ij)}$ and $\lambda^{(i+1,j)}$ differ, provided the relevant partitions are in the partial interval type table.

The condition $\boldsymbol{P}_{i+1} = \widetilde{\Delta} \boldsymbol{P}_i$ may be expressed as follows: $\boldsymbol{P}_i$ is a tableau on $i+1, \ldots, n$, while $\boldsymbol{P}_{i+1}$ is a tableau on $i+2, \ldots, n$; $\operatorname{sh} \boldsymbol{P}_{i+1} \lessdot \operatorname{sh} \boldsymbol{P}_i$; and $0 \le \operatorname{col}(P_i, j) - \operatorname{col}(P_{i+1}, j) \le 1$ for all $j$. A transposed version of this for the $\boldsymbol{P}^*_j$'s is equivalent. These conditions generalize as follows.

**Theorem 5.15.** *If an interval type table is realizable, then for all $0 \le i \le j \le k \le n$, we have $0 \le r_{ik} - r_{jk} \le \lambda^{(ij)}_1$, and dually, $0 \le r^*_{ik} - r^*_{ij} \le \lambda^{(jk)}_1$.*

*Proof.* We show the first statement. Since the types of $[f_j, f_k]$ and $[f_j, f_{k-1}]$ differ in column $r_{jk}$, we have $A^r f_j \wedge f_{k-1} = A^r f_j \wedge f_k$ iff $r < r_{jk}$, and $A^r f_j \wedge f_{k-1} \lessdot A^r f_j \wedge f_k$ iff $r \ge r_{jk}$. Similar statements hold for $i, j$ and $i, k$ in place of $j, k$. So $A^{r_{jk}-1} f_j \wedge f_{k-1} = A^{r_{jk}-1} f_j \wedge f_k$. Meet both sides of this with $A^{r_{jk}-1} f_i$ (which is weakly smaller than

$A^{r_{jk}-1}f_j$) to obtain $A^{r_{jk}-1}f_i \wedge f_{k-1} = A^{r_{jk}-1}f_i \wedge f_k$. Thus $r_{jk}-1 < r_{ik}$, or equivalently, $r_{ik} - r_{jk} \geq 0$.

Next, let $t = \lambda_1^{(ij)}$. If $r_{ik} - t - 1 < 0$ then $r_{ik} - t - 1 < r_{jk}$ so $r_{ik} - r_{jk} \leq t$. If $r_{ik} - t - 1 \geq 0$, consider the equality $A^{r_{ik}-1}f_i \wedge f_{k-1} = A^{r_{ik}-1}f_i \wedge f_k$. We have $A^t f_i \geq f_j$ so we obtain an inequality $A^{r_{ik}-1}f_i \geq A^{r_{ik}-t-1}f_j$. Meet both sides of the equality with $A^{r_{ik}-t-1}f_j$ to obtain $A^{r_{ik}-t-1}f_j \wedge f_{k-1} = A^{r_{ik}-t-1}f_j \wedge f_k$, whence $r_{ik} - t - 1 < r_{jk}$, and finally, $r_{ik} - r_{jk} \leq t$. $\square$

**Example 5.16.** No flag can have interval type table

$$\begin{array}{|c|c|c|c|}\hline 1&4&5&6\\ \hline 2\\ \cline{1-1} 3\\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|c|c|}\hline 2&4&5&6\\ \hline 3\\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 3&5&6\\ \hline 4\\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 4&6\\ \hline 5\\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 5&6\\ \hline\end{array}\quad \begin{array}{|c|}\hline 6\\ \hline\end{array}\quad \emptyset$$

because 5 moves left $\mathrm{col}(P_1, 5) - \mathrm{col}(P_3, 5) = 2$ columns from $\boldsymbol{P}_1$ to $\boldsymbol{P}_3$, but the top row of $\mathrm{type}[f_1, f_3] = \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$ has width 1, and $1 < 2$.

**Theorem 5.17.** *If the interval type table $\Lambda$ is realizable, then for $0 \leq i \leq j \leq k \leq n$, if $d^{n-k}\boldsymbol{P}_i = \boldsymbol{A} + \boldsymbol{B}$ where $\boldsymbol{A}$ is a standard tableau on $\{i+1, \ldots, j\}$ and $\boldsymbol{B}$ is a row and column strict composition tableau on $\{j+1, \ldots, k\}$, then $d^{n-k}\boldsymbol{P}_j = S(\boldsymbol{B})$.*

*Proof.* Let $f$ realize $\Lambda$. By Theorem 4.93 applied to $\partial^i d^{n-k} f = (f_i, \ldots, f_k)$, if $\mathrm{ftype}\, \partial^i d^{n-k} f = \boldsymbol{A} + \boldsymbol{B}$ then $\mathrm{ftype}\, \partial^j d^{n-k} f = S(\boldsymbol{B})$. Since $\mathrm{ftype}\, \partial^i f = \boldsymbol{P}_i$ and $\mathrm{ftype}\, \partial^j f = \boldsymbol{P}_j$, the result follows. $\square$

**Example 5.18.** If $\boldsymbol{P}_0 = \begin{array}{|c|c|c|c|}\hline 1&3&4&8\\ \hline 2&5\\ \cline{1-2} 6&7\\ \cline{1-2}\end{array}$ then

$$\boldsymbol{P}_2 = \begin{array}{|c|c|c|}\hline 3&4&8\\ \hline 5&7\\ \cline{1-2} 6\\ \cline{1-1}\end{array}\qquad d\boldsymbol{P}_3 = \begin{array}{|c|c|}\hline 4&7\\ \hline 5\\ \cline{1-1} 6\\ \cline{1-1}\end{array}\qquad d\boldsymbol{P}_4 = \begin{array}{|c|c|}\hline 5&7\\ \hline 6\\ \cline{1-1}\end{array}\qquad d\boldsymbol{P}_5 = \begin{array}{|c|c|}\hline 6&7\\ \hline\end{array}.$$

See Hesselink [**10**, §6] for stronger necessary conditions for an interval type table to be realizable.

Now we enumerate the number of flags realizing certain partial interval type tables. Initially, we are given $f_0 = \hat{0}$ and $f_n = \hat{1}$. Given a partial flag and certain of the interval types in it, we want to know how to select other elements satisfying other interval types. Throughout, $f$ is a partial flag, with $f_i$ the component of rank $i$, if present.

Let $f_\Lambda(q)$ be the number of flags realizing partial interval type table $\Lambda$ in a $q$-regular semi-primary lattice, when this is well defined for sufficiently large $q$.

**Proposition 5.19.** *Let $\Lambda = (\lambda^{(ij)})_{0\leq i\leq j\leq n}$ and $M = (\mu^{(ij)})_{0\leq i\leq j\leq m}$ be partial interval type tables of respective shapes $\lambda = \lambda^{(0n)}$, $\mu = \mu^{(0m)}$. Let $\nu \vdash m + n$. Let $N$ be a partial interval type table with indices $0$ through $m + n$ and components $\nu^{(ij)} = \lambda^{(ij)}$ (when defined) for $0 \leq i \leq j \leq n$; $\nu^{(i+n,j+n)} = \mu^{(ij)}$ (when defined) for $0 \leq i \leq j \leq m$; and $\nu^{(0,m+n)} = \nu$. Then $f_N(q) = g^\nu_{\lambda\mu}(q) f_\Lambda(q) f_M(q)$ when $f_\Lambda(q)$ and $f_M(q)$ exist.*

*Proof.* Choose $f_n$ of type $\lambda$ and cotype $\nu$ in one of $g^\nu_{\lambda\mu}(q)$ ways. Then choose $(f_0, \ldots, f_n)$ in $f_\Lambda(q)$ ways and $(f_n, \ldots, f_{n+m})$ in $f_M(q)$ ways. $\square$

**Definition 5.20.** The **transpose** of interval type table $\Lambda$ is $\Lambda' = (\lambda^{(n-j,n-i)})_{0\le i\le j\le n}$.

**Proposition 5.21.** *If either $f_\Lambda(q)$ or $f_{\Lambda'}(q)$ exists, both do and are equal.*

*Proof.* If either exists for sufficiently large $q$, all $q$-regular semi-primary lattices of shape $\lambda^{(0n)}$ have the same number of flags satisfying the partial interval type table $\Lambda$. Each such flag satisfies the partial interval type table $\Lambda'$ in the dual lattice. $\square$

Suppose $f_0 \lessdot \ldots \lessdot f_k \le f_n$ satisfies a partial interval type table, and we want to extend this to $f_{n-1}$. For now, abbreviate $r = r_{0n}$ and $r_i = r_{in}$. By Theorem 5.15, $r_0 \gtrsim r_1 \gtrsim \cdots \gtrsim r_k$. This splits $\{0,\ldots,k\}$ into intervals according to where the $\gtrsim$ is $\gtrdot$: $r_i = r - j$ on $t_{u-1} < i \le t_j$, where $t_{-1} = -1$ and $t_m = k$ (with $m = r_k - r_0$).

For each $i$, the interval $[f_i, f_{n-1}]$ has type $\lambda^{(i,n-1)}$ precisely when $f_{n-1}$ is a coatom of $I_i = [A^{r_i-1} f_i, f_n]$ not in $J_i = [A^{r_i} f_i, f_n]$. For $i$ with $t_{j-1} < i \le t_j$, we have $r_i = r-j$, so $I_i = [A^{r-j-1} f_i, f_n]$ and $J_i = [A^{r-j} f_i, f_n]$. The lower bound of each interval weakly increases as $i$ increases from $t_{j-1}+1$ up to $t_j$, so we have $I_{t_{j-1}+1} \subseteq \cdots \subseteq I_{t_j}$ and $J_{t_{j-1}+1} \subseteq \cdots \subseteq J_{t_j}$. Thus, the condition that $f_{n-1}$ is in all the $I_i$'s is equivalent to being in all of $I_{t_0}, I_{t_1}, \ldots, I_{t_m}$, while the condition that it is in none of the $J_i$'s is equivalent to it being in none of $J_{t_{-1}+1}, \ldots, J_{t_{m-1}+1}$.

Thus, $f_{n-1}$ is any lower cover of $f_n$ in

$$\begin{aligned}(I_{t_0} \cap \cdots \cap I_{t_m}) &\setminus (J_{t_{-1}+1} \cup \cdots \cup J_{t_{m-1}+1}) \\ &= [A^{r-0-1} f_{t_0} \vee \cdots \vee A^{r-m-1} f_{t_m}, f_n] \setminus \left( \bigcup_{j=-1}^{m-1} [A^{r-j-1} f_{t_j+1}, f_n] \right).\end{aligned}$$

Note that in all lattices, $[a,b] \cap [c,d] = [a \vee c, b \wedge d]$, but in general, $[a,b] \cup [c,d]$ is not an interval.

Now we apply inclusion-exclusion. The number of choices of $f_{n-1}$ is

$$\sum_{S \subseteq \{-1,\ldots,m-1\}} (-1)^{|S|} \cdot \#\text{ of coatoms in } \left[ \left( \bigvee_{j=0}^{m} A^{r-j-1} f_{t_j} \right) \vee \left( \bigvee_{j\in S} A^{r-j-1} f_{t_j+1} \right), f_n \right].$$

For each $j$ with $0 \le j < m$, if $j \in S$ then the term $A^{r-j-1} f_{t_j+1}$ in the right join absorbs the term $A^{r-j-1} f_{t_j}$ in the left join. Let $\alpha_j = 1$ if $j \in S$ and 0 otherwise. The sum simplifies to

$$(1) \qquad \sum_{S \subseteq \{-1,\ldots,m-1\}} (-1)^{|S|} \cdot \#\text{ of coatoms in } \left[ \bigvee_{j=-\alpha_{-1}}^{m} A^{r_0-j-1} f_{t_j+\alpha_j}, f_n \right].$$

Once we have chosen $f_{n-1}$ to satisfy the specified types of $[f_0, f_{n-1}], \ldots, [f_k, f_{n-1}]$, we can then choose $f_{n-2}$ to satisfy specified types of $[f_0, f_{n-2}], \ldots, [f_k, f_{n-2}]$ in a similar fashion, by replacing $n$ with $n-1$ throughout, and interpreting $A$ as $A_{f_n}$, which we replace with $A_{f_{n-1}}$. We could continue, choosing all of $f_n, \ldots, f_{k+1}$ to satisfy specified types with $f_0, \ldots, f_k$ (note we haven't imposed any types on intervals within $f_{k+1}, \ldots, f_n$). However, this equation is generally not computable from the partial interval type table; the terms of this equation usually depend on the specific values of $f_0, \ldots, f_k$, not just on types in the partial interval type table. When $[f_0, f_k]$ is elementary, we can compute it.

**Lemma 5.22.** *Pick $0 \le k \le n$. Consider a partial interval type table with $\lambda^{(ij)} = (1^{j-i})$ for $0 \le i \le j \le k$, and also with partitions $\lambda^{(i,n-1)}$ and $\lambda^{(in)}$ specified for $0 \le i \le k$, so that these all satisfy Theorem 5.15. Consider a partial flag $f_0 < \cdots < f_k \le f_n$ in $\mathcal{L}$ such that* $\operatorname{type}[f_i, f_n] = \lambda^{(in)}$ *for $i = 0, \ldots, k$, and $[f_0, f_k]$ is elementary. The number of extensions of $f$ by a coatom $f_{n-1}$ of $[f_k, f_n]$ such that* $\operatorname{type}[f_i, f_{n-1}] = \lambda^{(i,n-1)}$ *for $i = 0, \ldots, k$ is*

$$\begin{cases} \left\langle \lambda_r^{(kn)'} \right\rangle - \left\langle \lambda_{r+1}^{(0n)'} \right\rangle & \textit{if } r_{kn} = r_{0n}; \\ \left\langle \lambda_r^{(t_0,n)'} \right\rangle - \left\langle \lambda_r^{(t_0+1,n)'} \right\rangle & \textit{if } r_{kn} < r_{0n}, \end{cases}$$

*where $r = r_{0n}$.*

*Proof.* Since $[f_0, f_k]$ is elementary, we have $r_{0n} \ge r_{kn}$ by Theorem 5.15. First consider $r = r_{0n} = r_{kn}$. Then $m = 0$ and $t_m = k$. The number of choices of $f_{n-1}$ is

$$\#\text{ of coatoms in } [A^{r-1} f_k, \hat{1}] - \#\text{ of coatoms in } [A^r f_0 \vee A^{r-1} f_k, \hat{1}]$$

and since $[f_0, f_k]$ is elementary, we have $A f_0 \ge f_k$ so $A^r f_0 \ge A^{r-1} f_k$, so this reduces to

$$\#\text{ of coatoms in } [A^{r-1} f_k, \hat{1}] - \#\text{ of coatoms in } [A^r f_0, \hat{1}] = \left\langle \lambda_r^{(kn)'} \right\rangle - \left\langle \lambda_{r+1}^{(0n)'} \right\rangle.$$

Next, if $r = r_{0n} > r_{kn}$, then $m = 1$, and the number of choices of $f_{n-1}$ is obtained from Equation (1):

$$\sum_{S \subseteq \{-1,0\}} (-1)^{|S|} \cdot \#\text{ of coatoms in } \left[ \bigvee_{j=-\alpha_{-1}}^{1} A^{r-j-1} f_{t_j+\alpha_j}, f_n \right].$$

Now $[f_0, f_k]$ is elementary, so the subinterval $[f_{t_j+\alpha_j}, f_{t_{j+1}+\alpha_{j+1}}]$ is too. Thus

$$A^{r-j-1} f_{t_j+\alpha_j} \ge A^{r-j+1-1} f_{t_{j+1}+\alpha_{j+1}}$$

whence the $j = 1$ term of the join is always absorbed by the $j = 0$ term, so we may let $j$ run up to 0 instead of 1. Also, if $-1 \in S$, the $j = -1$ term absorbs the $j = 0$ term, so the terms for $S = \{-1, 0\}$ and $S = \{-1\}$ are equal except for sign, and hence cancel. We are then left with the terms $S = \emptyset$ and $S = \{0\}$, yielding

$$\begin{aligned} &\#\text{ of coatoms in } [A^{r-1} f_{t_0}, f_n] - \#\text{ of coatoms in } [A^{r-1} f_{t_0+1}, f_n] \\ &\qquad = \left\langle \lambda_r^{(t_0,n)'} \right\rangle - \left\langle \lambda_r^{(t_0+1,n)'} \right\rangle. \quad \square \end{aligned}$$

We now generalize Proposition 5.7.

**Proposition 5.23.** *Given partial interval type table $\boldsymbol{P}_0, \ldots, \boldsymbol{P}_k$ where $\boldsymbol{P}_{i+1} = \widetilde{\Delta} \boldsymbol{P}_i$ and $1, \ldots, k$ are all in the first column of $\boldsymbol{P}_0$, the number of flags $f$ in a q-regular semi-primary lattice with* $\operatorname{ftype} \partial^i f = \boldsymbol{P}_i$ *(for $i = 0, \ldots, k$) is*

$$\prod_{j=0}^{k-1} \Big( \langle l_1 \rangle - \langle l_2 \rangle \Big) \cdot \prod_{j=k+1}^{n} \begin{cases} \langle \operatorname{row}(\boldsymbol{P}_k, j) \rangle - \langle \operatorname{row}_1(\boldsymbol{P}_0, j) \rangle & \textit{if } \operatorname{col}(\boldsymbol{P}_k, j) = \operatorname{col}(\boldsymbol{P}_0, j); \\ \langle \operatorname{row}(\boldsymbol{P}_t, j) \rangle - \langle \operatorname{row}_1(\boldsymbol{P}_{t+1}, j) \rangle & \textit{if } \operatorname{col}(\boldsymbol{P}_k, j) < \operatorname{col}(\boldsymbol{P}_0, j), \end{cases}$$

*where for each* $j$*,*

$$\begin{aligned}
t & \quad \text{is the unique value for which } \operatorname{col}(\boldsymbol{P}_0, j) = \operatorname{col}(\boldsymbol{P}_t, j) > \operatorname{col}(\boldsymbol{P}_{t+1}, j); \\
l_1 &= \#\ \text{row-ends of } \boldsymbol{P}_n^* \text{ weakly right of } k \text{ containing any of } j^*, \ldots, (n-1)^*, \bullet; \\
l_2 &= \#\ \text{row-ends of } \boldsymbol{P}_n^* \text{ strictly right of } k \text{ containing any of } j^*, \ldots, (n-1)^*, \bullet.
\end{aligned}$$

*Proof.* Because $1, \ldots, k$ are in the first column of $\boldsymbol{P}_0$, the interval $[f_0, f_k]$ is elementary. To obtain the first product, choose $f_1, \ldots, f_k$ in the number of ways given by the dual to Theorem 4.69. To obtain the second, choose $f_{n-1}, f_{n-2}, \ldots, f_k$ in the number of ways given by Lemma 5.22. □

**5.5. Counting the number of flags achieving a full interval type table.** The previous section provides necessary but insufficient conditions for partial interval type tables to be realizable, and provides means of counting the number of flags achieving certain partial interval type tables. A conjectured extension to full interval type tables is as follows.

**Definition 5.24.** Consider an interval type table $\Lambda = (\lambda^{(ij)})_{0 \le i \le j \le n}$ of shape $\lambda \vdash n$. If there is $q_0$ depending on $\Lambda$ such that for all $q \ge q_0$, the number of flags realizing $\Lambda$ in $q$-regular semi-primary lattices is a only function of $q$ and not of the particular lattice, let $f_\Lambda(q)$ be this number of flags. If there is no such $q_0$, let $q_0 = \infty$.

**Conjecture 5.25.** *The functions* $f_\Lambda(q)$ *have the following properties.*

1. $f_\Lambda(q) \in \mathbb{Z}[q]$.
2. *As a polynomial,* $f_\Lambda(q)$ *is divisible by* $M_\lambda(q)$*, and when the polynomial is nonzero, the degree is in the range* $n(\lambda) - n_0(\lambda) \le \deg(f_\Lambda) \le n(\lambda)$*.*
3. *The maximum degree* $n(\lambda)$ *is attained exactly in the circumstance that* $\Lambda$ *is the evacuation table of a standard tableau, that is, for some standard tableau* $\boldsymbol{P}$*, we have* $\boldsymbol{P}_i = \Delta^i \boldsymbol{P}$ *for* $i = 0, \ldots, n$*.*
4. *When* $f_\Lambda(t)$ *is identically 0, no semi-primary lattice with at least* $q_0 + 1$ *atoms in each interval of type* $\boxminus$ *realizes* $\Lambda$*, including irregular semi-primary lattices.*

*Partial proof.* When (1) holds for some $\Lambda$, (2) and (3) do as well:

(2) By Theorem 4.42, the orbit of a flag under the automorphism group of certain $q$-regular semi-primary lattices has cardinality divisible by $M_\lambda(q)$. The interval type table of a flag is invariant under lattice automorphisms, so the set of flags with itype $f = \Lambda$ is a union of orbits of flags. Thus, $f_\Lambda(q)$ is divisible by $M_\lambda(q)$. The degree of $M_\lambda(q)$ is $n(\lambda) - n_0(\lambda)$, and the maximum degree of $f_\Lambda(q)$ is the degree of the number of flags of type $\boldsymbol{P}_0$, namely $n(\lambda)$.

(3) By Theorem 5.14, all but a fraction $O(q^{-1})$ of flags with ftype $f = \boldsymbol{P}$ have interval type table $\boldsymbol{P}_i = \Delta^i \boldsymbol{P}$, so given $\boldsymbol{P}_0 = \boldsymbol{P}$, only this interval type table achieves the same degree $n(\lambda)$ as $F^{\boldsymbol{P}}(q)$. □

The polynomials for all interval type tables with $n \le 8$, and for some larger cases, were computed using Mathematica by the method to be described in this section. In the computed cases, all parts of the conjecture hold, and $q_0 = 1$. A summary of the results is in Appendix A. In most of the computed cases, the polynomial is monic and is of the form, $M_\lambda(q)$ times a power of $q$ times products of falling factorials

$(q-1)\cdots(q-m)$. However, for lattices of type $(3,2,2,1)$, there is one polynomial with leading coefficient 2, and for lattices of certain other types, there are sometimes a few polynomials that do not factor in this fashion. There is a refinement of the decomposition of flags into interval type tables that decomposes these classes even further, and it is possible that these finer classes have polynomials that are of this form; see Section 8.5 for a description.

The method is as follows. Fix a partition $\lambda \vdash n$. We will consider all the possible interval type tables and the number of flags realizing them in semi-primary lattices of type $\lambda$. Let $L$ be a superset of the realizable type tables of shape $\lambda$. For the computations summarized in the appendix, $L$ was obtained by taking all interval type tables of shape $\lambda$ that satisfy Theorems 5.15 and 5.17, and whose transpose $\Lambda'$ also satisfies these. These conditions are necessary but not sufficient for a type table to be realizable; thus, all realizable type tables are in $L$, but there may be nonrealizable ones as well. A smaller collection of possibly realizable interval type tables could be obtained by finding stronger necessary conditions for a type table to be realizable, but in the cases computed, the further steps in this method weeded out all the nonrealizable ones.

For each interval type table $\Lambda$ in $L$, introduce a variable $N_\Lambda$, which will represent the number of flags achieving $\Lambda$ in a semi-primary lattice of type $\lambda$. We develop equations in the $N_\Lambda$'s from the assumption that we are working in a $q$-regular semi-primary lattice. Some of the equations will fail to hold for small $q$, but they will all hold for all $q$-regular semi-primary lattices with $q$ sufficiently large. Hence, if it happens that there are enough equations to solve for some of the unknowns $N_\Lambda$, we will have a count of how many flags achieve those $\Lambda$ depending only on $q$ and $\Lambda$, and not on any further structure of the semi-primary lattice. This $N_\Lambda$ is the $f_\Lambda(q)$ described in the conjecture. If $\Lambda$ is not realizable, then $N_\Lambda = 0$. Beyond the possible constraint on small $q$, the equations in $N_\Lambda$'s to be developed are true for all $q$-regular semi-primary lattices, even if two different semi-primary lattices with the same $q$ have different values of a particular $N_\Lambda$. It is tempting to introduce equations $N_\Lambda = N_{\Lambda'}$, but we can not, because a $q$-regular semi-primary lattice needn't be self-dual, and the proof in Proposition 5.21 that $f_\Lambda(q) = f_{\Lambda'}(q)$ has the caveat that this is true provided $f_\Lambda(q)$ exists, meaning it depends only on $q$, not the particular $q$-regular lattice. Since we do not know whether the polynomials exist, we cannot apply this.

Given a partial interval type table $\Gamma$ of shape $\lambda$, if we can count how many saturated flags achieve $\Gamma$ (for example, by applying the theorems of the previous section), we can obtain an equation in the $N_\Lambda$'s by setting this equal to the sum of $N_\Lambda$ over all extensions $\Lambda$ of $\Gamma$ in $L$. For example, given a tableau $\boldsymbol{P}$, the number of flags of type $\boldsymbol{P}$ is $F^{\boldsymbol{P}}(q)$, so

$$F^{\boldsymbol{P}}(q) = \sum_{\{\,\Lambda \in L \;:\; \boldsymbol{P}_0 = \boldsymbol{P}\,\}} N_\Lambda .$$

We now develop equations relating flags that differ in only one element. Choose $k$ with $0 < k < n$. Let $\Gamma$ be a partial interval type table in which $\lambda^{(ij)}$ is defined for all $0 \le i \le j \le n$ unless $i = k$ or $j = k$, and in which $\lambda^{(k-1,k+1)} = \begin{smallmatrix}\square\\\square\end{smallmatrix}$. It is sometimes possible to generate equations involving all extensions of $\Gamma$ to full interval type tables.

**Definition 5.26.** Two saturated flags $f$ and $f'$ are **$k$-adjacent** if $f_i = f_i'$ whenever $i \neq k$. The relation of being $k$-adjacent is an equivalence relation on the set of all flags in a lattice. Each $k$-adjacency class of flags has a partial interval type table $\Gamma$ associated to it, namely $\gamma^{(ij)} = \operatorname{type}[f_i, f_j]$ for $i \neq k$ and $j \neq k$. Two interval type tables $(\lambda^{(ij)})$ and $(\gamma^{(ij)})$ in $L$ are **$k$-adjacent** if $\lambda^{(ij)} = \gamma^{(ij)}$ whenever $i \neq k$ and $j \neq k$, and each $k$-adjacency class of interval type tables has a partial interval type table $\Gamma$ associated to it.

A **$k$-partial flag** is a flag $(f_0 \lessdot \cdots \lessdot f_{k-1} < f_{k+1} \lessdot \cdots \lessdot f_n)$ where $\rho(f_i) = i$ for $i \neq k$. A **$k$-partial interval type table** is a partial interval type table $\Gamma = (\gamma^{(ij)})$ defined for all $0 \leq i \leq j \leq n$ with $i \neq k$ and $j \neq k$. Let $f$ be a $k$-partial flag with $\operatorname{type}[f_{k-1}, f_{k+1}] = \begin{smallmatrix}\square\\\square\end{smallmatrix}$. Let $\Gamma$ be the $k$-partial interval type table of $f$. Let $\Lambda_1, \ldots, \Lambda_m$ be the $k$-adjacency class of $\Gamma$ in $L$. For some (but not all) $\Gamma$, the number of extensions of $f$ to a saturated flag achieving each of $\Lambda_1, \ldots, \Lambda_m$ is the same for all $f$ realizing $\Gamma$. If for all $f$ realizing $\Gamma$, exactly $n_t$ extensions of $f$ have interval type table $\Lambda_t$ for $t = 1, \ldots, m$, we have the equation $N_{\Lambda_1}/n_1 = \cdots = N_{\Lambda_m}/n_m$.

**Lemma 5.27.** *Fix $k$. Let $f$ be a $k$-partial flag with $\operatorname{type}[f_{k-1}, f_{k+1}] = \begin{smallmatrix}\square\\\square\end{smallmatrix}$. Let $0 \leq i < k$ and set $\alpha = \operatorname{type}[f_i, f_{k-1}]$ and $\gamma = \operatorname{type}[f_i, f_{k+1}]$.*

(1) *$\gamma/\alpha$ is not a horizontal brick. It consists of two squares, $x$ and $y$, with $\operatorname{row}(x) < \operatorname{row}(y)$ and $\operatorname{col}(x) \geq \operatorname{col}(y)$.*
(2) *Choose $f_k \in (f_{k-1}, f_{k+1})$ and let $\beta = \operatorname{type}[f_i, f_k]$. If $x$ and $y$ are vertically adjacent then $\beta \gtrdot_x \alpha$. Otherwise, all but one $f_k \in (f_{k-1}, f_{k+1})$ yield $\beta \gtrdot_x \alpha$, and the unique $f_k$ with $\beta \gtrdot_y \alpha$ is $f_k = f_{k-1} \vee A^{\operatorname{col}(y)}_{f_{k+1}} f_i$.*

*Now let $k < i \leq n$. Similar statements hold with $\alpha = \operatorname{type}[f_{k+1}, f_i]$, $\beta = \operatorname{type}[f_k, f_i]$ and $\gamma = \operatorname{type}[f_{k-1}, f_i]$, and the unique $f_k$ with $\beta \gtrdot_y \alpha$ is $f_k = f_{k+1} \wedge C^{\operatorname{col}(y)}_{f_{k-1}} f_i$.*

*Proof.* First we consider $0 \leq i < k$. If we extend $f$ to a saturated flag by choosing some $f_k \in (f_{k-1}, f_{k+1})$, what can $\beta = \operatorname{type}[f_i, f_k]$ be? Let the two squares of $\gamma/\alpha$ be denoted $x$ and $y$. Either $\beta \gtrdot_x \alpha$, or $\beta \gtrdot_y \alpha$. By Corollary 4.72, if $\gamma/\alpha$ were a horizontal brick, we would have $\operatorname{type}[f_{k-1}, f_{k+1}] = \square\square$, so it is not a horizontal brick. So assume $\operatorname{row}(x) < \operatorname{row}(y)$ (and hence $\operatorname{col}(x) \geq \operatorname{col}(y)$). If $\gamma/\alpha$ is a vertical brick, then all saturated extensions of $f$ by a choice of $f_k$ have $\beta \gtrdot_x \alpha$. If $\gamma/\alpha$ is not a brick, exactly one choice of $f_k$ has $\beta \gtrdot_y \alpha$, namely $f_k = f_{k-1} \vee A^{\operatorname{col}(y)}_{f_{k+1}} f_i$. For suppose $\operatorname{type} f_k \gtrdot_y \alpha$. Let $z = A^{\operatorname{col}(y)}_{f_{k+1}} f_i$. Since $\operatorname{type} f_k$ and $\operatorname{type} f_{k+1}$ agree in the first $\operatorname{col}(y)$ columns, $f_k \geq z$; and since $\operatorname{type}[f_i, f_{k-1}]$ and $\operatorname{type}[f_i, f_{k+1}]$ do not agree in these columns, $f_{k-1} \not\geq z$. Since $f_k \geq f_{k-1}$, we get $f_k \geq f_{k-1} \vee z$. Since $[f_{k-1}, f_{k+1}]$ is elementary, we may, by Theorem 4.62 or Corollary 3.10, compute

$$\operatorname{type}[f_i, f_{k-1} \vee A^{\operatorname{col}(y)}_{f_{k+1}} f_i] = \operatorname{type}[f_i, f_{k-1}] \vee \mathcal{A}^{\operatorname{col}(y)}_{\gamma} \emptyset = \alpha \oplus y.$$

Next, let $k < i \leq n$ and $\alpha = \operatorname{type}[f_{k+1}, f_i]$ and $\gamma = \operatorname{type}[f_{k-1}, f_i]$. If we extend $f$ to a saturated flag by choosing some $f_k \in (f_{k-1}, f_{k-1})$, what can $\beta = \operatorname{type}[f_i, f_k]$ be? Again, $\gamma/\alpha$ consists of two squares $x$ and $y$ with $\operatorname{row}(x) < \operatorname{row}(y)$. If $\operatorname{col}(x) = \operatorname{col}(y)$ then all such extensions have $\beta \gtrdot_x \alpha$, while if $\operatorname{col}(y) < \operatorname{col}(x)$, there is exactly one extension with $\beta \gtrdot_y \alpha$, namely $f_k = f_{k+1} \wedge C^{\operatorname{col}(y)}_{f_{k-1}} f_i$, and all other extensions have $\beta \gtrdot_x \alpha$. $\square$

**Definition 5.28.** Let $\Lambda$ be a interval type table, and fix $k = 1, \ldots, n-1$ and $i \neq k$. When $0 \leq i < k$, say $i$ is **specializable** when $k$ and $k+1$ are not in adjacent cells in $\boldsymbol{P}_i$; that $\Lambda$ is **$i$-special** when $\operatorname{col}(\boldsymbol{P}_i, k) < \operatorname{col}(\boldsymbol{P}_i, k+1)$; and that $\Lambda$ is **$i$-generic** when $\operatorname{col}(\boldsymbol{P}_i, k) \geq \operatorname{col}(\boldsymbol{P}_i, k+1)$. Dually, when $n \geq i > k$, say $i$ is specializable when $n-k$ and $n-k+1$ are not in adjacent cells in $\boldsymbol{P}_i^*$; that $\Lambda$ is $i$-special when $\operatorname{col}(\boldsymbol{P}_i^*, n-k) < \operatorname{col}(\boldsymbol{P}_i^*, n-k+1)$; and that $\Lambda$ is $i$-generic when $\operatorname{col}(\boldsymbol{P}_i^*, n-k) \geq \operatorname{col}(\boldsymbol{P}_i^*, n-k+1)$. In the notation of the preceding lemma, $i$ is specializable when $\gamma/\alpha$ is not a brick, while $\Lambda$ is $i$-special when $\beta \gtrdot_y \alpha$ and is $i$-generic when $\beta \gtrdot_x \alpha$.

A $k$-partial interval type table has at most one extension that is $i$-generic for all $i$; such an extension is called **generic** and is denoted $\Lambda_0$.

See Figure 5 for an example.

We consider several possibilities about the distribution of $i$-special and $i$-generic extensions of $\Gamma$. Throughout, $\Gamma$ and $k$ are fixed, and $\gamma^{(k-1,k+1)} = \begin{smallmatrix}\square\\\square\end{smallmatrix}$. In terms of the tableau representation of an interval type table, $\boldsymbol{P}_{k-1}$ has $\begin{array}{|c|c|}\hline k & k+1 \\ \hline\end{array}$ in its upper left corner iff $\gamma^{(k-1,k+1)} = \square\square$, while it has $\begin{array}{|c|}\hline k \\ \hline k+1 \\ \hline\end{array}$ in its upper left corner iff $\gamma^{(k-1,k+1)} = \begin{smallmatrix}\square\\\square\end{smallmatrix}$.

**Proposition 5.29.** *Let $i \neq k$, and $L$ be a list of interval type tables of shape $\lambda$ that includes all realizable ones, and possibly some unrealizable ones. Consider the $k$-adjacency class of $\Gamma$ in $L$. The following hold for all semi-primary lattices (including irregular ones).*

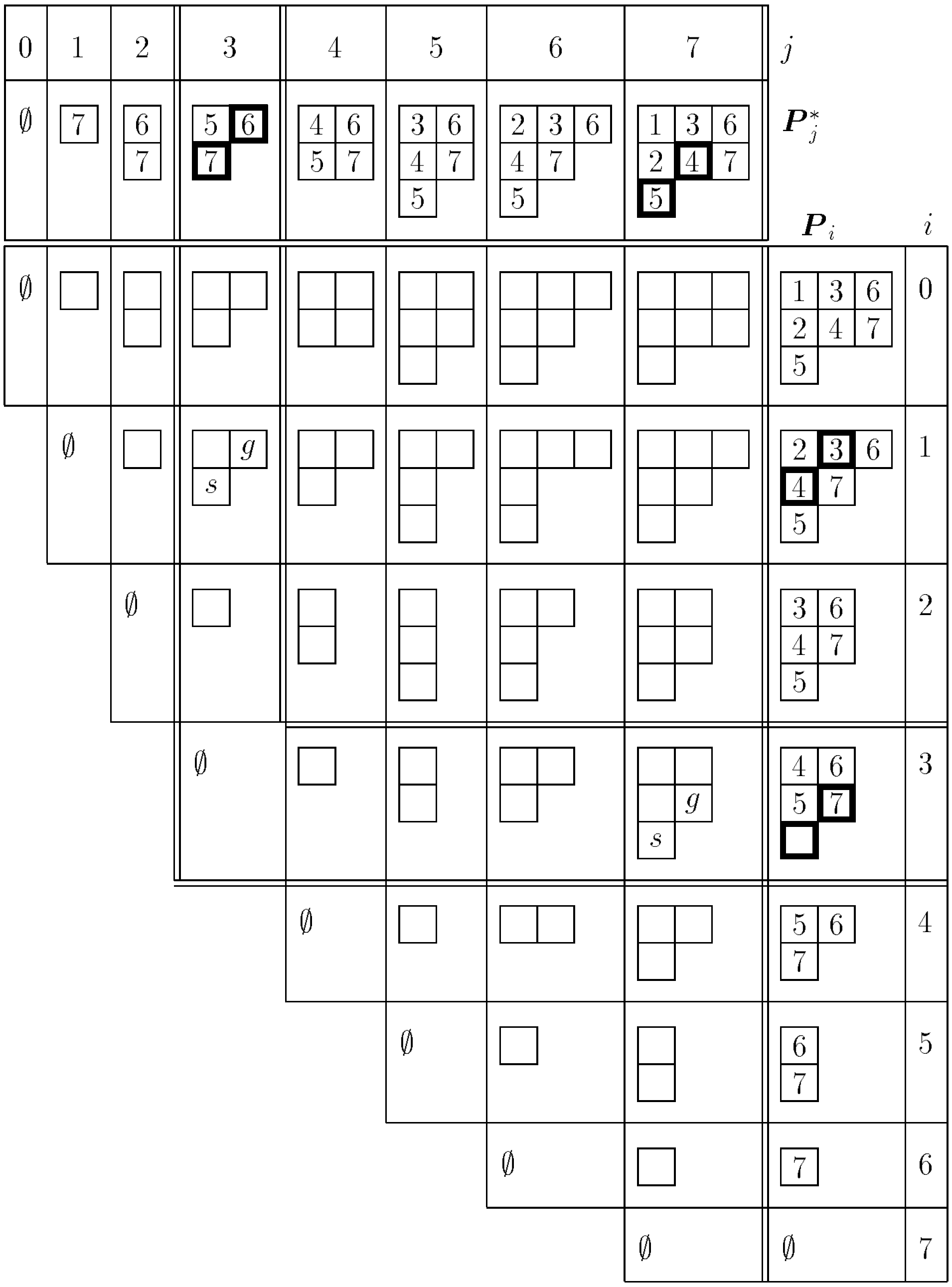
(1) *If $i$ is specializable but there are no $i$-special extensions of $\Gamma$ in $L$, then all extensions of $\Gamma$ in $L$ are nonrealizable, and so should be removed from $L$.*
(2) *If no extensions of $\Gamma$ are $i$-generic, then again, no extensions of $\Gamma$ in $L$ are realizable, and so all such extensions should be removed.*

*Proof.*

(1) If $\Gamma$ is realized by flag $f$, then by Lemma 5.27, $f_k$ can be replaced so as to achieve an $i$-special extension of $\Gamma$. Since $L$ is a superset of all realizable interval type tables, $\operatorname{itype} f$ is in $L$. Thus, if $L$ has no $i$-special extensions of $\Gamma$, no flag realizes $\Gamma$, so no flag realizes any extension of $\Gamma$, and all extensions of $\Gamma$ can be removed from $L$.
(2) If $\Gamma$ is realized by a flag $f$, the interval $[f_{k-1}, f_{k+1}]$ has type $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, so it has more than one element on the middle level, but at most one element yields an $i$-special interval type table, so all the others must yield an $i$-generic interval type table. $\square$

Let $\Lambda_0$ be the unique extension of $\Gamma$ that is $i$-generic for all $i \neq k$. Given any $k$-partial flag $f$ realizing $\Gamma$, at most $n-2$ choices of $f_k \in (f_{k-1}, f_{k+1})$ are $i$-special for some $i$'s, and the rest are $i$-generic for all $i$; thus, all but at most $n-2$ elements in $(f_{k-1}, f_{k+1})$ achieve $\Lambda_0$. The bound $n-2$ can be lowered on a case by case basis, because sometimes, the special elements for different $i$ coincide. We now show that if any extensions of $\Gamma$ are realizable, then $\Lambda_0$ is realizable, except in pathological cases.

**Pathological case.** Suppose the $k$-adjacency class of $\Gamma$ does not contain $\Lambda_0$, but does contain an $i$-special extension of $\Gamma$ for each specializable $i$, and an $i$-generic extension of $\Gamma$ for every $i \neq k$. Then it is conceivable that some $q$-regular lattice



Both 1 and 7 are specializable. Choose the square marked $g$ for a generic partition, or the one marked $s$ for a special partition. For the tableaux, the generic value is shown, and the special value is obtained by swapping the contents of the highlighted cells. When $i < k$ is specializable, there are two possibilities for $\boldsymbol{P}_i$; if there are $c$ specializable values of $i < k$, there are $2^c$ possibilities for $\boldsymbol{P}_k$; and in all other cases, there is only one value of $\boldsymbol{P}_i$. Dual statements hold for $\boldsymbol{P}^*_j$. It turns out that all four combinations of special and generic choices in this example are realizable.

FIGURE 5. An interval type table adjacency class with $k = 3$.

with $q \leq n-2$ exists in which $\Gamma$ is realizable. It is also conceivable that an irregular lattice exists in which some interval $[f_{k-1}, f_{k+1}]$ of type $\boxminus$ happens to have fewer than $n$ atoms, and that $\Gamma$ is realized with this $f_{k-1}$ and $f_{k+1}$.

**Usual case.** For all $q > n-2$, if $\Gamma$ is realizable, it is necessary that $\Lambda_0$ be realizable, so if $\Lambda_0 \notin L$, the class is not realizable. So assume $\Lambda_0$ is realizable. Then for each specializable $i$, there is a unique $i$-special extension of each $k$-partial flag realizing $\Gamma$. However, the $i$-special extensions of two different flags realizing $\Gamma$ do not have to have the same interval type tables, as in Example 5.36 at the end of this section. If the $i$-special extensions of all flags realizing $\Gamma$ happen to have the same interval type table, we can generate an equation in the $N_\Lambda$'s.

**Theorem 5.30.** *Suppose the $k$-adjacency class of $\Gamma$ in $L$ has $\Lambda_0$, and has exactly one $i$-special member for each $i \neq k$ (these can coincide for different $i$). Let the class be $\Lambda_0, \ldots, \Lambda_m$. If $\mathcal{L}$ is $q$-regular with $q \geq m$, then*

$$\frac{N_{\Lambda_0}}{q+1-m} = N_{\Lambda_1} = \cdots = N_{\Lambda_m};$$

*if $q = m-1$ then $N_{\Lambda_0} = 0$ while $N_{\Lambda_1} = \cdots = N_{\Lambda_m}$, and if $q < m-1$, then $N_{\Lambda_0} = \cdots = N_{\Lambda_m} = 0$.*

*Proof.* Each of $\Lambda_1, \ldots, \Lambda_m$ may be $i$-special for multiple values of $i$, but by assumption, no two of them are special for the same value of $i$. Then every $k$-partial flag $f$ realizing $\Gamma$ has one extension realizing each of $\Lambda_1, \ldots, \Lambda_m$, and the remaining extensions realize $\Lambda_0$. If the lattice is $q$-regular, then $q+1-m$ extensions realize $\Lambda_0$. The equation for $q \geq m$ and the $q = m-1$ case both follow.

If $q+1 < m$, we see there are not enough elements in $(f_{k-1}, f_{k+1})$ to realize each of $\Lambda_1, \ldots, \Lambda_m$, but if any extension of $\Gamma$ can be realized, all of these can. Thus, none of them are realizable, so $N_{\Lambda_0} = \cdots = N_{\Lambda_m} = 0$. □

*Note.*

1. This theorem is true even when none of $\Lambda_0, \ldots, \Lambda_m$ are realizable; if they are not realizable, $N_{\Lambda_0} = \cdots = N_{\Lambda_m} = 0$, which is consistent with all the cases in the theorem.
2. It is possible that we may usually not need to consider $q < m$ as a special case; in the known cases, summarized in Appendix A, the $N_\Lambda$'s are polynomials in $q$, and the polynomials in this theorem have $N_{\Lambda_0}$ divisible by $(q-1)\cdots(q-m+1)$ and $N_{\Lambda_1}, \ldots, N_{\Lambda_m}$ divisible by $(q-1)\cdots(q-m+2)$, so they vanish when required without introducing $q < m$ as a special case in the equations.

We will work out some examples, according to the following procedure.

**Procedure 5.31.** Fix a partition $\lambda$.

1. Create a superset $L$ of all realizable interval type tables of shape $\lambda$. This can be done by taking all interval type tables of shape $\lambda$ which satisfy Theorems 5.15 and 5.17, and whose transpose also satisfies these.
2. For each $k = 1, \ldots, n-1$, consider the interval type tables from step (1) that satisfy $\lambda^{(k-1,k+1)} = \boxminus$ (or equivalently, the interval type tables for which $\boldsymbol{P}_{k-1}$ has $\begin{array}{|c|}\hline k \\ \hline k+1 \\ \hline\end{array}$ in the upper left corner). Split these into $k$-adjacency

classes: two of these interval type tables are $k$-adjacent provided their respective $\boldsymbol{P}_0, \ldots, \boldsymbol{P}_{k-1}$ agree in all cells except possibly have some $k$ and $(k+1)$'s reversed, and their respective $\boldsymbol{P}_{k+1}, \ldots, \boldsymbol{P}_n$ are equal. They may differ on $\boldsymbol{P}_k$.

(3) For each $k$-adjacency class formed in step (2), consider
   (a) Does the class contain an $i$-generic and at least one $i$-special interval type table for each specializable $i$? If not, then by Proposition 5.29, none of the interval type tables in this class ever occur, so they should be removed from $L$.
   (b) Does the class contain a generic interval type table? If not, the interval type tables in it never occur for sufficiently large $q$, but may occur for small $q$ in certain pathological cases.
   (c) Does the class contain exactly one $i$-special interval type table for each specializable $i$? If so, denote the generic interval type table by $\Lambda_0$ and the others by $\Lambda_1, \ldots, \Lambda_r$. We then obtain $N_0/(q+1-r) = N_1 = \ldots = N_r$ by Theorem 5.30.
   (d) When there is a generic interval type table; at least one $i$-special table for each specializable $i$; and more than one for some $i$, we do nothing with the class.

(4) Combine the equations in step (3c) from different adjacency classes to express as many different $N_\Lambda$'s as possible as a rational function of $q$ times a particular $N_\Lambda$.

(5) When $\boldsymbol{P}$ is the superstandard tableau of shape $\lambda$, all flags of type $\boldsymbol{P}$ have a hereditary decomposition by Theorem 4.70. Thus the interval type tables of all such flags are the same, say $\tilde{\Lambda}$. We thus have $N_{\tilde{\Lambda}} = F^{\boldsymbol{P}}(q)$, and now all other $N_\Lambda$'s that can be expressed as multiples of $N_{\tilde{\Lambda}}$ can be computed as well.

(6) For most $\lambda$ listed in Appendix A, all $N_\Lambda$ are computed by the previous step. In most of the computed cases, even when a $\Lambda$ is in a class of form (3d) for some $k$, it usually is also in a class of form (3c) for other $k$'s, so there are equations in it. However, there are cases when $\Lambda$ only occurs in adjacency classes of form (3d), and cases when the equations generated by (3c) do not express all $N_\Lambda$ as multiples of $N_{\tilde{\Lambda}}$. Additional equations can sometimes be generated to deal with this; see Example 5.36.

**Theorem 5.32.** *In a semi-primary lattice of type $(n)$, there is only one flag, and it has interval type table $\lambda^{(ij)} = (j-i)$. In a semi-primary lattice of type $(1^n)$, there are $\langle n\rangle!$ flags, all with interval type table $\lambda^{(ij)} = (j-i)$.*

*Proof.* For either $\lambda = (n)$ or $\lambda = (1^n)$, there is only one subpartition of $\lambda$ of each rank, so the type tables are as indicated. For $\lambda = (n)$, the only flag is $f_i = \hat{1}[i]$ (for $0 \le i \le n$). For $\lambda = (1^n)$, we set $f_n = \hat{1}$, and choose $f_{i-1}$ to be a coatom of $[\hat{0}, f_i]$ in one of $\langle i\rangle$ ways, for $i = n, n-1, \ldots, 1$. $\square$

**Example 5.33.** Let $\lambda = (2,1)$. The interval type tables satisfying $\boldsymbol{P}_{i+1} = \widetilde{\Delta}\boldsymbol{P}_i$ are as follows.

| | | | | |
|---|---|---|---|---|
| $\Lambda_1 = \Lambda_2'$ | $\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ | $\begin{array}{\|c\|}\hline 2\\\hline 3\\\hline\end{array}$ | $\boxed{3}$ | $\emptyset$ |
| $\Lambda_2 = \Lambda_1'$ | $\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ | $\begin{array}{\|c\|c\|}\hline 2&3\\\hline\end{array}$ | $\boxed{3}$ | $\emptyset$ |
| $\Lambda_3 = \Lambda_3'$ | $\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ | $\begin{array}{\|c\|}\hline 2\\\hline 3\\\hline\end{array}$ | $\boxed{3}$ | $\emptyset$ |

Let $N_i = N_{\Lambda_i}$ denote the number of flags with interval type table $\Lambda_i$. For each $k = 1,\ldots,n-1$, we partition the interval type tables with $\lambda^{(k-1,k+1)} = \begin{array}{|c|}\hline \ \\\hline \ \\\hline\end{array}$ into $k$-adjacency classes, and apply Theorem 5.30 when possible to compute the ratios of the $N_i$'s within each class.

First consider $k=1$. The interval type tables with $\boldsymbol{P}_0$ having $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$ in the upper left corner are $\Lambda_2$ and $\Lambda_3$. These are $k$-adjacent to each other because they only differ in $\boldsymbol{P}_k$. The partition $\lambda^{(13)}$ is $\begin{array}{|c|c|}\hline \ &\ \\\hline\end{array}$ in $\Lambda_2$ and $\begin{array}{|c|}\hline \ \\\hline \ \\\hline\end{array}$ in $\Lambda_3$, so $\Lambda_2$ is generic while $\Lambda_3$ is 3-special. Thus the ratio $N_2 : N_3$ is $(q+1)-1 : 1 = q : 1$.

Next consider $k=2$. The interval type tables with $\boldsymbol{P}_1$ having $\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}$ in the upper left corner are $\Lambda_1$ and $\Lambda_3$. These differ only in the positions of 2's and 3's, so they are 2-adjacent. Note that $\Lambda_1$ is generic while $\Lambda_3$ is 0-special, so the ratio $N_1 : N_3$ is $(q+1)-1 : 1 = q : 1$.

Thus, the ratio $N_1 : N_2 : N_3$ is $q : q : 1$. All flags of type $\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ have type table $\Lambda_1$, and by Theorem 4.38, the number of flags of this type is

$$\Big(\langle 2\rangle - \langle 1\rangle\Big)\cdot\Big(\langle 1\rangle - \langle 0\rangle\Big)\cdot\Big(\langle 1\rangle - \langle 0\rangle\Big) = q.$$

Thus $N_1 = N_2 = q$ and $N_3 = 1$.

**Example 5.34.** Let $\lambda = (2,2)$. The interval type tables satisfying $\boldsymbol{P}_{i+1} = \widetilde{\Delta}\boldsymbol{P}_i$ are as follows. Again, let $N_i$ be the number of flags with type table $\Lambda_i$.

| | | | | | |
|---|---|---|---|---|---|
| $\Lambda_1 = \Lambda_1'$ | $\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3&4\\\hline\end{array}$ | $\begin{array}{\|c\|c\|}\hline 2&4\\\hline 3\\\cline{1-1}\end{array}$ | $\begin{array}{\|c\|c\|}\hline 3&4\\\hline\end{array}$ | $\boxed{4}$ | $\emptyset$ |
| $\Lambda_2 = \Lambda_4'$ | $\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3&4\\\hline\end{array}$ | $\begin{array}{\|c\|c\|}\hline 2&4\\\hline 3\\\cline{1-1}\end{array}$ | $\begin{array}{\|c\|}\hline 3\\\hline 4\\\hline\end{array}$ | $\boxed{4}$ | $\emptyset$ |
| $\Lambda_3 = \Lambda_3'$ | $\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2&4\\\hline\end{array}$ | $\begin{array}{\|c\|c\|}\hline 2&3\\\hline 4\\\cline{1-1}\end{array}$ | $\begin{array}{\|c\|}\hline 3\\\hline 4\\\hline\end{array}$ | $\boxed{4}$ | $\emptyset$ |
| $\Lambda_4 = \Lambda_2'$ | $\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2&4\\\hline\end{array}$ | $\begin{array}{\|c\|c\|}\hline 2&4\\\hline 3\\\cline{1-1}\end{array}$ | $\begin{array}{\|c\|c\|}\hline 3&4\\\hline\end{array}$ | $\boxed{4}$ | $\emptyset$ |
| $\Lambda_5 = \Lambda_5'$ | $\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2&4\\\hline\end{array}$ | $\begin{array}{\|c\|c\|}\hline 2&4\\\hline 3\\\cline{1-1}\end{array}$ | $\begin{array}{\|c\|}\hline 3\\\hline 4\\\hline\end{array}$ | $\boxed{4}$ | $\emptyset$ |

If

$$\boldsymbol{P}_0 = \begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau} = \begin{ytableau} 1 & 2 \end{ytableau} + \begin{ytableau} 3 & 4 \end{ytableau}$$

then $\boldsymbol{P}_2 = \begin{ytableau} 3 & 4 \end{ytableau}$ by Theorem 4.93, so $\Lambda_2$ is not realizable. Also, if

$$\boldsymbol{P}_0 = \begin{ytableau} 1 & 3 \\ 2 & 4 \end{ytableau} = \begin{ytableau} 1 \\ 2 \end{ytableau} + \begin{ytableau} 3 \\ 4 \end{ytableau}$$

then $\boldsymbol{P}_2 = \begin{ytableau} 3 \\ 4 \end{ytableau}$, so $\Lambda_4$ is not realizable. Thus $N_2 = N_4 = 0$.

Next, we partition the remaining interval type tables into $k$-adjacency classes, and apply Theorem 5.30. For $k = 1$ we have $N_3 : N_5 = q : 1$; for $k = 2$ we have $N_1 : N_5 = q : 1$; for $k = 3$ we have $N_3 : N_5 = q : 1$. So $N_1 : N_3 : N_5 = q : q : 1$. Also, $N_1$ is the total number of flags of type $\begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau}$, which is

$$\Big(\langle 2\rangle - \langle 0\rangle\Big) \cdot \Big(\langle 2\rangle - \langle 1\rangle\Big) \cdot \Big(\langle 1\rangle - \langle 0\rangle\Big) \cdot \Big(\langle 1\rangle - \langle 0\rangle\Big) = (q+1)q,$$

so $N_1 = N_3 = q(q+1)$ and $N_5 = q+1$.

**Example 5.35.** Let $\lambda = (2,2,1)$. We list all interval type tables satisfying the various theorems listed in step (1) of our procedure, but omit the constant values $\boldsymbol{P}_4 = \begin{ytableau} 5 \end{ytableau}$ and $\boldsymbol{P}_5 = \emptyset$ from each.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\Lambda_1{=}\Lambda'_6$ | $\begin{ytableau} 1 & 2 \\ 3 & 4 \\ 5 \end{ytableau}$ | $\begin{ytableau} 2 & 4 \\ 3 \\ 5 \end{ytableau}$ | $\begin{ytableau} 3 & 4 \\ 5 \end{ytableau}$ | $\begin{ytableau} 4 \\ 5 \end{ytableau}$ | $\Lambda_9{=}\Lambda'_9$ | $\begin{ytableau} 1 & 3 \\ 2 & 5 \\ 4 \end{ytableau}$ | $\begin{ytableau} 2 & 5 \\ 3 \\ 4 \end{ytableau}$ | $\begin{ytableau} 3 & 5 \\ 4 \end{ytableau}$ | $\begin{ytableau} 4 \\ 5 \end{ytableau}$ |
| $\Lambda_2{=}\Lambda'_2$ | $\begin{ytableau} 1 & 2 \\ 3 & 5 \\ 4 \end{ytableau}$ | $\begin{ytableau} 2 & 5 \\ 3 \\ 4 \end{ytableau}$ | $\begin{ytableau} 3 & 5 \\ 4 \end{ytableau}$ | $\begin{ytableau} 4 & 5 \end{ytableau}$ | $\Lambda_{10}{=}\Lambda'_{15}$ | $\begin{ytableau} 1 & 3 \\ 2 & 5 \\ 4 \end{ytableau}$ | $\begin{ytableau} 2 & 5 \\ 3 \\ 4 \end{ytableau}$ | $\begin{ytableau} 3 \\ 4 \\ 5 \end{ytableau}$ | $\begin{ytableau} 4 \\ 5 \end{ytableau}$ |
| $\Lambda_3{=}\Lambda'_8$ | $\begin{ytableau} 1 & 2 \\ 3 & 5 \\ 4 \end{ytableau}$ | $\begin{ytableau} 2 & 5 \\ 3 \\ 4 \end{ytableau}$ | $\begin{ytableau} 3 & 5 \\ 4 \end{ytableau}$ | $\begin{ytableau} 4 \\ 5 \end{ytableau}$ | $\Lambda_{11}{=}\Lambda'_4$ | $\begin{ytableau} 1 & 4 \\ 2 & 5 \\ 3 \end{ytableau}$ | $\begin{ytableau} 2 & 4 \\ 3 & 5 \end{ytableau}$ | $\begin{ytableau} 3 & 4 \\ 5 \end{ytableau}$ | $\begin{ytableau} 4 \\ 5 \end{ytableau}$ |
| $\Lambda_4{=}\Lambda'_{11}$ | $\begin{ytableau} 1 & 3 \\ 2 & 4 \\ 5 \end{ytableau}$ | $\begin{ytableau} 2 & 3 \\ 4 \\ 5 \end{ytableau}$ | $\begin{ytableau} 3 \\ 4 \\ 5 \end{ytableau}$ | $\begin{ytableau} 4 \\ 5 \end{ytableau}$ | $\Lambda_{12}{=}\Lambda'_5$ | $\begin{ytableau} 1 & 4 \\ 2 & 5 \\ 3 \end{ytableau}$ | $\begin{ytableau} 2 & 4 \\ 3 & 5 \end{ytableau}$ | $\begin{ytableau} 3 & 5 \\ 4 \end{ytableau}$ | $\begin{ytableau} 4 \\ 5 \end{ytableau}$ |
| $\Lambda_5{=}\Lambda'_{12}$ | $\begin{ytableau} 1 & 3 \\ 2 & 4 \\ 5 \end{ytableau}$ | $\begin{ytableau} 2 & 4 \\ 3 \\ 5 \end{ytableau}$ | $\begin{ytableau} 3 \\ 4 \\ 5 \end{ytableau}$ | $\begin{ytableau} 4 \\ 5 \end{ytableau}$ | $\Lambda_{13}{=}\Lambda'_7$ | $\begin{ytableau} 1 & 4 \\ 2 & 5 \\ 3 \end{ytableau}$ | $\begin{ytableau} 2 & 4 \\ 3 \\ 5 \end{ytableau}$ | $\begin{ytableau} 3 & 4 \\ 5 \end{ytableau}$ | $\begin{ytableau} 4 \\ 5 \end{ytableau}$ |
| $\Lambda_6{=}\Lambda'_1$ | $\begin{ytableau} 1 & 3 \\ 2 & 5 \\ 4 \end{ytableau}$ | $\begin{ytableau} 2 & 3 \\ 4 & 5 \end{ytableau}$ | $\begin{ytableau} 3 & 5 \\ 4 \end{ytableau}$ | $\begin{ytableau} 4 & 5 \end{ytableau}$ | $\Lambda_{14}{=}\Lambda'_{14}$ | $\begin{ytableau} 1 & 4 \\ 2 & 5 \\ 3 \end{ytableau}$ | $\begin{ytableau} 2 & 4 \\ 3 \\ 5 \end{ytableau}$ | $\begin{ytableau} 3 \\ 4 \\ 5 \end{ytableau}$ | $\begin{ytableau} 4 \\ 5 \end{ytableau}$ |
| $\Lambda_7{=}\Lambda'_{13}$ | $\begin{ytableau} 1 & 3 \\ 2 & 5 \\ 4 \end{ytableau}$ | $\begin{ytableau} 2 & 3 \\ 4 \\ 5 \end{ytableau}$ | $\begin{ytableau} 3 \\ 4 \\ 5 \end{ytableau}$ | $\begin{ytableau} 4 \\ 5 \end{ytableau}$ | $\Lambda_{15}{=}\Lambda'_{10}$ | $\begin{ytableau} 1 & 4 \\ 2 & 5 \\ 3 \end{ytableau}$ | $\begin{ytableau} 2 & 5 \\ 3 \\ 4 \end{ytableau}$ | $\begin{ytableau} 3 & 5 \\ 4 \end{ytableau}$ | $\begin{ytableau} 4 \\ 5 \end{ytableau}$ |
| $\Lambda_8{=}\Lambda'_3$ | $\begin{ytableau} 1 & 3 \\ 2 & 5 \\ 4 \end{ytableau}$ | $\begin{ytableau} 2 & 5 \\ 3 \\ 4 \end{ytableau}$ | $\begin{ytableau} 3 & 5 \\ 4 \end{ytableau}$ | $\begin{ytableau} 4 & 5 \end{ytableau}$ | $\Lambda_{16}{=}\Lambda'_{16}$ | $\begin{ytableau} 1 & 4 \\ 2 & 5 \\ 3 \end{ytableau}$ | $\begin{ytableau} 2 & 5 \\ 3 \\ 4 \end{ytableau}$ | $\begin{ytableau} 3 \\ 4 \\ 5 \end{ytableau}$ | $\begin{ytableau} 4 \\ 5 \end{ytableau}$ |

Next, for each $k = 1, 2, 3, 4$, we form the $k$-adjacency classes. In the table below, $\Lambda_i$ is abbreviated $i$. The column $\square\square$ lists those $\Lambda_i$ for which $\lambda^{(k-1,k+1)} = \square\square$ (so these $\Lambda_i$ are in singleton $k$-adjacency classes). The generic member of each class is listed first. The classes listed in "Bad classes" either don't have a generic member, or don't have precisely one $i$-special member for each specializable $i$. The classes listed in "Good classes" have a generic member, and for each specializable $i$, precisely one member is $i$-special. In general, for $i_1 \neq i_2$, we could have the $i_1$-special and $i_2$-special members being different, but this example is sufficiently small that the same nongeneric member is $i$-special for all specializable $i$.

| $k$ | $\square\square$ | Bad classes | Good classes |
|---|---|---|---|
| 1 | $1, 2, 3$ | $(9)$ | $(4, 5), (6, 8), (7, 10), (11, 13), (12, 15), (14, 16)$ |
| 2 | $4, 6, 7$ | $(3, 9, 10)$ | $(1, 5), (2, 8), (11, 12), (13, 14), (15, 16)$ |
| 3 | $1, 11, 13$ | $(8, 9, 15)$ | $(2, 3), (4, 5), (6, 12), (7, 14), (10, 16)$ |
| 4 | $2, 6, 8$ | $(9)$ | $(1, 3), (4, 7), (5, 10), (11, 12), (13, 15), (14, 16)$ |

First we explain the bad classes. For $k = 2$, both $\Lambda_9$ and $\Lambda_{10}$ are 0-specializations of $\Lambda_3$. Since there is more than one 0-specialization, we cannot compute anything from this class. For $k = 3$, both $\Lambda_9$ and $\Lambda_{15}$ are 5-specializations of $\Lambda_8$, so again, we cannot compute anything from this class.

For $k = 4$, the generic interval type table that corresponds to $\Lambda_9$ is obtained from $\Lambda_9$ by examining each $\boldsymbol{P}_i$ with $i < 4$ and swapping 4 and 5 if 4 is strictly below and left of 5. In general, we might also have to modify $\boldsymbol{P}_4$, but $\boldsymbol{P}_4 = \boxed{5}$ is forced in this case. So the generic interval type table corresponding to $\Lambda_9$ is

$$\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 2&4\\\hline 3\\\cline{1-1} 5\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 3&4\\\hline 5\\\cline{1-1}\end{array}\quad\begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}\quad\begin{array}{|c|}\hline 5\\\hline\end{array}\quad\emptyset$$

which is not in the list of interval type tables that passed all the necessity tests of step (1) of the procedure; specifically, since $\boldsymbol{P}_0 = \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 4\\\hline 5\\\hline\end{array}$, we must have $\boldsymbol{P}_2 = \begin{array}{|c|}\hline 3\\\hline 4\\\hline 5\\\hline\end{array}$ by Theorem 5.17. So the class with 9 does not have a generic interval type table, and hence we cannot use it to compute any ratios.

The transpose of $\Lambda_9$ is itself $\Lambda_9$, and the trouble with $k = 1$ is the transpose of the $k = 4$ problem.

We can actually show that $\Lambda_9$ is never realized in any semi-primary lattice, including irregular ones. The 4-partial interval type table corresponding to $\Lambda_9$ is

$$\Gamma \;=\; \begin{array}{|c|c|}\hline 1&3\\\hline 2&?\\\hline ?\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 2&?\\\hline 3\\\cline{1-1} ?\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 3&?\\\hline ?\\\cline{1-1}\end{array}\quad\begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}\quad\begin{array}{|c|}\hline 5\\\hline\end{array}\quad\emptyset$$

The only interval type table in the list that extends $\Gamma$ is $\Lambda_9$. Thus, there is no 0, 1, or 2-generic extension of $\Gamma$, so by Proposition 5.29, no extension of $\Gamma$ is realizable in any semi-primary lattice. Thus, $N_9 = 0$, and $\Lambda_9$ should be removed from the list. Once it is removed, $(3, 10)$ becomes a good class for $k = 2$ and $(8, 15)$ becomes a good class for $k = 3$, but we will see that we have enough equations to solve for all the $N_i$ without these two good classes.

The good classes all have the form $(x, y)$ where $x$ is a generic interval type table and $y$ is a special one. Since there is just one special interval type table in each class, we obtain $N_x = q \cdot N_y$. By looking at the last class for each $k$, we see that 16 happens always to be special, so we can relate the ratios with $N_{16}$ for the different $k$'s, obtaining $N_{10} = N_{14} = N_{15} = q \cdot N_{16}$. We next see that 10, 14, 15 occur as special elements in conjunction with generic elements 5, 7, 12, 13 for various $k$, so we get $N_5 = N_7 = N_{12} = N_{13} = q \cdot N_{10} = q^2 \cdot N_{16}$. Next, 5, 7, 12, 13, occur as special elements in conjunction with generic elements 1, 4, 6, 11, so $N_1 = N_4 = N_6 = N_{11} = q \cdot N_5 = q^3 \cdot N_{16}$. From $(1, 3)$, we obtain $N_3 = N_1/q = q^2 N_{16}$, and from $(6, 8)$ we obtain $N_8 = N_6/q = q^2 N_{16}$. From $(2, 3)$ we obtain $N_2 = q \cdot N_3 = q^3 N_{16}$, and now we have expressed all but $N_9$ as multiples of $N_{16}$.

Next we note that all flags of type $\boldsymbol{P} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1} \end{array}$ have interval type table $\Lambda_1$, since this tableau is superstandard. Thus, $N_1 = F^{\boldsymbol{P}}(q) = q^3(q + 1)$, so $N_1 = N_2 = N_4 = N_6 = N_{11} = q^3(q + 1)$; $N_3 = N_5 = N_7 = N_8 = N_{12} = N_{13} = q^2(q + 1)$; $N_{10} = N_{14} = N_{15} = q(q + 1)$; and $N_{16} = q + 1$.

With larger $\lambda$, we will generally find that the good classes have different numbers of elements in them, so that the ratios are not always of the form $N_0 = qN_1$, but more generally of the form $N_0/(q + 1 - r) = N_1 = \cdots = N_r$. Also, in this example we were able to express all the $N_i$ as multiples of a single $N_i$, because the graph determined by $\{x, y\}$ is an edge iff $x$ and $y$ are in a good class together, has a single component. In general, the graph may have multiple components, and within each component, we can express the ratios of the $N_i$'s by Theorem 5.30, but then we need additional equations to completely solve the problem. For the lattice types computed in Appendix A, the graph had multiple components for types $(3, 3, 1)$, $(3, 3, 2)$, $(3, 3, 3)$, $(4, 2, 2)$, $(4, 3, 1)$, $(3, 2, 2, 1)$, $(3, 3, 1, 1)$, and $(4, 2, 1, 1)$; there was one large component with almost all of the type tables, and several small components with just a handful of type tables. The large component contained the type table for the superstandard tableau of shape $\lambda$, so most values of $N_\Lambda$ could be computed as described above. We illustrate how to choose additional equations by considering $(3, 3, 1)$.

**Example 5.36.** Let $\lambda = (3, 3, 1)$ and perform Procedure 5.31 to determine all the interval type tables and the values of $N_\Lambda$. At the end, it turns out there is a single type table, $\Lambda_0$ below, which has neither been shown to be realizable or not realizable, and which always occurs in adjacency classes of the form (3d) of the procedure. The

$k = 3$ adjacency class of $\Lambda_0$ is as follows, and is also shown in Figure 5.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\Lambda_0$ | $\begin{smallmatrix}1&3&6\\2&4&7\\5\end{smallmatrix}$ | $\begin{smallmatrix}2&3&6\\4&7\\5\end{smallmatrix}$ | $\begin{smallmatrix}3&6\\4&7\\5\end{smallmatrix}$ | $\begin{smallmatrix}4&6\\5&7\end{smallmatrix}$ | $\begin{smallmatrix}5&6\\7\end{smallmatrix}$ | $\begin{smallmatrix}6\\7\end{smallmatrix}$ | $\begin{smallmatrix}7\end{smallmatrix}$ | $\emptyset$ | generic |
| $\Lambda_1$ | $\begin{smallmatrix}1&3&6\\2&4&7\\5\end{smallmatrix}$ | $\begin{smallmatrix}2&4&6\\3&7\\5\end{smallmatrix}$ | $\begin{smallmatrix}3&6\\4&7\\5\end{smallmatrix}$ | $\begin{smallmatrix}4&6\\5&7\end{smallmatrix}$ | $\begin{smallmatrix}5&6\\7\end{smallmatrix}$ | $\begin{smallmatrix}6\\7\end{smallmatrix}$ | $\begin{smallmatrix}7\end{smallmatrix}$ | $\emptyset$ | 1-special |
| $\Lambda_2$ | $\begin{smallmatrix}1&3&6\\2&4&7\\5\end{smallmatrix}$ | $\begin{smallmatrix}2&3&6\\4&7\\5\end{smallmatrix}$ | $\begin{smallmatrix}3&6\\4&7\\5\end{smallmatrix}$ | $\begin{smallmatrix}4&6\\5\\7\end{smallmatrix}$ | $\begin{smallmatrix}5&6\\7\end{smallmatrix}$ | $\begin{smallmatrix}6\\7\end{smallmatrix}$ | $\begin{smallmatrix}7\end{smallmatrix}$ | $\emptyset$ | 7-special |
| $\Lambda_3$ | $\begin{smallmatrix}1&3&6\\2&4&7\\5\end{smallmatrix}$ | $\begin{smallmatrix}2&4&6\\3&7\\5\end{smallmatrix}$ | $\begin{smallmatrix}3&6\\4&7\\5\end{smallmatrix}$ | $\begin{smallmatrix}4&6\\5\\7\end{smallmatrix}$ | $\begin{smallmatrix}5&6\\7\end{smallmatrix}$ | $\begin{smallmatrix}6\\7\end{smallmatrix}$ | $\begin{smallmatrix}7\end{smallmatrix}$ | $\emptyset$ | 1,7-special |

The values of $N_\Lambda$ are known from the procedure for all $\Lambda$ of shape $(3,3,1)$ except $\Lambda_0$. Here are two different ways to find $N_{\Lambda_0}$.

(1) The number of flags of type $\boldsymbol{P} = \begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2&4&7\\\hline 5\\\cline{1-1}\end{array}$ is $F^{\boldsymbol{P}}(q)$. In the equation

$$F^{\boldsymbol{P}}(q) = \sum_{\{\,\Lambda \in L \;:\; \boldsymbol{P}_0 = \boldsymbol{P}\,\}} N_\Lambda,$$

the only unknown is $N_{\Lambda_0}$, so

$$N_{\Lambda_0} = F^{\boldsymbol{P}}(q) - \sum_{\{\,\Lambda \in L \;:\; \boldsymbol{P}_0 = \boldsymbol{P} \text{ and } \Lambda \neq \Lambda_0\,\}} N_\Lambda.$$

This is similar to step (5) of the procedure.

(2) A 3-partial flag either has one extension to $\Lambda_3$ and $q$ to $\Lambda_0$, or one to $\Lambda_1$, one to $\Lambda_2$, and $q-1$ to $\Lambda_0$. Thus $N_{\Lambda_1} = N_{\Lambda_2}$ and

$$N_{\Lambda_0} = qN_{\Lambda_3} + (q-1)N_{\Lambda_1}.$$

This is similar to step (3c) of the procedure.

For the other exceptional partitions listed before the example, there were sometimes several interval type tables not in the large component of the graph. In all such cases, the first method above was used to find some $N_\Lambda$ within each component, and then the ratio of this to other $N_\Lambda$ within the component was used to find the remaining ones.

*Note.* Hesselink [**10**] analyzes necessary and sufficient conditions for the realizability of "typricies," which are equivalent to interval type tables, but with a different formalism. He has several interesting pathological examples, on which some light may be shed by our enumerative approach.

(1) In [**10**, §5.7,§9.3], he gives typricies that are realized in all invariant subspace lattices over fields of characteristic greater than 2, but not in fields of characteristic 2. At least one of these is not a pathological case in our formalism, but rather, an interval type table whose polynomial $f_\Lambda(q)$ is divisible by $q-2$.

The polynomial for the example in [**10**, §5.7], where $\lambda = (4,2)$, turns out to be $(q-1)(q-2)$. The example in [**10**, §9.3] has $n = 21$, which is much larger than the polynomials that were computed with Mathematica.

(2) Also in [**10**, §9.3], there is a typrix that is only realized in fields of characteristic 2. In our treatment, it is a pathological case, and may correspond to a polynomial $f_\Lambda(q) = 0$ when $q \geq q_0 = 3$, provided other $q$-regular semi-primary lattices with $q \geq 3$ also do not realize it. Again, $n = 21$, so it was too large to compute.

(3) Finally, in [**10**, §9.4], there is a typrix that is realizable in an invariant subspace lattice over the field $\mathbb{C}$ but not $\mathbb{R}$, and a similar construction for certain other pairs of fields in [**10**, §9.5–9.6]. These correspond to the "escape clause" $q_0 = \infty$ of Definition 5.24. In the graph relating the ratios of $N_\Lambda$'s, described above Example 5.36, such an interval type table is not in the main component, or in any other component where the polynomials are known to exist.

### 5.6. Semi-primary flags indexed by set partitions.

**Definition 5.37.** A **partition** $\pi$ of a set $S$ is a set $\{B_1, \ldots, B_k\}$ of nonempty subsets of $S$, with $B_1 \cup \cdots \cup B_k = S$ and $B_i \cap B_j = \emptyset$ when $i \neq j$. An **ordered partition** $\pi$ of $S$ is a tuple $(B_1, \ldots, B_k)$ of subsets (which may be empty) of $S$ whose union is $S$ and whose pairwise intersections are $\emptyset$. In both cases, each $B_i$ is called a **block** of $\pi$.

Partitions have been well studied; see, for example, [**27**].

**Definition 5.38.** The **type** of a set partition $\pi = \{B_1, \ldots, B_k\}$ is the integer partition

$$(\#B_1, \ldots, \#B_k),$$

where we arrange the blocks so these numbers are in weakly decreasing order. For an $n$-element set, the number of partitions of type $\lambda$ is

$$c_\lambda = \frac{n!}{\lambda_1!\lambda_2!\cdots m_1!m_2!\cdots}.$$

The total number of partitions of an $n$-element set is the $n$**th Bell number** $B(n)$.

The **type** of an ordered partition $\pi = (B_1, \ldots, B_k)$ is the weak composition

$$(\#B_1, \ldots, \#B_k).$$

For a weak composition $\lambda \vdash n$, the number of ordered partitions of type $\lambda$ of an $n$-set is the multinomial coefficient

$$\binom{n}{\lambda_1, \lambda_2, \ldots} = \frac{n!}{\lambda_1!\lambda_2!\cdots}.$$

**Definition 5.39.** A **standard (skew) composition tableau** is a row strict tableau $\boldsymbol{P}$ of (skew) weak composition shape, with distinct entries. Almost equivalently, it is a multichain $(\lambda^{(l)} \leq \cdots \leq \lambda^{(h)})$ of weak compositions. When entry $j$ is in row $i$ of $\boldsymbol{P}$, the composition $\lambda^{(j)}$ is obtained from $\lambda^{(j-1)}$ by adding a square to row $i$, and when $j$ does not appear, $\lambda^{(j)} = \lambda^{(j-1)}$. The shape of $\boldsymbol{P}$ is $\lambda^{(h)}/\lambda^{(l)}$, which is skew when $\lambda^{(l)} \neq \emptyset$. The parameters $l = \mathrm{low}(\boldsymbol{P})$ and $h = \mathrm{high}(\boldsymbol{P})$ are a minor additional structure we require that is not ordinarily present in a tableau.

We encode the ordered partition $(B_1, \ldots, B_k)$ by a standard composition tableau $\boldsymbol{P}$ of shape $\lambda$ whose $i$th row has the entries of $B_i$ written in increasing order from left to right.

*Notation.* For a composition $\lambda$, the partition obtained by sorting the parts into decreasing order is called the **straightening** of the composition, and is denoted $S(\lambda)$. For compositions $\lambda$ and $\mu$, write $\lambda \equiv \mu$ to mean $S(\lambda) = S(\mu)$. The straighening of a skew composition tableau $\boldsymbol{P} = (\lambda^{(l)} \leq \cdots \leq \lambda^{(h)})$ is the skew Young tableau $S(\boldsymbol{P}) = (S(\lambda^{(l)}), \ldots, S(\lambda^{(h)}))$. For composition tableaux, write $\boldsymbol{P} \equiv \boldsymbol{Q}$ to mean $S(\boldsymbol{P}) = S(\boldsymbol{Q})$.

**Definition 5.40.** Let $\boldsymbol{P} = (\lambda^{(l)} \leq \cdots \leq \lambda^{(h)})$ be any standard composition tableau, with $l = \mathrm{low}(\boldsymbol{P})$ and $h = \mathrm{high}(\boldsymbol{P})$. The **interval type table** of $\boldsymbol{P}$ is

$$\mathrm{itype}\,\boldsymbol{P} = \left(S(\lambda^{(j)} - \lambda^{(i)})\right)_{l \leq i \leq j \leq h}.$$

Two standard composition tableau have the same interval type table iff they encode the same unordered set partition, that is, if we permute the order of the rows of one to obtain the other. Thus, the numbers of these interval type tables are enumerated by $c_\lambda$ and $B(n)$.

These interval type tables were introduced by Hesselink [**10**, p. 103], under the name "elementary typrices," and with a different formalism. He showed that all such interval type tables are realized in all invariant subspace lattices of shape $S(\mathrm{sh}\,\boldsymbol{P})$. We will enumerate how many flags realize each one in $q$-regular semi-primary lattices.

**Definition 5.41.** Let $\mathcal{L}$ be a semi-primary lattice, $\vec{x}$ be a join decomposition of $\hat{1}$, and $\boldsymbol{P} = (\lambda^{(l)}, \ldots, \lambda^{(h)})$ be a skew composition tableau, with $\mathrm{osh}\,\boldsymbol{P} = (\rho(x_1), \rho(x_2), \ldots) \equiv \mathrm{type}\,\mathcal{L}$. The flag $f(\boldsymbol{P}, \vec{x})$ with components $f_i = \vec{x}[[\lambda^{(i)}]]$ for $i = l, \ldots, h$ is a **hereditary flag**.

These flags are interesting because all (multisaturated) flags in a lattice that is a product of chains are hereditary flags; all flags in a semi-primary lattice whose type has two columns are hereditary flags; and the computed data in Appendix A shows empirically that at least for small $n$, the interval type tables of hereditary flags form a large class of the realized interval type tables.

**Proposition 5.42.** *Let $\vec{x}$ be a decomposition of $\hat{1}$ into independent join-irreducibles and $\boldsymbol{P}$ be a standard composition tableau, with* $\mathrm{sh}\,\boldsymbol{P} = (\rho(x_1), \rho(x_2), \ldots)$. *Then* $\mathrm{itype}\,\boldsymbol{P} = \mathrm{itype}\, f(\boldsymbol{P}, \vec{x})$.

*Proof.* This follows from Corollary 4.53. □

**Definition 5.43.** A **special composition tableau** is a standard skew composition tableau of shape $\lambda/\mu$ with $\mu = \lambda[k]$ for some $k$; in other words, no entries occur in columns $1, \ldots, k$, and no $\bullet$'s appear in columns $k+1, k+2, \ldots$.

**Definition 5.44.** Let $f = (f_l, \ldots, f_h)$ be a flag. Recall that $\partial f = (f_{l+1}, \ldots, f_h)$ and $df = (f_l, \ldots, f_{h-1})$. Define $Af = (A_{f_h} f_l, \ldots, A_{f_h} f_h)$. For an integer $k = l, \ldots, h$, define $A_k f = (A_{f_k} f_l, \ldots, A_{f_k} f_k) = Ad^{h-k} f$.

**Definition 5.45.** For a skew composition tableau $\boldsymbol{P} = (\lambda^{(l)}, \ldots, \lambda^{(h)})$, define $d\boldsymbol{P} = (\lambda^{(l)}, \ldots, \lambda^{(h-1)})$, $\mathcal{C}\boldsymbol{P} = (\mathcal{C}\lambda^{(l)}, \ldots, \mathcal{C}\lambda^{(h)})$ and $\mathcal{A}\boldsymbol{P} = (\mathcal{A}_\lambda\lambda^{(l)}, \ldots, \mathcal{A}_\lambda\lambda^{(h)})$, where $\lambda = \lambda^{(h)}$. For $k = l, \ldots, h$, define $\mathcal{A}_k\boldsymbol{P} = \mathcal{A}d^{h-k}\boldsymbol{P} = (\mathcal{A}_\lambda\lambda^{(l)}, \ldots, \mathcal{A}_\lambda\lambda^{(k)})$ where $\lambda = \lambda^{(k)}$. The tableau $d\boldsymbol{P}$ deletes all $h$'s from $\boldsymbol{P}$. The tableau $\mathcal{C}\boldsymbol{P}$ is $\boldsymbol{P}$ with the first column deleted. The tableau $\mathcal{A}\boldsymbol{P}$ is obtained from $\boldsymbol{P}$ by performing the following operation on each nonempty row: if there are no numeric entries, the row stays intact, and if there are, insert $\bullet$ at the left, slide the row one cell right, and delete the rightmost entry. For $\mathcal{A}_k\boldsymbol{P}$, first delete all cells whose entries are larger than $k$, and evaluate $\mathcal{A}$ on the tableau so obtained. Note that if $\boldsymbol{P}$ is special, then so are $\mathcal{C}\boldsymbol{P}$, $\mathcal{A}\boldsymbol{P}$, and $\mathcal{A}_k\boldsymbol{P}$.

**Example 5.46.**

$$\boldsymbol{P} = \begin{matrix} 2 & 5 & 6 & \\ 1 & 4 & 7 & 8 \\ 3 & & & \end{matrix} \qquad \mathcal{C}\boldsymbol{P} = \begin{matrix} 5 & 6 & \\ 4 & 7 & 8 \end{matrix} \qquad \mathcal{A}\boldsymbol{P} = \begin{matrix} \bullet & 2 & 5 & \\ \bullet & 1 & 4 & 7 \\ \bullet & & & \end{matrix} \qquad \mathcal{A}_5\boldsymbol{P} = \begin{matrix} \bullet & 2 \\ \bullet & 1 \\ \bullet & \end{matrix}$$

$$d\boldsymbol{P} = \begin{matrix} 2 & 5 & 6 \\ 1 & 4 & 7 \\ 3 & & \end{matrix} \quad \text{if } h = 8, \qquad \text{or} \qquad d\boldsymbol{P} = \begin{matrix} 2 & 5 & 6 & \\ 1 & 4 & 7 & 8 \\ 3 & & & \end{matrix} \quad \text{if } h > 8.$$

Since our tableaux are actually chains of compositions rather than fillings of subsets of $\mathbb{P} \times \mathbb{P}$ with entries, there is an ambiguity in evaluating $d\boldsymbol{P}$ and some other operators. There is no ambiguity in the chain form, but in the pictorial form, the hidden values of $\operatorname{low}(\boldsymbol{P})$ and $\operatorname{high}(\boldsymbol{P})$ must be tracked through careful bookkeeping. Note $\mathcal{C}$ and $\mathcal{A}$ affect neither the low nor high value, and $d$ lowers the high value by 1. Be very careful in evaluating compositions of these operators pictorially: since $\operatorname{high}(\boldsymbol{P}) = \operatorname{high}(\mathcal{A}\boldsymbol{P})$, but $\operatorname{high}(\boldsymbol{P}) \notin \mathcal{A}\boldsymbol{P}$, the composition $d\mathcal{A}\boldsymbol{P}$ has the same pictorial appearance as $\mathcal{A}\boldsymbol{P}$, rather than removing the highest occurring entry in it. The chain form of these two are different, because $\mathcal{A}\boldsymbol{P}$ has one more term than $d\mathcal{A}\boldsymbol{P}$.

**Proposition 5.47.** *If* $\boldsymbol{P}$ *is a special composition tableau,* $f$ *is a flag, and* $\operatorname{itype} f = \operatorname{itype} \boldsymbol{P}$*, then* $\operatorname{itype} Af = \operatorname{itype} \mathcal{A}\boldsymbol{P}$.

*Proof.* We may assume that $\boldsymbol{P}$ is a tableau of ordinary composition shape: if $\operatorname{sh}\boldsymbol{P} = \lambda/\mu$ with $\mu = \lambda[k]$, then apply this proposition to the flag $f$ and tableau $\mathcal{C}^k\boldsymbol{P}$, and note that $\operatorname{itype}\boldsymbol{P} = \operatorname{itype}\mathcal{C}^k\boldsymbol{P}$ and $\operatorname{itype}\mathcal{A}\boldsymbol{P} = \operatorname{itype}\mathcal{A}\mathcal{C}^k\boldsymbol{P}$.

Without loss of generality, the flag is indexed $f = (f_0, \ldots, f_n)$, and since $A$ will only be applied to flags whose top element is $f_n$, yielding the operator $A_{f_n}$, we assume $f_n = \hat{1}$. We induct on the length of the flag. For flags of length 0 or 1 it is trivially true. Now consider any flag $f$ with $\operatorname{itype} f = \operatorname{itype}\boldsymbol{P}$ for some $\boldsymbol{P} = (\lambda^{(0)}, \ldots, \lambda^{(n)})$, and assume the theorem holds for all flags of smaller length. Let $\lambda = \lambda^{(n)}$. Let $\boldsymbol{Q} = (\lambda^{(j)} - \lambda^{(1)})_{1 \le j \le n}$ and $\nu = \lambda^{(n)} - \lambda^{(1)}$. The smaller flag $\partial f$ has $\operatorname{itype}\partial f = (\lambda^{(j)} - \lambda^{(i)})_{1 \le i \le j \le n} = \operatorname{itype}\boldsymbol{Q}$, so inductively applying the proposition to $\partial f$,

$$\begin{aligned} \operatorname{itype} A\partial f &= \operatorname{itype}\mathcal{A}\boldsymbol{Q} \\ &\equiv \Big(\mathcal{A}_\nu(\lambda^{(j)} - \lambda^{(1)}) - \mathcal{A}_\nu(\lambda^{(i)} - \lambda^{(1)})\Big)_{1 \le i \le j \le n} \\ &= \Big(\mathcal{A}_\lambda\lambda^{(j)} - \mathcal{A}_\lambda\lambda^{(i)}\Big)_{1 \le i \le j \le n}. \end{aligned}$$

It remains to show that $\operatorname{ftype} Af \equiv (\mathcal{A}_\lambda\lambda^{(j)} - \mathcal{A}_\lambda\lambda^{(0)})_{0\leq j\leq n}$, which is equivalent to showing $\operatorname{ftype} Af \equiv \mathcal{C}\mathcal{A}\boldsymbol{P}$ because $\lambda^{(0)} = \emptyset$ so that $\mathcal{A}_\lambda\lambda^{(0)}$ is the first column of $\mathcal{A}_\lambda\lambda^{(j)}$ for all $j$.

If $Af_0 = Af_1$ then $\operatorname{ftype} Af$ and $\operatorname{ftype} A\partial f$ agree (prepend $\emptyset$ to the latter as the 0th term to obtain the former). When this happens, $\operatorname{type}[f_0, f_n] \equiv \lambda^{(n)} - \lambda^{(0)}$ and $\operatorname{type}[f_1, f_n] \equiv \lambda^{(n)} - \lambda^{(1)}$ are either equal, so $1 \notin \boldsymbol{P}$, or differ in their first column, so 1 is the only entry in its row of $\boldsymbol{P}$.

So assume 1 is in $\boldsymbol{P}$ on a row of length at least two; without loss of generality, it is the first row.

By restricting the flag, $\operatorname{itype} \partial f = \operatorname{itype} \boldsymbol{Q}$, and inductively, $\operatorname{itype} A\partial f = \operatorname{itype} \mathcal{A}\boldsymbol{Q}$. Consider the following ordinary Young tableaux:

$$\begin{array}{lcll} \boldsymbol{P}^{(1)} & = & \operatorname{ftype} f & \equiv \boldsymbol{P} \\ \boldsymbol{P}^{(2)} & = & \operatorname{ftype} \partial f & \equiv \boldsymbol{Q} \\ \boldsymbol{P}^{(3)} & = & \operatorname{ftype} A\partial f & \equiv \mathcal{C}\mathcal{A}\boldsymbol{Q} \\ \boldsymbol{P}^{(4)} & = & \operatorname{ftype} Af & \stackrel{?}{\equiv} \mathcal{C}\mathcal{A}\boldsymbol{P} \end{array}$$

A standard Young tableau is uniquely determined by specifying its entries and their columns (though not all such specifications yield standard Young tableau). The entries in $\boldsymbol{P}^{(4)}$ are the entries of $\boldsymbol{P}^{(3)}$ and a 1, because in the two flags $A\partial f = (Af_1 \leq f_2 \leq \ldots \leq Af_n)$ and $Af = (Af_0 \lessdot Af_1 \leq Af_2 \leq \ldots \leq Af_n)$, the cover relations and equalities are the same, except for the addition of $Af_0 \lessdot Af_1$; an entry $e$ occurs in the tableau when $Af_{e-1} \lessdot Af_e$. The proposed value of $\boldsymbol{P}^{(4)}$ has the correct entries, and has the correct shape because $\operatorname{type}[Af_0, Af_n] = \mathcal{C}(\operatorname{type}\mathcal{L}) \equiv \mathcal{C}\lambda$.

Let $e^{(m)} = \operatorname{col}(\boldsymbol{P}^{(m)}, e)$, and $a_e = 1$ if $\operatorname{row}(\boldsymbol{P}, e) = 1$ and $a_e = 0$ otherwise. The entry 1 does not occur in $\boldsymbol{P}^{(2)}$, and for other entries $e$ of $\boldsymbol{P}^{(1)}$, we have $e^{(2)} = e^{(1)} - a_e$.

By inductively applying the proposition to $\partial f$, we have $\operatorname{ftype} A\partial f = \mathcal{C}\mathcal{A}\boldsymbol{P}$, so $e^{(3)} = e^{(2)}$ when $1 \neq e \in \mathcal{A}\boldsymbol{P}$, and no other values of $e$ occur in $\boldsymbol{P}^{(3)}$.

Since $\partial Af = A\partial f$ holds for all flags of positive length, $\boldsymbol{P}^{(3)} = \operatorname{ftype} \partial Af = \widetilde{\Delta}\boldsymbol{P}^{(4)}$. See Section 5.2 for a description of $\widetilde{\Delta}$. One way to describe the tableau game $\boldsymbol{P}^{(3)} = \widetilde{\Delta}\boldsymbol{P}^{(4)}$ is that the columns of all entries stay the same, except for some sequence of entries $e_1 < e_2 < \ldots < e_k$, where $e_i$ is in column $i$ of $\boldsymbol{P}^{(4)}$ for $1 \leq i \leq k$; in column $i-1$ of $\boldsymbol{P}^{(3)}$ for $2 \leq i \leq k$; and $e_1 = 1$ isn't in $\boldsymbol{P}^{(3)}$. The column in which $\operatorname{sh}\boldsymbol{P}^{(4)} = \operatorname{cotype} Af_0$ and $\operatorname{sh}\boldsymbol{P}^{(3)} = \operatorname{cotype} Af_1$ differ is $\lambda_1 - 1$, so $k = \lambda_1 - 1$. Let $b_e = 1$ if $e \in \{e_2, \ldots, e_k\}$ and 0 otherwise. Then $e^{(4)} = e^{(3)} + b_e$ when $1 < e \in \mathcal{A}\boldsymbol{P}$, and $1^{(4)} = 1$. So $e^{(4)} = e^{(1)} - a_e + b_e$ when $1 < e \in \mathcal{A}\boldsymbol{P}$.

The tableaux $\boldsymbol{P}^{(4)}$ and $\mathcal{A}\boldsymbol{P}$ have the same entries. Let $e \in \mathcal{A}\boldsymbol{P}$. Then

$$\begin{aligned} e^{(4)} &= \min\{\, r : A^r(Af_0) \wedge Af_e > A^r(Af_0) \wedge Af_{e-1} \,\} \\ &= \min\left\{ r : A(A^r\hat{0} \wedge f_e) > A(A^r\hat{0} \wedge f_{e-1}) \right\} \\ &\geq \min\left\{ r : A^r\hat{0} \wedge f_e > A^r\hat{0} \wedge f_{e-1} \right\} \\ &= e^{(1)} \end{aligned}$$

Thus, for $e > 1$, we must have $b_e = 1$ whenever $a_e = 1$. Since exactly $k$ values of $e > 1$ have $a_e = 1$ and the same number have $b_e = 1$, this determines the values of all $a_e$ and $b_e$, so $e^{(4)} = e^{(1)}$ for all $e \in \mathcal{A}\boldsymbol{P}$, whence $\boldsymbol{P}^{(4)} \equiv \mathcal{C}\mathcal{A}\boldsymbol{P}$. $\square$

**Proposition 5.48.** *Suppose $(f, \boldsymbol{P})$ satisfies $f$ is a multisaturated flag; $\boldsymbol{P}$ is a special composition tableau;* $\operatorname{ftype}_{\hat{0}} f \equiv \boldsymbol{P}$*; and* $\operatorname{itype} f = \operatorname{itype} \boldsymbol{P}$*. Then $(df, d\boldsymbol{P})$ and $(Af, \mathcal{A}\boldsymbol{P})$ do too.*

*Proof.* Let $l = \operatorname{low}(f) = \operatorname{low}(\boldsymbol{P})$ and $h = \operatorname{high}(f) = \operatorname{high}(\boldsymbol{P})$. If $f$ is a multisaturated flag, so are $df$ and $Af$. We always have $\operatorname{ftype}_{\hat{0}} df = d(\operatorname{ftype}_{\hat{0}} f)$, and we delete column $h$ of the arrays $\operatorname{itype} f = \operatorname{itype} \boldsymbol{P}$ to obtain $\operatorname{itype} df = \operatorname{itype} d\boldsymbol{P}$.

By the preceding proposition, $\operatorname{itype} Af = \operatorname{itype} \mathcal{A}\boldsymbol{P}$. Let $A = A_{f_h}$. Let $k$ be the maximum component of $\lambda^{(l)}$. Since $\operatorname{ftype}_{\hat{0}} f \equiv \boldsymbol{P}$ where $\boldsymbol{P}$ is a special composition tableau, we have $f_l = f_h[k]$, so $Af_l = f_h[k+1]$. Also, $\operatorname{type}[\hat{0}, f_i] = \operatorname{type} f_l + \operatorname{type}[f_l, f_i]$ and $\operatorname{type}[\hat{0}, Af_i] = \operatorname{type} Af_l + \operatorname{type}[Af_l, Af_i]$ for each $i = l, \ldots, h$. So we form $\operatorname{ftype}_{\hat{0}} Af$ by prepending the partition $\operatorname{type} Af_l$ to the left side of the tableau $\operatorname{ftype} Af$. This partition is formed from $\boldsymbol{P}$ by taking its outer shape, restricting each row to length $k+1$, and sorting the lengths into decreasing order. The result agrees with $\mathcal{A}\boldsymbol{P}$. $\square$

**Definition 5.49.** Let $A = \{a_1 < \cdots < a_r\}$ and $B = \{b_1 < \cdots < b_s\}$ be sets of integers, with either $A = B$ or with $A$ and $B$ disjoint. When $A = B$, let $\operatorname{dom}(A, A) = 0$. When $A \cap B = \emptyset$, let $\operatorname{dom}(A, B)$ be the largest number $t \le \min\{r, s\}$ for which $a_{r-h} > b_{t-h}$ for $h = 0, \ldots, t-1$, i.e., the last $t$ entries of $A$ are consecutively greater than the first $t$ entries of $B$. Clearly all numbers less than $t$ have this property and no number larger than $t$ does, for if $t'$ possesses this property and we replace $t'$ by $t'-1$, the right side of each inequality is lowered (and the last inequality is dropped). Finally, for a standard composition tableau $\boldsymbol{P}$ whose set of entries on the $i$th row is $B_i$, let $\operatorname{dom}(\boldsymbol{P}) = \sum_{i,j} \operatorname{dom}(B_i, B_j)$.

**Example 5.50.**

$$\boldsymbol{P} = \begin{array}{|c|c|c|c|} \hline 1 & 5 & 7 & 10 \\ \hline 2 & 8 & 9 \\ \cline{1-3} 3 \\ \cline{1-1} 4 & 6 & 11 \\ \cline{1-3} \end{array}$$

dom$(B_i, B_j)$

| $i \backslash j$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0 | 2 | 1 | 2 |
| 2 | 3 | 0 | 1 | 2 |
| 3 | 1 | 1 | 0 | 0 |
| 4 | 3 | 2 | 1 | 0 |

The sum of all $\operatorname{dom}(B_i, B_j)$ is $\operatorname{dom}(\boldsymbol{P}) = 19$. For instance, the entry $\operatorname{dom}(B_1, B_4) = 2$ is computed as follows.

$$\begin{array}{cccc} \multicolumn{4}{c}{\operatorname{dom} = 3 \text{ fails}} \\ 1 & 5 & 7 & 10 \\ & \vee & \vee & \not\vee \\ & 4 & 6 & 11 \end{array} \qquad \begin{array}{ccccc} \multicolumn{5}{c}{\operatorname{dom} = 2 \text{ works}} \\ 1 & 5 & 7 & 10 & \\ & & \vee & \vee & \\ & & 4 & 6 & 11 \end{array}$$

**Lemma 5.51.** *Let $\boldsymbol{P}$ be a standard composition tableau, and the set of entries on the $j$th row be $B_j$. Fix an integer $k$, and denote the entries on the $k$th row $B_k = \{a_1 < \cdots < a_r\}$. Let $\mu$ be the composition $\mu_j = \operatorname{dom}(B_k, B_j)$ when $j \neq k$, and $\mu_k = r$. In a semi-primary lattice, for any flag $f$ with* $\operatorname{itype} f = \operatorname{itype} \boldsymbol{P}$*, we have*

$$\operatorname{type} f_{a_r} \wedge Af_{a_{r-1}} \wedge \cdots \wedge A^{r-1} f_{a_1} \wedge A^r f_0 \equiv \mu.$$

*If we replace one or more of the $a_i$'s by $a_i - 1$ on the left side, we replace $\mu_k$ with $r-1$ on the right side.*

*Proof.* Let $z_m = f_{a_r} \wedge Af_{a_{r-1}} \wedge \cdots \wedge A^{r-m} f_{a_m}$. Using the relations $A(x \wedge y) = Ax \wedge Ay$ and $A_y x = Ax \wedge y$ for elements $x, y$ of a modular lattice, it is easy to verify that $A_{a_m} A_{a_{m+1}} \cdots A_{a_r} f = (A^{r-m+1} f_0 \wedge z_m, \ldots, A^{r-m+1} f_{a_m} \wedge z_m)$, and that the $a_m$th component of this is $z_m$ and the $a_{m-1}$th component is $z_{m-1}$. We seek $\nu = \operatorname{type} z_0$, which by Proposition 5.48, is the straightening of the inner shape of $\operatorname{ftype}_{\hat{0}} g = \mathcal{A}_{a_1} \mathcal{A}_{a_2} \cdots \mathcal{A}_{a_r} \boldsymbol{P}$.

Let $t = \operatorname{dom}(B_k, B_j)$. In the expression for $\operatorname{ftype}_{\hat{0}} g$, the effect of $\mathcal{A}_{a_i}$ on a row is to delete all entries larger than $a_i$, and if any numerical entries remain, insert a $\bullet$ on the left, slide all entries right one cell, and delete the highest remaining entry. Write $B_j = \{b_1 < \cdots < b_s\}$. Since $a_{r-h} \geq b_{t-h}$ for $0 \leq h < t$, when we apply $\mathcal{A}_{a_{r-h}}$, row $j$ has $h+1$ $\bullet$'s followed by at least $b_1, \ldots, b_{t-h}$. Thus, row $j$ of $\operatorname{ftype}_{\hat{0}} g$ has at least $t$ $\bullet$'s, so $\nu_j \geq t$.

If $t = r$ or $t = s$, this row is exactly $t$ $\bullet$'s, so $\nu_j = t$. We now consider $t < \min\{r, s\}$. For some $h = 0, \ldots, t$ we have $a_{r-h} < b_{(t+1)-h}$. When we apply $\mathcal{A}_{r-h}$, we first delete all entries larger than $a_{r-h}$, leaving at most $b_1, \ldots, b_{t-h}$, then delete one more entry, and insert $\bullet$. So after applying $\mathcal{A}_{r-h}$, the entries on row $j$ are $h+1$ $\bullet$'s followed by at most $b_1, \ldots, b_{t-1-h}$, for a maximum length of $t$. The maximum length of the row after applying the remaining operators $\mathcal{A}_{a_{r-h-1}}, \ldots, \mathcal{A}_{a_1}$ is $t$, so $\nu_j \leq t$. As the reverse inequality was already shown to hold, $\nu_j = t$ in all cases when $j \neq k$.

When $j = k$, applying $\mathcal{A}_{a_{r-h}}$ yields $h+1$ $\bullet$'s followed by $a_1, \ldots, a_{r-h-1}$ when $h < r$, and yields $r$ $\bullet$'s when $h \geq r$. So $\nu_k = r$.

Now replace one or more $a_i$'s by $a_i - 1$ in the expression $z' = f_{a_r} \wedge Af_{a_{r-1}} \wedge \cdots \wedge A^{r-1} f_{a_1} \wedge A^r f_0$, and evaluate $\operatorname{type} z'$ similarly to the evaluation of $\operatorname{type} z_1$ above. When $j \neq k$, we have $b_m \leq a_i$ iff $b_m \leq a_i - 1$ for all $m$ and $i$ because $a_i \neq b_m$ (they're on different rows). Since all comparisons of entries on row $j$ with either $a_i$ or $a_i - 1$ are the same, the rows besides the $k$th in $\operatorname{ftype}_{\hat{0}}(\mathcal{A}_{a_1} \ldots \mathcal{A}_{a_r} f)$ are the same upon any substitutions $a_i \to a_i - 1$ in the subscripts of $\mathcal{A}$'s. But row $k$ shrinks by one $\bullet$ when one or more such substitutions are made. $\square$

We now classify and enumerate how many flags realize each type table of the form $\operatorname{itype} \boldsymbol{P}$ for each standard composition tableau $\boldsymbol{P}$.

**Theorem 5.52.** *If* $\operatorname{itype} f = \operatorname{itype} \boldsymbol{P}$ *for some standard composition tableau* $\boldsymbol{P}$*, the flag is hereditary. The number of hereditary decompositions of a given hereditary flag is* $q^{\operatorname{dom}(\boldsymbol{P})}$*, and the number of flags with interval type table* $\operatorname{itype} \boldsymbol{P}$ *is*

$$f_{\operatorname{itype} \boldsymbol{P}}(q) = q^{2n(\lambda) - \operatorname{dom}(\boldsymbol{P})} M_\lambda(q^{-1}).$$

*Synthetic proof.* Let $f$ be a flag and $\boldsymbol{P}$ be a standard composition tableau of shape $\lambda$ such that $\operatorname{itype} f = \operatorname{itype} \boldsymbol{P}$. Let $B_i$ be the set of entries on the $i$th row of $\boldsymbol{P}$.

We seek a join decomposition $\vec{y}$ of $f$. Fix $i$ and let the entries on the $i$th row of $\boldsymbol{P}$ be $a_1 < a_2 < \cdots < a_r$ where $r = \lambda_i$. If such a decomposition exists, the cycle $y_i$ is complemented, as the join of the other cycles is its complement, and it satisfies $C^{r-m} y_i \leq f_{a_m}$ but $C^{r-m} y_i \not\leq f_{a_m - 1}$ for $m = 1, \ldots, r$. Conversely, a simple induction on the successive elements $f_0, \ldots, f_n$ shows that all independent selections of $y_1, \ldots, y_k$ satisfying these conditions are hereditary decompositions of $f$.

We have $C^{r-m} y_i \leq f_a$ iff $y_i \leq A^{r-m} f_a$ by Theorem 3.3(6), so we need $y_i \leq A^{r-m} f_a$ and $y_i \not\leq A^{r-m} f_{a-1}$ for $i, m, a$ as given above. Also, $y_i$ is a cycle of rank $r$, so $y_i \leq A^r f_0$.

Let

$$\begin{aligned} z &= A^0 f_{a_m} \wedge \cdots \wedge A^{r-1} f_{a_1} \wedge A^r f_0 \\ z' &= A^0 f_{a_m-1} \wedge \cdots \wedge A^r f_{a_1-1} \wedge A^r f_0 \end{aligned}$$

It is necessary that $y_i \le z$, and it is sufficient, though not *a priori* necessary, that $y_i \not\le z'$. Let $\mu_j = \nu_j = \operatorname{dom}(B_i, B_j)$ for $1 \le j \le k$, except $\mu_i = \lambda_i$ and $\nu_i = \lambda_i - 1$. By the preceding lemma, $\operatorname{type} z \equiv \mu$ and $\operatorname{type} z' \equiv \nu$, so that $z' \lessdot z$, and also, $\operatorname{type}(z \wedge A^{r-m} f_{b_m-1}) \equiv \nu$ for $m = 0, \ldots, r$, so that $z \wedge A^{r-m} f_{b_m-1} = z'$. Thus in all cases when $y_i \le z$ and $y_i \le A^{r-m} f_{a-1}$, we do in fact have $y_i \le z'$ as well.

Thus, we require $y_i$ to be a complemented cycle in $[\hat{0}, z]$ not in $[\hat{0}, z']$. By Corollary 4.64, there are $q^{|\mu|-\lambda_i}$ ways to choose such a cycle when the lattice is $q$-regular; when plugging into the corollary, note that $\mu$ has exactly $\lambda_i$ columns, so that the cells in the first $\lambda_i$ columns are all of the squares.

The exponent is $|\mu| - \lambda_i = \sum_j \operatorname{dom}(B_i, B_j)$. Choose all of $y_1, \ldots, y_k$ in this fashion to yield

$$q^{\sum_{i,j} \operatorname{dom}(B_i,B_j)} = q^{\operatorname{dom}(\boldsymbol{P})}$$

different hereditary decompositions of $f$. Since there are $q^{2n(\lambda)} M_\lambda(q^{-1})$ join decompositions $\vec{y}$ of $\hat{1}$, which split into classes according to the value of $f(\boldsymbol{P}, \vec{y})$, the quotient of these is $f_{\operatorname{itype} \boldsymbol{P}} = q^{2n(\lambda)-\operatorname{dom}(\boldsymbol{P})} M_\lambda(q^{-1})$. $\square$

*Analytic proof of the enumeration only.* Let $f$ be a saturated hereditary flag and $\boldsymbol{P} = (\lambda^{(0)}, \ldots, \lambda^{(n)})$ a standard composition tableau with $\operatorname{itype} f = \operatorname{itype} \boldsymbol{P}$. Let $V$ be a vector space with a nilpotent action $N$ of type $\lambda = S(\lambda^{(n)})$. Any hereditary decomposition of $f$ gives rise to bases of $V$ with elements and action

$$e_{ij} \text{ with } 1 \le j \le \lambda_i \qquad N e_{i1} = 0 \quad N e_{ij} = e_{i,j-1} \text{ for } j > 1.$$

For convenience, we also set $e_{ij} = 0$ when $j < 1$, but these of course do not contribute to a basis.

By the synthetic proof above, any flag $f$ with interval type table $\operatorname{itype} \boldsymbol{P}$ can be represented by some such basis, with $f_m$ the span of all $e_{ij}$ with $(i,j) \in \operatorname{sh} \boldsymbol{P}$. For a given $f$, how many such canonical bases are there? If $m$ is in cell $(i,j)$, then the coset $e_{ij} + f_{m-1}$ of $V$ spans $f_m$ modulo $f_{m-1}$. So we can find the most general form of another such basis of $f$ as follows. The $i$th row of $\boldsymbol{P}$ has entries $B_i = \{b_{i1} < b_{i2} < \ldots < b_{i,\lambda_i}\}$. For $i = 1, \ldots, k$, let

$$\begin{aligned} e'_{ij} &= N^{\lambda_i - j} \left( e_{i,\lambda_i} + \sum_{(i',j') \in I_i} c^{(i)}_{i',j'} \, e_{i',j'} \right) \\ &= e_{ij} + \sum_{(i',j') \in I_i} c^{(i)}_{i',j'} \, e_{i',j'+j-\lambda_i} \\ y_i &= \operatorname{span} e'_{i1}, \ldots, e'_{i,\lambda_i} \end{aligned}$$

where the $I_i$ and $c^{(i)}{}_{(i',j')}$ are to be determined so that $f_m = \operatorname{span}\left\{ e'_{ij} \,:\, (i,j) \in \lambda^{(m)} \right\}$ in some canonical fashion. In general, the $e_{ij}$ term could have any nonzero coefficient besides 1, but we will normalize so that it is 1. The following conditions ensure we obtain distinct bases in a canonical fashion.

(1) $I_i \subset \lambda[\lambda_i]$ because $e'_{ij}$ is annihilated by $N^j$. The coefficient 1 on $e_{ij}$ both ensures it is not annihilated by $N^{j-1}$, and provides a normalization.
(2) $I_i$ has no points $(i', j')$ with $i' = i$, for by subtracting a linear combination of $Ne'_{ij}, N^2e'_{ij}, \ldots$ from $e'_{ij}$, we can erase any apparent coefficient of $e_{i',j'}$ with $i' = i$ and $j' < j$ from $e'_{ij}$.
(3) $(i', j') \in I_i$ requires $b_{i',j'+j-\lambda_i} < b_{ij}$ whenever $\lambda_i \geq j > \lambda_i - j'$. For pick $i$, and consider $m = b_{ij} \in B_i$. Assume that $f_{m-1} = f(\boldsymbol{P}, \vec{y})_{m-1}$ holds. The span of $f_{m-1}$ and $e'_{ij}$ is $f_m$ provided that all the nonzero $e_{i',j'}$ occurring with nonzero coefficients in $e'_{ij}$ have $b_{i',j'} < m = b_{ij}$, i.e, when $(i', j') \in I_i$ and $j' + j - \lambda_i > 0$, we have $b_{i',j'+j-\lambda_i} < b_{ij}$.

Conditions (1)–(3) are succinctly summarized by

$$I_i = \{ (i', j') : j' \leq \mathrm{dom}(B_i, B_{i'}) \},$$

because (1) is equivalent to $\mathrm{dom}(B_i, B_{i'}) \leq \min\{\lambda_i, \lambda_{i'}\}$, which always is true; (2) is equivalent to $\mathrm{dom}(B_i, B_{i'}) = 0$ when $i' = i$; and these conditions with (3) essentially define $\mathrm{dom}(B_i, B_{i'})$.

We can independently choose $c^{(i)}_{i',j'} \in K$ for $(i', j') \in I_i$. Thus, there are $q^{\mathrm{dom}(\boldsymbol{P})}$ canonical bases yielding $f$.

There are $q^{2n(\lambda)} M_\lambda(q^{-1})$ join decompositions of $V$, each of which yields a canonical basis of the above form, and so we obtain the result in the theorem. □

**Theorem 5.53.** *In a semi-primary lattice whose type is a two column partition, all flags are hereditary.*

*Proof.* Let $f$ be a saturated flag in the lattice. By Theorem 5.52, it suffices to show that $\mathrm{itype}\, f = \mathrm{itype}\, \boldsymbol{P}$ for some standard composition tableau $\boldsymbol{P}$. Let $\boldsymbol{P}_i = \mathrm{ftype}\, \partial^i f$ for $0 \leq i \leq n$. Let $e$ be an entry in the second column of $\boldsymbol{P}_0$. Let $i$ be minimal with $e$ in the first column of $\boldsymbol{P}_i$; at most one entry from $\boldsymbol{P}_{i-1}$ moves into column 1 of $\boldsymbol{P}_i$, so each $i$ has at most one $e$ associated to it. We will show that $\boldsymbol{P}_e$ is obtained from $\boldsymbol{P}_{e-1}$ by deleting $e$ and sliding the first column up to fill the hole; no entry moves from the second column to the first. Because this is true for all $e$ in the second column of $\boldsymbol{P}_0$, we thus have that $\boldsymbol{P}$ has rows $\boxed{i \mid e}$ for each $e$ in the second column of $\boldsymbol{P}_0$, and rows $\boxed{i}$ for each remaining $i$ in the first column of $\boldsymbol{P}_0$.

The tableaux $\boldsymbol{P}_{e-1}$ and $\boldsymbol{P}_e$ have the same second column iff $Af_{e-1} \wedge f_j = Af_e \wedge f_j$ for all $j \geq e$ iff this holds for maximum $j$ (when $f_j = \hat{1}$) iff $Af_{e-1} = Af_e$. Since $e$ is in the second column of $\boldsymbol{P}_0$, we have $A\hat{0} \wedge f_{e-1} = A\hat{0} \wedge f_e$. Apply $A$ to this to obtain $A^2\hat{0} \wedge Af_{e-1} = A^2\hat{0} \wedge Af_e$. Since $\mathrm{type}\, \boldsymbol{\mathcal{L}}$ has two columns, substitute $A^2\hat{0} = \hat{1}$ to obtain $Af_{e-1} = Af_e$. □

**Definition 5.54.** Let $\sigma$ be a permutation and $\boldsymbol{P}$ a composition tableau. Then $\sigma\boldsymbol{P}$ is the composition tableau of the same shape as $\boldsymbol{P}$ whose entries on a row are $\{\sigma(a_1), \ldots, \sigma(a_k)\}$ arranged in order, where $a_1, \ldots, a_k$ are the entries on that row of $\boldsymbol{P}$. Below, $(i, i+1)$ denotes the permutation exchanging $i$ and $i+1$.

**Theorem 5.55.** *Let $\lambda$ be an integer partition. Let $\boldsymbol{P}$ be a standard composition tableau of type $\lambda$.*

(1) $n(\lambda) \leq \mathrm{dom}(\boldsymbol{P}) \leq 2n(\lambda) - n_0(\lambda)$.
(2) $\left|\mathrm{dom}(\boldsymbol{P}) - \mathrm{dom}\big((i, i+1)\boldsymbol{P}\big)\right| \leq 1$.

(3) *As $\boldsymbol{P}$ ranges over all standard composition tableaux of shape $\lambda$, $\operatorname{dom}(\boldsymbol{P})$ takes on every value in the interval given in (1).*

*Proof.*

(1) Let $A = \{a_1 < \cdots < a_r\}$ and $B = \{b_1 < \cdots < b_s\}$ be disjoint sets of integers with $r \geq s$. We will show that

$$s \leq \operatorname{dom}(A, B) + \operatorname{dom}(B, A) \leq 2s - \delta_{rs}.$$

First we establish the lower bound. Let $t = \operatorname{dom}(A, B)$ and $u = \operatorname{dom}(B, A)$. We want to show that $t+u \geq s$. If $t = s$, we are done. Otherwise, $t < s \leq r$, so we have $b_{t+1-h} > a_{r-h}$ for some $h \in \{0, \ldots, t\}$, so that $b_{t+1} \geq b_{t+1-h} > a_{r-h} \geq a_{s-t}$ (as $r \geq s$ and $h \leq t$, so $r - h \geq s - t$). Then for $k = 0, \ldots, s - t - 1$, we have $b_{s-k} \geq b_{s-(s-t-1)} = b_{t+1} > a_{s-t} \geq a_{s-t-k}$ so $\operatorname{dom}(B, A) \geq s - t$. Thus $u \geq s - t$ so $u + t \geq s$.

Now we establish the upper bound. Each of $\operatorname{dom}(A, B)$ and $\operatorname{dom}(B, A)$ is less than both $r$ and $s$, the smaller of which is $s$, so $\operatorname{dom}(A, B) + \operatorname{dom}(B, A) \leq 2s$. Now suppose $r = s = t$. Then $a_i > b_i$ for all $i = 1, \ldots, r$, so we do not have $b_s > a_r$; thus, $t' < t$, so $\operatorname{dom}(A, B) + \operatorname{dom}(B, A) \leq 2s - 1$.

Now let $\lambda$ be a partition and $\boldsymbol{P}$ be a standard composition tableau of shape $\lambda$ whose set of entries on the $i$th row is $B_i$. The lower bound on $\operatorname{dom}(\boldsymbol{P})$ is obtained by summing the lower bound for each pair of rows.

$$\begin{aligned}
\operatorname{dom}(\boldsymbol{P}) &= \sum_{(i,j):\, 0<i<j\leq\lambda'_1} \operatorname{dom}(B_i, B_j) + \operatorname{dom}(B_j, B_i) \\
&\geq \sum_{(i,j):\, 0<i<j\leq\lambda'_1} \lambda_j \;=\; \sum_j (j-1)\lambda_j \;=\; n(\lambda)
\end{aligned}$$

The upper bound is similarly obtained by summing the upper bound for pairs of rows.

$$\begin{aligned}
\operatorname{dom}(\boldsymbol{P}) &= \sum_{(i,j):\, 0<i<j\leq\lambda'_1} \operatorname{dom}(B_i, B_j) + \operatorname{dom}(B_j, B_i) \\
&\leq \sum_{(i,j):\, 0<i<j\leq\lambda'_1} 2\lambda_j - \delta_{\lambda_i,\lambda_j} \\
&= 2n(\lambda) - \sum_{r>0} \binom{m_r}{2} \;=\; 2n(\lambda) - n_0(\lambda)
\end{aligned}$$

The sum of $2\lambda_j$ is double the sum computed for the lower bound. The sum of the $\delta$ terms is the number of pairs of parts of $\lambda$ of equal length, which is as indicated.

(2) Let $\boldsymbol{P}$ have rows $B_1, \ldots, B_k$, and $\boldsymbol{Q} = (i, i+1)\boldsymbol{P}$ have rows $B'_1, \ldots, B'_k$. If neither or only one of $i$ and $i+1$ occurs in $\boldsymbol{P}$, then $\operatorname{dom}(\boldsymbol{P}) = \operatorname{dom}(\boldsymbol{Q})$ because all comparisons with $i$ and with $i+1$ are equivalent. If both are on the same row then $\boldsymbol{Q} = \boldsymbol{P}$, so $\operatorname{dom}(\boldsymbol{P}) = \operatorname{dom}(\boldsymbol{Q})$.

We are left with $i$ and $i+1$ occurring in $\boldsymbol{P}$ on different rows. Assume without loss of generality that $i$ is in $B_1$ and $i+1$ is in $B_2$. Then

$$\operatorname{dom}(\boldsymbol{P}) - \operatorname{dom}(\boldsymbol{Q}) = \operatorname{dom}(B_1, B_2) + \operatorname{dom}(B_2, B_1) - \operatorname{dom}(B'_1, B'_2) - \operatorname{dom}(B'_2, B'_1)$$

because all comparisons of an element from the other rows with $i$ or with $i+1$ are the same; only when $i$ and $i+1$ are compared can the sense of the comparison be reversed.

Let $B_1 = \{a_1 < a_2 < \cdots < a_r\}$ and $B_2 = \{b_1 < b_2 < \cdots < b_s\}$, and $t = \mathrm{dom}(B_1, B_2)$. Let $B'_1 = \{a'_1 < a'_2 < \cdots < a'_r\}$ and $B_2 = \{b'_1 < b'_2 < \cdots < b'_s\}$, and $t' = \mathrm{dom}(B'_1, B'_2)$. Let $a_j = i$ and $b_k = i+1$, so that $a'_k = i+1$ and $b'_j = i$ and in all other cases, $a_h = a'_h$ and $b_h = b'_h$. We have $a_{r-h} > b_{t-h}$ for $h = 0, \ldots, t-1$, but this fails if $t$ is replaced by $t+1$. If the only failure in the latter case is of the form $i > i+1$, then when we swap $i$ and $i+1$ to obtain $B'_1$ and $B'_2$, this failure vanishes, so that $t' \geq t+1$. The failure $i > i+1$ is replaced by success $i+1 > i$, or rather $a'_k > b'_j$. We cannot have $t' > t+1$, because we would either exceed the length of $B'_2$, or would obtain a comparison $i+1 = a'_k > b'_{j+t'-(t+1)} > i$ (for if $t' > t+1$ then $j + t' - (t+1) > j$ so $b'_{j+t'-(t+1)} > b'_j = i$); but that requires there to be an integer strictly between $i$ and $i+1$, and there isn't one. Thus, $t' = t$ or $t+1$, i.e., $\mathrm{dom}(B_1, B_2) - \mathrm{dom}(B'_1, B'_2) = 0$ or $-1$.

Similarly, $\mathrm{dom}(B_2, B_1) - \mathrm{dom}(B'_2, B'_1) = 0$ or $+1$; the opposite sign arises from the fact that the rows $i$ and $i+1$ are reversed.

Thus, $\mathrm{dom}(\boldsymbol{P}) - \mathrm{dom}(\boldsymbol{Q}) = -1$, 0, or 1.

(3) Let $\boldsymbol{P}_1$ be the superstandard tableau of shape $\lambda$. Then $\mathrm{dom}(B_i, B_j) = 0$ when $i < j$, and $\lambda_i$ when $i > j$. There are $i-1$ values of $j$ less than each $i$, so $\mathrm{dom}(\boldsymbol{P}) = \sum_i (i-1)\lambda_i = n(\lambda)$.

Now let $\boldsymbol{P}_2$ be transpose of the superstandard tableau of shape $\lambda'$. Then for $i \neq j$,

$$\mathrm{dom}(B_i, B_j) = \begin{cases} \lambda_j & \text{if } \lambda_i > \lambda_j \text{ (so } i < j\text{);} \\ \lambda_j - 1 & \text{if } \lambda_i = \lambda_j \text{ and } i < j; \\ \lambda_i & \text{if } i > j. \end{cases}$$

The sum of these over all $i$ and $j$ is the same as the upper bound given in part (1) of this theorem.

Any permutation $\sigma$ for which $\boldsymbol{P}_1 = \sigma \boldsymbol{P}_2$ can be decomposed as a product of elementary transpositions $(i, i+1)$, and so we can obtain a sequence of composition tableaux from $\boldsymbol{P}_1$ to $\boldsymbol{P}_2$ differing by transpositions; by (2), the successive tableaux in this sequence have values of dom that either are equal or go up or down by 1, so all the values from the lower bound to the upper bound are obtained. $\square$

Recall the polynomial $f_\Lambda(q)$ and parameter $q_0$ of Conjecture 5.25; the polynomial is the number of flags with type table $\Lambda$ in a $q$-regular semi-primary lattice when $q \geq q_0$.

**Corollary 5.56.** *Let $\lambda$ be an integer partition. Type tables $\Lambda$ with $f_\Lambda(q) = q^d M_\lambda(q)$ exist for each $0 \leq d \leq n(\lambda) - n_0(\lambda)$.*

*Proof.* This follows from Theorems 5.52 and 5.55, and the fact that $M_\lambda(q^{-1}) = q^{-n_0(\lambda)} M_\lambda(q)$. $\square$

**Theorem 5.57.** *Let $\Lambda$ be an interval type table of shape $\lambda$ for which the polynomial $f_\Lambda(q)$ and parameter $q_0$ of Conjecture 5.25 exist. If there is no standard composition* *tableau* $\boldsymbol{P}$ *for which* $\Lambda$ *has the form* $\operatorname{itype}\boldsymbol{P}$*, then either* $q_0 > 1$*, or* $f_\Lambda(q)$ *does not have the form* $q^d M_\lambda(q)$*.*

*Proof.* Consider the 1-regular semi-primary lattice which is the direct product of chains of lengths $\lambda_1, \lambda_2, \ldots$. All its flags are hereditary flags, and so if $\Lambda$ is not of the form $\operatorname{itype}\boldsymbol{P}$ for some $\boldsymbol{P}$, there are no flags with interval type table $\Lambda$. Thus, either $q_0 > 1$, or $q-1$ is a divisor of $f_\Lambda(q)$. □

*Note.* For all the polynomials listed in Appendix A, $q_0 = 1$ and $q-1$ is a divisor of all polynomials except those of the form $\Lambda = \operatorname{itype}\boldsymbol{P}$.

## 6. Tableaux games describing operations on flags

**6.1. Operations on flags.** We now consider a number of operations on multisaturated flags. We will work in the generality of multisaturated flags rather than saturated flags because certain operations, such as $A$ and $C$, turn saturated flags into multisaturated flags, and we want to look at compositions of operations. The indexing of the elements of flags and tableaux is important as well. For a flag $f = (f_0, \ldots, f_n)$, we have $\partial f = (f_1, \ldots, f_n)$, whose components are indexed $1, \ldots, n$, not $0, \ldots, n-1$; the tableau game $\operatorname{ftype}\partial f = \widetilde{\Delta}(\operatorname{ftype} f)$ is nicer this way, as it can be expressed by moving numbers along a single path and leaving the rest of the tableau intact, rather than also decrementing every entry of the tableau, and a similar phenomenon occurs with other games. The price to pay for these saturation and indexing issues is bookkeeping to track the indices used in flags and tableau, as already seen in Example 5.46. For multisaturated flags, there are only two numbers that must be tracked: the low index and the high index of a flag or tableau. The indices between them do not change. For nonsaturated flags (Section 6.8), the intermediate indices will be changed as well.

Here are all the operations on multisaturated flags.

| flag $g$ | $\operatorname{low}(g)$ | $\operatorname{high}(g)$ |
|---|---|---|
| $f = (f_l, f_{l+1}, \ldots, f_h)$ | $l$ | $h$ |
| $\partial f = (f_{l+1}, \ldots, f_h)$ | $l+1$ | $h$ |
| $df = (f_l, \ldots, f_{h-1})$ | $l$ | $h-1$ |
| $Cf = (C_{f_l} f_l, \ldots, C_{f_l} f_h)$ | $l$ | $h$ |
| $C_x f = (C_x f_l, \ldots, C_x f_h)$ | $l$ | $h$ |
| $Af = (A_{f_h} f_l, \ldots, A_{f_h} f_h)$ | $l$ | $h$ |
| $A_x f = (A_x f_l, \ldots, A_x f_h)$ | $l$ | $h$ |
| $f^* = (f_h^*, \ldots, f_l^*)$ | $n-h$ | $n-l$ |
| $f \vee x = (f_l \vee x, \ldots, f_h \vee x)$ | $l$ | $h$ |
| $f \wedge x = (f_l \wedge x, \ldots, f_h \wedge x)$ | $l$ | $h$ |

Certain pairs of operators commute: $\partial, d$; $\partial, C_x$; $\partial, A$; $\partial, A_x$; $d, C$; $d, C_x$; $d, A_x$. The pairs $\partial, d$; $C, A$; $C_x, A_{x^*}$; $\vee x, \wedge x^*$ are dual, in the sense that $(\partial(f^*))^* = df$ and $(d(f^*))^* = \partial f$, and similarly for the others.

The value of $n$ must be specified, but generally, we start with a flag $f = (f_0, \ldots, f_n)$ and apply some sequence of operators to it; the high index may no longer be $n$ when dualizing, but we still use the initial $n$ rather than the new high index.

For each operator, we would like to determine the following, to the extent possible. We may ask the same questions with ftype replaced by $\mathrm{ftype}_y$.

(1) What are the possible values of $\mathrm{ftype}\, g$ given $\mathrm{ftype}\, f$, and which value, if any, is generic or uniform?
(2) How many flags $f$ (or pairs $f$ and $x$) achieve specified values of $\mathrm{ftype}\, f$ and $\mathrm{ftype}\, g$?
(3) Inversely, given $g$ and a tableau $\boldsymbol{P}$, how many flags $f$ are there with $\mathrm{ftype}\, f = \boldsymbol{P}$ such that applying the operator yields $g$?

For the various operations, we will develop the **tableau game** $\mathcal{G}$ transforming $\boldsymbol{P} = \mathrm{ftype}\, f = (\lambda^{(l)} \leq \cdots \leq \lambda^{(h)})$ to $\mathcal{G}\boldsymbol{P} = \mathrm{ftype}\, g = (\mu^{(l')} \leq \cdots \leq \mu^{(h')})$. Either or both ftype's may be replaced by $\mathrm{ftype}_y$, as indicated in the particular game. There are two equivalent formulations of each game: there are rules describing how to manipulate the numbers and cells of the tableau $\boldsymbol{P}$ to form $\mathcal{G}\boldsymbol{P}$, and there are rules describing how to compute the partitions $\mu^{(k)}$ from the partitions $\lambda^{(k)}$. In the latter formulation, we write the partitions $\lambda^{(l)}, \ldots, \lambda^{(h)}$ in columns $l$ through $h$ on one line, and then apply rules that ultimately yield partitions $\mu^{(l')}, \ldots, \mu^{(h')}$ in columns $l'$ through $h'$ on the line below it. We can compose operations by using that line as the input to yet another operator, whose result is written beneath it; the evacuation of Figure 3 shows the computation of successive powers of $\Delta$ in this format.

The tableau and partition formulations of each game are equivalent, provided we keep track of $\mathrm{low}(\boldsymbol{P})$, $\mathrm{high}(\boldsymbol{P})$, $\mathrm{low}(\mathcal{G}\boldsymbol{P})$, and $\mathrm{high}(\mathcal{G}\boldsymbol{P})$ through careful bookkeeping in the tableau formulation of the game; in the partition formulation, we keep track of these quantities automatically.

Depending on the operation, the value of $\mathrm{ftype}\, g$ may not be uniquely determined by the value of $\mathrm{ftype}\, f$. If multiple values are possible, we define two similar games: $\tilde{\mathcal{G}}\boldsymbol{P}$ is a nondeterministic game yielding all possible values of $\mathrm{ftype}\, g$ given $\boldsymbol{P} = \mathrm{ftype}\, f$, and $\mathcal{G}\boldsymbol{P}$ is a deterministic game, usually yielding the generic value. For most games (the Robinson-Schensted games being the important exception), the generic value is uniform, so that the uniform value of a composition of operators on a flag is described by composing the generic games.

**6.2. The backwards jeu de taquin slide, $j^c(\boldsymbol{P})$.** Let $\mathrm{sh}\, \boldsymbol{P} = \lambda/\mu$; the parameter $c$ is an inner corner of $\mu$.

Let $\hat{0} < x \leq f_l$. Uniformly,

$$j^c, \tilde{\jmath}^c : \mathrm{ftype}_{\hat{0}}\, f \longmapsto \mathrm{ftype}_x\, f \qquad c = \mathrm{type}[\hat{0}, f_l] / \mathrm{type}[x, f_l].$$

(That is, the game $j^c$ uniformly maps $\{\, f \,:\, \mathrm{ftype}_{\hat{0}}\, f = \boldsymbol{P} \,\}$ to $\{\, f \,:\, \mathrm{ftype}_x\, f = j^c(\boldsymbol{P}) \,\}$, and in all degenerate cases, $\mathrm{ftype}_x\, f = \tilde{\jmath}^c(\mathrm{ftype}_{\hat{0}}\, f)$.)

*Tableau Rules.* The items tagged with $j$ or $\tilde{\jmath}$ are performed only in one or the other game, and the untagged items are done in both games.

(1) Place $\star$ at $c$ and let $e = \bullet$ be the value that was there.

($2j$) If $\star$ is at an inner corner, delete it, and we are done. Otherwise, choose the smaller or only of the cells just below and just right of $\star$, and slide that entry into $\star$. Repeat step (2).

($2\tilde{\jmath}$) Choose one of these alternatives, when the provision allows.

(a) Slide the entry $e'$ below $\star$ into $\star$, provided the numeric entries on the rows continue to increase left to right.

(b) Slide any entry $e' > e$ from the column left of $\star$ into $\star$, provided the rows continue to increase left to right and the columns top to bottom.

(c) Delete $\star$ and we are done, provided $\star$ is at an inner corner.

Then let $e := e'$ and repeat step ($2\tilde{\jmath}$) until alternative (c) is chosen.

See Figure 6.

*Local Rules.* Place $\lambda^{(l)} \ominus c$ in column $l$. Then for $k = l+1, l+2, \ldots, h$, propagate as follows.

$$\begin{array}{r|c|c|c|c|l} \text{column} & l & & k-1 & k & \\ \hline \boldsymbol{P} & \lambda^{(l)} & & \gamma & \delta & \\ \tilde{\jmath}^c\boldsymbol{P} & (\lambda^{(l)} \ominus c) & & \alpha & (\beta) & \longrightarrow \\ \hline \end{array}$$

The nonparenthesized values are given, and the parenthesized values are computed, according to which of several cases occurs.

**$\boldsymbol{\alpha = \gamma}$:** Let $\beta := \delta$.

**$\boldsymbol{\delta = \gamma}$:** Let $\beta := \alpha$.

**$\boldsymbol{\delta/\alpha}$ is a brick:** Let $\beta$ be the unique partition between $\alpha$ and $\delta$.

**$\boldsymbol{\delta/\alpha}$ is two nonadjacent squares ($\boldsymbol{j^c}$):** There are two partitions between $\alpha$ and $\delta$, and $\beta$ is the one that's not $\gamma$.

**$\boldsymbol{\delta/\alpha}$ is two nonadjacent squares ($\boldsymbol{\tilde{\jmath}^c}$):** If $\operatorname{col}(\delta/\gamma) > \operatorname{col}(\gamma/\alpha)$, choose either partition between $\alpha$ and $\delta$ to be $\beta$, and otherwise, choose the partition unequal to $\gamma$ to be $\beta$.

**Theorem 6.1.** *Let $\boldsymbol{P}$ be a tableau of shape $\lambda/\mu$. Let $\nu < \mu$ and $c = \mu/\nu$. For all flags $f$ with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$, and all atoms $x$ of $[\hat{0}, f_l]$ with $\operatorname{type}[x, f_l] = \nu$, we have $\operatorname{ftype}_x f = \tilde{\jmath}^c(\operatorname{ftype}_{\hat{0}} f)$.*

*Fix $\boldsymbol{Q} = \tilde{\jmath}^c\boldsymbol{P}$. Let $r = \operatorname{col}(\lambda/\mu^{(h)})$. In a q-regular semi-primary lattice of type $\lambda$, the number of pairs $(x, f)$ where $f$ is a flag with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ and $x$ is an atom of $[\hat{0}, f_l]$ with $\operatorname{type}[x, f_l] = \nu$, and $\operatorname{ftype}_x f = \boldsymbol{Q}$, is*

$$\left(\langle\lambda'_r\rangle - \left\langle\lambda'_{r+1}\right\rangle\right) \cdot \prod_{k \,\in\, \boldsymbol{P}} \begin{cases} \langle\operatorname{row}(\boldsymbol{Q}, k)\rangle - \langle\operatorname{row}_1(\boldsymbol{P}, k)\rangle & \text{if } \operatorname{col}(\boldsymbol{P}, k) = \operatorname{col}(\boldsymbol{Q}, k); \\ \langle\operatorname{row}(\boldsymbol{P}, k)\rangle - \langle\operatorname{row}_1(\boldsymbol{Q}, k)\rangle & \text{if } \operatorname{col}(\boldsymbol{P}, k) > \operatorname{col}(\boldsymbol{Q}, k). \end{cases}$$

*This is a polynomial of degree*

$$n(\operatorname{sh}\boldsymbol{P}) + n(\operatorname{osh}\boldsymbol{P}/\operatorname{osh}\boldsymbol{Q}) - \#\ \textit{of entries that move up one cell from } \boldsymbol{P} \textit{ to } \boldsymbol{Q},$$

*which is uniquely maximized in the uniform game $\boldsymbol{P} = j^c(\boldsymbol{P})$.*

*Proof.* See Proposition 5.7, which counts the number of flags $f$ with specified values of $\operatorname{ftype} f$ and $\operatorname{ftype} \partial f$. The proof is essentially the same, except that here we allow the flag to be multisaturated by setting $f_{k-1} := f_k$ when $k \notin \boldsymbol{Q}$ (which simply introduces factors of 1 that can be ignored), and we included the number $\langle\lambda'_r\rangle - \left\langle\lambda'_{r+1}\right\rangle$ of choices of $x$. Also, the final cell $c$ of the deletion path was fixed at $(1,1)$ in the proposition, but is given here as a parameter. $\square$

$$\boldsymbol{P} = \begin{array}{|c|c|c|} \hline \bullet & \bullet & 3 \\ \hline {}_{c}\bullet & 2 & 5 \\ \hline 1 & 4 & 8 \\ \hline 6 & 7 & \\ \cline{1-2} \end{array} \qquad\qquad c = (2,1)$$

The possible computations of $\tilde{\jmath}^c(\boldsymbol{P})$ are as follows.

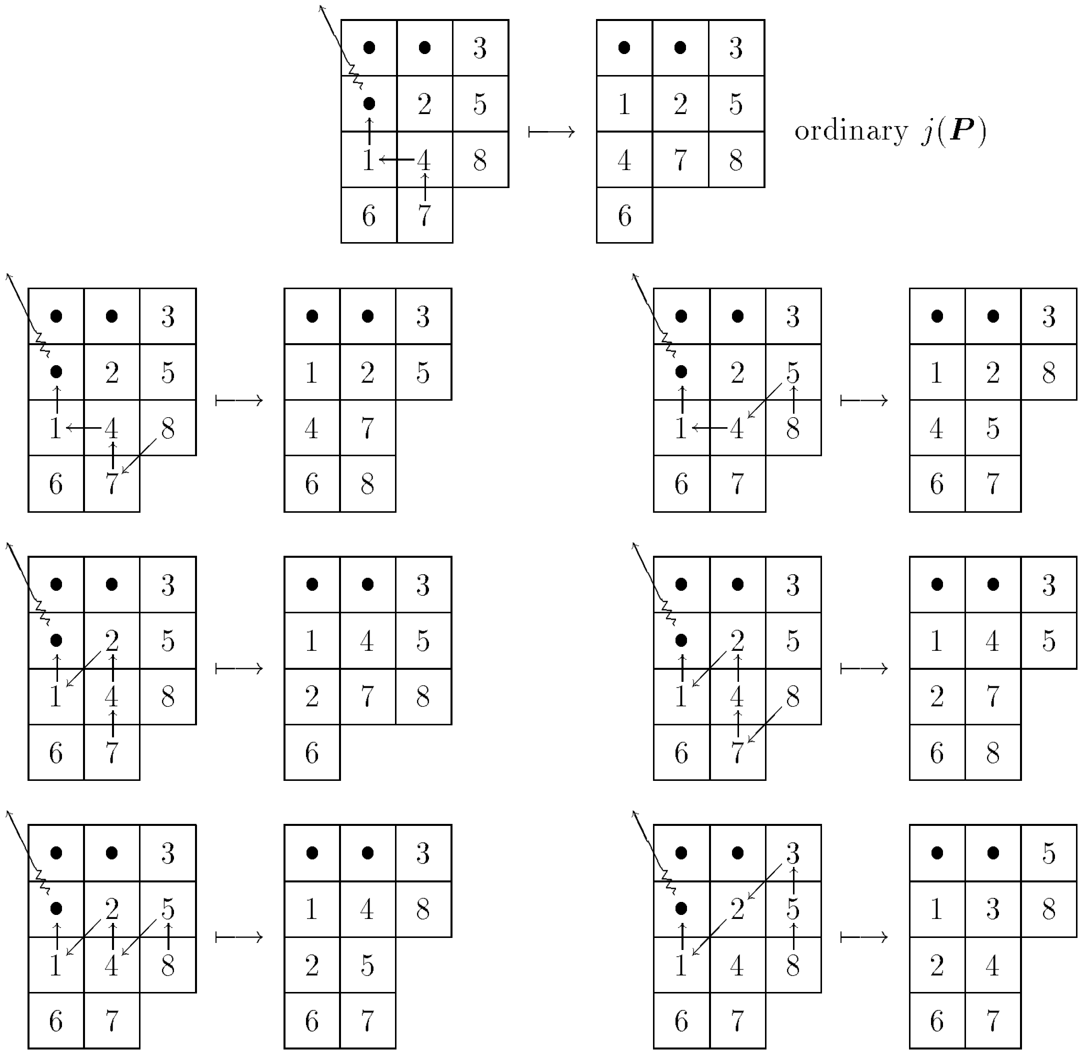



FIGURE 6. All possible degenerate *jeu de taquin* slides.

*Note.* The following conventions explicitly described in this game will be adopted for all games without further explicit descriptions.

(1) The notation
$$\mathcal{G}, \widetilde{\mathcal{G}} : \operatorname{ftype}_x f \longmapsto \operatorname{ftype}_y \phi(f)$$
means $\widetilde{\mathcal{G}}$ is a nondeterministic game with $\operatorname{ftype}_y \phi(f) = \widetilde{\mathcal{G}}(\operatorname{ftype}_x f)$ for all flags $f$, and that the game $\mathcal{G}$ is a deterministic version of it, usually giving a uniform or generic value, as noted.

(2) Differences in the rules for $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are noted by tagging the rules that differ with $\mathcal{G}$ or $\widetilde{\mathcal{G}}$; untagged rules are the same in both.

(3) In the local section shown of the grid of partitions for a game, the unparenthesized partitions are given, and the parenthesized ones are computed as a (possibly nondeterministic) function of them. If there is a simple formula for a parenthesized one, it is given, and otherwise, it is given a name, such as "$\beta$," and then rules for computing $\beta$ are presented.

**6.3. The *jeu de taquin*, $j(\boldsymbol{P})$.** Uniformly,
$$j, \tilde{\jmath} : \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype} f.$$

*Tableau Rules.* Let $\boldsymbol{P}$ be a skew tableau. Choose any inner corner $c$ of the inner shape of $\boldsymbol{P}$ and apply the backwards slide $j^c$ (for $j$), or the degenerate backwards slide $\tilde{\jmath}^c$ (for $\tilde{\jmath}$), and repeat again with another inner corner of the resulting tableau, until a tableau of ordinary shape is achieved.

The tableau $j(\boldsymbol{P})$ is independent of the sequence of inner corners chosen! For a detailed analysis of this game and a combinatorial proof of the uniqueness of $j(\boldsymbol{P})$, see the original source [**26**] or an exposition [**22**, §3.9]. We provide the following algebraic proof.

**Definition 6.2.** For a standard tableau $\boldsymbol{R}$ on $1, \ldots, m$ with $i$ in cell $c_i$, let $j^{\boldsymbol{R}} = j^{c_1} \cdots j^{c_m}$.

**Theorem 6.3.**

(1) *Let $\mu$ be a partition. The tableau operator $j^{\boldsymbol{R}}$ is the same for all standard tableau $\boldsymbol{R}$ of shape $\mu$. Simply denote this operator by $j$.*

(2) *Generically,* $\operatorname{ftype} f = j(\operatorname{ftype}_{\hat{0}} f)$ *in $q$-regular semi-primary lattices.*

*Proof.* Let $\boldsymbol{P}$ be a standard skew tableau of shape $\lambda/\mu$ and $\boldsymbol{R}$ be a standard tableau of shape $\mu \vdash m$. Let $A$ be the set of flags $f$ with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$, and let $X_f$ be the set of flags $x$ with $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_m = f_l$ and $\operatorname{ftype} x^* = \boldsymbol{R}$. For all flags $f \in A$, there are $F^{\boldsymbol{R}}(q)$ flags in $X_f$. By genericity of the composition $j^{\boldsymbol{R}}$ of uniform games, all but a fraction $O(q^{-1})$ of pairs $\{\, (x, f) : f \in A \text{ and } x \in X_f \,\}$ satisfy $\operatorname{ftype} f = \operatorname{ftype}_{x_m} f = j^{\boldsymbol{R}}(\boldsymbol{P})$. This expression does not depend on which $x \in X_f$ was chosen, and the size of $X_f$ is independent of $f \in A$, so in fact, all but a fraction $O(q^{-1})$ of flags $f \in A$ satisfy $\operatorname{ftype} f = j^{\boldsymbol{R}}(\boldsymbol{P})$. Since this holds for any standard tableau $\boldsymbol{R}$ of shape $\mu$, in fact, $j^{\boldsymbol{R}}(\boldsymbol{P})$ is independent of $\boldsymbol{R}$. $\square$

**Corollary 6.4.** *Let $\mu$ be a partition and $\boldsymbol{P}$ be a standard tableau. Then $j(\mu + \boldsymbol{P}) = S(\boldsymbol{P})$.*

*Proof.* All flags $f$ with $\operatorname{ftype}_{\hat{0}} f = \mu + \boldsymbol{P}$ have $\operatorname{ftype} f = S(\boldsymbol{P})$ by Theorem 4.93, and generically have $\operatorname{ftype} f = j(\boldsymbol{P})$ by the preceding theorem, so $j(\boldsymbol{P}) = S(\boldsymbol{P})$. $\square$

This can also be proved without the machinery of flags by showing $j^c(\mu + \boldsymbol{P}) = (\mu \ominus c) + \boldsymbol{P}$, i.e., that the slide $j^c$ simply slides $c$'s row one cell left, deleting $\bullet$ from $c$, and does nothing to any other row.

**6.4. The elementary evacuation step, $\Delta\boldsymbol{P}$.** Uniformly,

$$\Delta, \widetilde{\Delta} : \operatorname{ftype} f \longmapsto \operatorname{ftype} \partial f.$$

*Tableau Rules.* If the top left corner of $\boldsymbol{P}$ is larger than $l+1$ then $\widetilde{\Delta}\boldsymbol{P} := \boldsymbol{P}$, and we are done. Otherwise, let $c = (1,1)$ be the upper left corner of $\boldsymbol{P}$, place $\bullet$ there, and perform the backwards slide $j^c$ (for $\Delta$) or $\tilde{\jmath}^c$ (for $\widetilde{\Delta}$) into it.

*Local Rules.* Place $\emptyset$ in column $l+1$. Then for $k = l+2, \ldots, h$, propagate by the same rule as in $j^c$ (for $\Delta$) or $\tilde{\jmath}^c$ (for $\widetilde{\Delta}$).

$$\begin{array}{r|c|c|c|c|c|l} \text{column} & l & l+1 & & k-1 & k & \\ \boldsymbol{P} & \emptyset & \lambda^{(l+1)} & & \gamma & \delta & \longrightarrow \\ \widetilde{\Delta}\boldsymbol{P} & & (\emptyset) & & \alpha & (\beta) & \end{array}$$

The game $\Delta$ is uniform; see Proposition 5.7 for further details, including an enumeration of the number of flags achieving $\operatorname{ftype} f = \boldsymbol{P}$ and $\operatorname{ftype} \partial f = \boldsymbol{Q}$. The proof there is for saturated flags, but is trivially extended to multisaturated flags, with the same enumerative formulas.

**6.5. Remove the low entry, $\partial\boldsymbol{P} = (\lambda^{(l+1)}, \ldots, \lambda^{(h)})$.** Uniformly (in fact, identically),

$$\begin{aligned} \partial &: \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}} \partial f \\ \partial &: \ \operatorname{ftype} f \longmapsto \operatorname{ftype}_{f_l} \partial f. \end{aligned}$$

*Tableau Rules.* Replace any occurrence of $l+1$ in $\boldsymbol{P}$ by $\bullet$. See Figure 7.

*Local Rules.* Place partitions $\lambda^{(l+1)}, \ldots, \lambda^{(h)}$ in columns $l+1$ through $h$.

$$\begin{array}{r|c|c|c|l} \text{column} & l & & k & \\ \boldsymbol{P} & \lambda^{(l)} & & \lambda^{(k)} & \qquad \text{for } l+1 \le k \le h \\ \partial\boldsymbol{P} & & & (\lambda^{(k)}) & \end{array}$$

All flags $f$ with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ have $\operatorname{ftype}_{\hat{0}} \partial f = \partial\boldsymbol{P}$. In a regular semi-primary lattice whose type is osh $\boldsymbol{P}$, all flags $g$ with $\operatorname{ftype}_{\hat{0}} g = \partial\boldsymbol{P}$ have the same number of extensions to a flag $f = (f_l, g_{l+1}, \ldots, g_h)$ with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$; if $\lambda^{(l)} = \lambda^{(l+1)}$, then $f_l := g_{l+1}$ in one way, and if $\lambda^{(l)} \lessdot \lambda^{(l+1)}$ then $f_l$ is any coatom of $[\hat{0}, f_{l+1}]$ of type $\lambda^{(l)}$.

All flags $f$ with $\operatorname{ftype} f = \boldsymbol{P}$ have $\operatorname{ftype}_{f_l} \partial f = \partial\boldsymbol{P}$. In a lattice whose type is sh $\boldsymbol{P}$, all flags $g$ up to $\hat{1}$ with with $\operatorname{ftype}_{\hat{0}} g = \partial\boldsymbol{P}$ have one extension, $f = (\hat{0}, g_{l+1}, \ldots, g_h)$, with $\operatorname{ftype} f = \boldsymbol{P}$.

$$\boldsymbol{P} = \begin{array}{|c|c|c|} \hline \bullet & \bullet & 4 \\ \hline 2 & 5 & 6 \\ \hline 3 & 9 \\ \cline{1-2} \end{array}$$

$$\partial\boldsymbol{P} = \begin{array}{|c|c|c|} \hline \bullet & \bullet & 4 \\ \hline \bullet & 5 & 6 \\ \hline 3 & 9 \\ \cline{1-2} \end{array} \quad \text{if } l = 1, \qquad \partial\boldsymbol{P} = \begin{array}{|c|c|c|} \hline \bullet & \bullet & 4 \\ \hline 2 & 5 & 6 \\ \hline 3 & 9 \\ \cline{1-2} \end{array} \quad \text{if } l < 1.$$

$$d\boldsymbol{P} = \begin{array}{|c|c|c|} \hline \bullet & \bullet & 4 \\ \hline 2 & 5 & 6 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \text{if } h = 9, \qquad d\boldsymbol{P} = \begin{array}{|c|c|c|} \hline \bullet & \bullet & 4 \\ \hline 2 & 5 & 6 \\ \hline 3 & 9 \\ \cline{1-2} \end{array} \quad \text{if } h > 9.$$

$$\mathcal{C}\boldsymbol{P} = \begin{array}{|c|c|} \hline \bullet & 4 \\ \hline 5 & 6 \\ \hline 9 \\ \cline{1-1} \end{array}$$

FIGURE 7. Deterministic games $\partial$, $d$, $\mathcal{C}$.

**6.6. Remove the high entry, $d\boldsymbol{P} = (\lambda^{(l)}, \ldots, \lambda^{(h-1)})$.** Uniformly (in fact, identically),

$$\begin{aligned} d : \ \operatorname{ftype} f &\longmapsto \operatorname{ftype} df \\ d : \operatorname{ftype}_{\hat{0}} f &\longmapsto \operatorname{ftype}_{\hat{0}} df. \end{aligned}$$

*Tableau Rules.* Delete any cell with $h$ from $\boldsymbol{P}$. See Figure 7.

*Local Rules.* Place partitions $\lambda^{(l)}, \ldots, \lambda^{(h-1)}$ in columns $l$ through $h-1$.

$$\begin{array}{rcc} \text{column} & k & h \\ \boldsymbol{P} & \boxed{\lambda^{(k)}} & \boxed{\lambda^{(h)}} \\ d\boldsymbol{P} & \boxed{(\lambda^{(k)})} & \boxed{\phantom{\lambda}} \end{array} \qquad \text{for } l \le k \le h-1$$

All flags $f$ with $\operatorname{ftype} f = \boldsymbol{P}$ have $\operatorname{ftype} df = d\boldsymbol{P}$, and all with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ have $\operatorname{ftype}_{\hat{0}} df = d\boldsymbol{P}$.

In a semi-primary lattice whose type is osh $\boldsymbol{P}$, all flags $g$ with $\operatorname{ftype}_{\hat{0}} g = d\boldsymbol{P}$ have exactly one extension $f = (g_l, \ldots, g_{h-1}, \hat{1})$ to a flag with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$, and all flags $g$ from $\hat{0}$ with $\operatorname{ftype} g = d\boldsymbol{P}$ have exactly one extension $f = (g_l, \ldots, g_{h-1}, \hat{1})$ to a flag with $\operatorname{ftype} f = \boldsymbol{P}$.

**6.7. Delete the left column, $\mathcal{C}\boldsymbol{P} = (\mathcal{C}\lambda^{(l)}, \ldots, \mathcal{C}\lambda^{(h)})$.** Uniformly,

$$\begin{aligned} \mathcal{C} : \ \operatorname{ftype} f &\longmapsto \operatorname{ftype} Cf \\ \mathcal{C} : \operatorname{ftype}_{\hat{0}} f &\longmapsto \operatorname{ftype}_{\hat{0}} C_{\hat{0}} f. \end{aligned}$$

*Tableau Rules.* Delete the first column of $\boldsymbol{P}$. See Figure 7.

*Local Rules.* Place partitions $\mathcal{C}\lambda^{(l)}, \ldots, \mathcal{C}\lambda^{(h)}$ in columns $l$ through $h$.

$$\begin{array}{rc} \text{column} & k \\ \boldsymbol{P} & \boxed{\lambda^{(k)}} \\ \mathcal{C}\boldsymbol{P} & \boxed{(\mathcal{C}\lambda^{(k)})} \end{array} \qquad \text{for } l \le k \le h$$

We will examine $\operatorname{ftype}_{\hat{0}}$, and ftype is treated by setting $\hat{0}$ to $f_l$.

All flags $f$ with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ yield $\operatorname{ftype}_{\hat{0}} f = \mathcal{C}\boldsymbol{P}$.

For the inverse problem, we show that all flags $g$ to $C\hat{1}$ with type $\mathcal{C}\boldsymbol{P}$ can be expressed $g = Cf$ where $f$ has type $\boldsymbol{P}$. Since $f_n = \hat{1}$, it is mandatory that $g_n = C\hat{1}$. So select any flag $g$ to $C\hat{1}$ with $\operatorname{ftype}_{\hat{0}} g = \mathcal{C}\boldsymbol{P}$. Given

$$\longleftarrow \text{type} \boxed{\begin{array}{ccc} (f_{k-1}) & \dot{\leq} & f_k \\ Cf_{k-1} & \dot{\leq} & Cf_k \end{array}} = \boxed{\begin{array}{ccc} \gamma & \dot{\leq} & \delta \\ \alpha & \dot{\leq} & \beta \end{array}}$$

(that is, the partitions $\alpha, \beta, \gamma, \delta$ are given; $f_k$, $Cf_k$, and the desired value of $Cf_{k-1}$ are given; but $f_{k-1}$ is not given), we want to find all possible $f_{k-1}$. Let $w$ be the given desired value of $Cf_{k-1}$.

If $\gamma = \delta$ then $f_{k-1} := f_k$ so $g_{k-1} := g_k$ and $\alpha := \beta$.

$$\boxed{\begin{array}{ccc} \gamma & = & \delta \\ \rotatebox[origin=c]{90}{$\dot{\leq}$} & & \rotatebox[origin=c]{90}{$\dot{\leq}$} \\ \mathcal{C}\gamma & = & \mathcal{C}\delta \end{array}} \qquad \text{1 choice}$$

So let $\gamma \lessdot_{|r|} \delta$. The intervals $[\hat{0}, Cf_k]$ and $[A\hat{0}, f_k]$ are isomorphic by Theorem 4.17. Since $w \leq Cf_k$, the type of $A_{f_k}w$ is $(\delta'_1, \alpha'_1, \alpha'_2, \alpha'_3, \ldots)'$, which agrees with $\gamma$ in all columns except possibly the first. Any $f_{k-1}$ with $Cf_{k-1} = w$ must have $f_{k-1} \leq A_{f_k}w$ because the interval $[w, f_{k-1}]$ is elementary. If $r = 1$ then $\alpha = \beta$ and $w = Cf_k$; set $f_{k-1}$ to be any lower cover of $f_k$ whose type is $\gamma$.

$$\boxed{\begin{array}{ccc} \gamma & \lessdot_{|1|} & \delta \\ \rotatebox[origin=c]{90}{$\dot{\leq}$} & & \rotatebox[origin=c]{90}{$\dot{\leq}$} \\ \mathcal{C}\gamma & = & \mathcal{C}\delta \end{array}} \qquad \begin{array}{l} \langle \delta'_1 \rangle - \langle \delta'_2 \rangle \;= \\ \quad \langle \operatorname{row}(\boldsymbol{P}, k) \rangle - \langle \operatorname{row}_1(\boldsymbol{P}, k) \rangle \\ \text{choices} \end{array}$$

If $r > 1$ then type $A_{f_k}w = \gamma$ so $f_{k-1} = A_{f_k}w$.

$$(r > 1) \quad \boxed{\begin{array}{ccc} \gamma & \lessdot_{|r|} & \delta \\ \rotatebox[origin=c]{90}{$\dot{\leq}$} & & \rotatebox[origin=c]{90}{$\dot{\leq}$} \\ \mathcal{C}\gamma & \lessdot_{|r-1|} & \mathcal{C}\delta \end{array}} \qquad \text{1 choice}$$

Written in terms of tableaux, given a flag $g$ up to $C\hat{1}$ with $\operatorname{ftype}_{\hat{0}} g = \mathcal{C}\boldsymbol{P}$, the number of flags $f$ with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ and $Cf = g$ is

$$\frac{F^{\boldsymbol{P}}(q)}{F^{\mathcal{C}\boldsymbol{P}}(q)} = \prod_{k\,:\ \operatorname{col}(\boldsymbol{P}, k) = 1} \Big( \langle \operatorname{row}(\boldsymbol{P}, k) \rangle - \langle \operatorname{row}_1(\boldsymbol{P}, k) \rangle \Big).$$

**6.8. Saturating a flag, $\mathcal{S}\boldsymbol{P}$.** Let $f = (f_l \leq \cdots \leq f_h)$, but no longer assume each $\leq$ is $\dot{\leq}$. The rules given previously for tableau games $\partial, d, \mathcal{C}$ do not change, and give the same transformations of flag types: for $\partial\boldsymbol{P}$, drop the low partition from $\boldsymbol{P}$, or equivalently, replace $\operatorname{low}(\boldsymbol{P})+1$ by $\bullet$; for $d\boldsymbol{P}$, drop the high partition, or equivalently, delete all cells whose entry is $\operatorname{high}(\boldsymbol{P})$; and for $\mathcal{C}\boldsymbol{P}$, apply $\mathcal{C}$ to each partition in $\boldsymbol{P}$, or equivalently, delete the first column of $\boldsymbol{P}$. The situation for most other games is more complicated, but can be handled in a systematic fashion.

Saturate the flag $f$ by adding additional elements, as follows. For some $i$, let $r = \rho(f_{i-1}, f_i)$, and introduce additional indicies ordered $i - 1 = i_0 < i_1 < i_2 < \cdots < i_r = i$. Choose an extension of $(f_{i-1} < f_i)$ to a saturated flag $(f_{i-1} = f_{i_0} \lessdot f_{i_1} \lessdot f_{i_2} \lessdot \cdots \lessdot f_{i_r} = f_i)$; there is always some way to do this, and usually there are many. After doing this for all $l < i \leq h$, let $f^s$ be the flag so obtained. We introduce a game that uniformly determines the type of the saturated flag when $\boldsymbol{P}$ is semistandard. When $\boldsymbol{P}$ is row and column weak but not necessarily semistandard, we shall see that something close to uniformity is obtained.

$$\begin{aligned} \mathcal{S}, \tilde{\mathcal{S}} :\ \operatorname{ftype} f &\longmapsto \operatorname{ftype} f^s \\ \mathcal{S}, \tilde{\mathcal{S}} : \operatorname{ftype}_{\hat{0}} f &\longmapsto \operatorname{ftype}_{\hat{0}} f^s \end{aligned}$$

*Tableau Rules.* For each symbol $i$ in $\boldsymbol{P}$, do the following. Suppose $\boldsymbol{P}$ has a total of $r$ entries equal to $i$.

$\boldsymbol{\mathcal{S}}$: Traverse $\boldsymbol{P}$ by going down the first column, then down the second column, and so on, and replace the occurrences of $i$ by $i_1, \ldots, i_r$ in the order they are encountered.

$\tilde{\boldsymbol{\mathcal{S}}}$: Replace the symbols $i$ by $i_1, \ldots, i_r$ in any fashion so that the subtableau formed by $i_1, \ldots, i_r$ is a standard skew tableau.

**Example 6.5.**

$$\boldsymbol{P} = \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & 1 & 1 \\ \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2} \end{array} \qquad \mathcal{S}\boldsymbol{P} = \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & 1_3 & 1_5 \\ \hline 1_1 & 1_2 & 1_4 & 2_3 \\ \hline 2_1 & 2_2 \\ \cline{1-2} \end{array}$$

There are 5 ways to place subscripts on the 1's so that the subscripts are row and column strict, and 3 ways to similarly place them on the 2's, so there are 15 possible values of $\tilde{\mathcal{S}}\boldsymbol{P}$.

In $\operatorname{ftype}_x f^s$, the cell of $\operatorname{type}[x, f_{i_k}]/\operatorname{type}[x, f_{i_{k-1}}]$ is filled with the symbol $i_k$, and the cells of $\operatorname{type}[x, f_i]/\operatorname{type}[x, f_{i-1}]$ are a standard skew tableau on entries $i_1, \ldots, i_r$. Not all $f_{i-1}$ and $f_i$ allow saturations permitting any given skew tableau filling, but given a filling, we can begin with arbitrary $f_i$ and form a descending chain of elements terminating in some $f_{i-1}$ so that the filling is achieved.

**Theorem 6.6.**

(1) *Given tableaux* $\boldsymbol{P}$ *and* $\boldsymbol{Q} = \tilde{\mathcal{S}}\boldsymbol{P}$, *the number of pairs of flags* $(f, f^s)$ *with* $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$, $\operatorname{ftype}_{\hat{0}} f^s = \boldsymbol{Q}$, *and* $f^s$ *a saturation of* $f$ *is* $F^{\boldsymbol{Q}}(q)$.

(2) *Let* $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ *and* $n_{ij}$ *be the number of entries in column* $j$ *of* $\boldsymbol{P}$ *equal to* $i$. *There are*
$$\prod_{l<j\le h} \langle n_{ij}\rangle!$$
*saturations* $f^s$ *of* $f$ *with* $\operatorname{ftype}_{\hat{0}} f = \mathcal{S}\boldsymbol{P}$.

(3) *When* $\boldsymbol{P}$ *is semistandard,* $\mathcal{S}$ *uniformly maps flags with* $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ *to saturations* $f^s$ *with* $\operatorname{ftype}_{\hat{0}} f^s = \mathcal{S}\boldsymbol{P}$.

*Proof.*

(1) Choose any of $F^{\boldsymbol{Q}}(q)$ flags $f^s$ with $\operatorname{ftype}_{\hat{0}} f^s = \boldsymbol{Q}$. The only possible choice of $f$ is the subflag $f = (f^s_l, f^s_{l+1}, \ldots, f^s_h)$.

(2) Choose $i$ with $l < i \le h$. Let $N_j = n_{i1} + \cdots + n_{ij}$ and $I_j = i_{N_j}$. Let $\mu = \lambda^{(i-1)}$ and $\lambda = \lambda^{(i)}$. Since the desired type of $f_{I_j}$ is $\mu \vee \lambda[j]$ and $f_{i-1} \le f_{I_j} \le f_i$, we have $f_{I_j} = f_{i-1} \vee f_i[j]$ for all $j$. Choose a saturation of the type $(1^{n_{ij}})$ interval $[f_{I_{j-1}}, f_{I_j}]$ to a flag $(f_{I_{j-1}} \lessdot f_{I_{j-1}+1} \lessdot \cdots \lessdot f_{I_j})$ in any of $\langle n_{ij}\rangle!$ ways. Multiply the number of choices over all $i$ and $j$ to obtain the total in the theorem.

(3) If $\boldsymbol{P}$ is semistandard, each $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip. We show that generically, $[f_{i-1}, f_i]$ is a chain, so that there is only one saturation of it. Let $y$ have type $\lambda^{(i)}$ and let $\mu = \lambda^{(i-1)}$. The number of $x \le y$ with type $\mu$ and cotype $\nu$ is the Hall polynomial $g^{\lambda}_{\mu\nu}(q)$, of degree $n(\lambda)-n(\mu)-n(\nu)$; we seek to maximize this degree, or equivalenty, minimize $n(\nu)$. Since $\lambda/\mu$ is a horizontal strip of some length $r$, the value $\nu = (r)$ with $n(\nu) = 0$ uniquely minimizes $n(\nu)$.

So generically, $[f_{i-1}, f_i]$ is a chain for each $i$, and $f$ has only one saturation, whose type is therefore $\mathcal{S}\boldsymbol{P}$. Every saturated flag of type $\mathcal{S}\boldsymbol{P}$ has exactly one inverse of type $\boldsymbol{P}$, so the map is uniform. $\square$

When $\boldsymbol{P}$ is not semistandard, there need not be a generic value of $\operatorname{ftype}_{\hat{0}} f^s$. For example, in $q$-regular semi-primary lattices of type $\lambda \vdash r$, if $f_0 = \hat{0}$ and $f_1 = \hat{1}$, there are $F^{\boldsymbol{Q}}(q)$ saturations of type $\boldsymbol{Q}$ for each standard tableau $\boldsymbol{Q}$ of shape $\lambda$ on $1_1, \ldots, 1_r$. These polynomials all have degree $n(\lambda)$, so asymptotically, each tableau has probability $1/f^{\lambda}$ of occurring as the type of a saturation.

More generally, by Theorem 4.92, if $\operatorname{type} f_i/\operatorname{type} f_{i-1} = \lambda/\mu$, then generically, $\operatorname{type}[f_{i-1}, f_i] = \nu$, where $\nu = (S(\lambda' - \mu'))'$. Let $r = |\nu|$. Let

$$T = \{\, \boldsymbol{Q} : \boldsymbol{Q} \text{ is standard on } i_1, \ldots, i_r, \text{ with } \operatorname{sh} \boldsymbol{Q} = \lambda/\mu \text{ and } \operatorname{sh} j(\boldsymbol{Q}) = \nu \,\}.$$

As $q \to \infty$, over all saturations $f^s$ of flags $f$ of type $\boldsymbol{P}$, each tableau in $T$ has probability $(\#T)^{-1} + O(q^{-1})$ of being the subtableau replacing the $i$'s, while any tableau not in $T$ has probability $o(q^{-1})$. Every flag $f^s$ of type $\boldsymbol{Q}$ arises from a single flag $f$ of type $\boldsymbol{P}$, so we have a condition similar to uniformity.

Note that $\mathcal{S}\boldsymbol{P} \in T$ because applying $j$ to the subtableau of $i_1, \ldots, i_r$ simply deletes the $\bullet$'s at the top of each column, pushes the columns up to be aligned at the top, and then pushes the entries on each row horizontally left to fill any holes, thus obtaining a tableau of shape $\nu$ (this is transpose to Corollary 6.4).

In the following, the type of a flag may be interpreted as $\operatorname{ftype} f$ or $\operatorname{ftype}_x f$, as appropriate.

**Theorem 6.7.** *Let $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ be a growth on semi-primary lattices, and extend it to an operator on flags by $\phi(f_l, \ldots, f_h) = (\phi(f_l), \ldots, \phi(f_h))$. Suppose $\phi$ generically (respectively, uniformly) transforms multisaturated flags of type $\boldsymbol{P}$ to flags of type $\mathcal{G}\boldsymbol{P}$. If $\boldsymbol{P}$ is row and column weak but not necessarily standard, $\phi$ generically (uniformly) maps flags of type $\boldsymbol{P}$ to flags of type $\mathcal{S}^{-1}\mathcal{G}\mathcal{S}(\boldsymbol{P})$, where $\mathcal{S}^{-1}$ drops the subscripts from the entries. The degenerate possibilities are all of the form $\mathcal{S}^{-1}\tilde{\mathcal{G}}\mathcal{S}(\boldsymbol{P})$.*

*A similar statement holds for order-reversing growths, that is, $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ such that if $x \le y$ then $\phi(y) \le \phi(x)$. Set $\phi(f_l, \ldots, f_h) = (\phi(f_h), \ldots, \phi(f_l))$ in the above.*

*Proof.* Since there are always saturations of type $\mathcal{S}\boldsymbol{P}$, the possible outcomes of a game on a nonsaturated flag may be determined by saturating it with type $\mathcal{S}\boldsymbol{P}$, applying further operators, and later "unsaturating" by dropping the subscripts from the entries; the value of $\phi(f)$ does not depend on the saturation, so the added elements can be removed.

Every flag of type $\boldsymbol{P}$ has the same number of saturations of type $\mathcal{S}\boldsymbol{P}$, and every saturated flag of type $\mathcal{S}\boldsymbol{P}$ has a single unsaturated inverse flag $f$ of type $\boldsymbol{P}$. This is quite similar to uniformity, but with a multivalued saturation function instead of a singly valued function, so saturation composes with other generic or uniform operators in a fashion similar to Theorem 5.12 by essentially the same proof. $\square$

All operators we describe on semi-primary lattices have the form described in the theorem, except for $d\boldsymbol{P}$ and $\partial\boldsymbol{P}$, which already have been described for nonsaturated flags. The theorem does not hold for $d\boldsymbol{P}$, for suppose $\rho(f_{h-1}, f_h) = r > 1$. We saturate $f$ to obtain $f^s = (\ldots, f_{h_{r-1}}, f_{h_r})$. Then $df^s = (\ldots, f_{h_{r-1}})$, and so the type of this still has subscripted $h$'s, which turn to ordinary $h$'s when we apply $\mathcal{S}^{-1}$. However, $df = (f_l, \ldots, f_{h-1})$, so there are no $h$'s in $\operatorname{ftype} df$. A similar problem occurs for $\partial f$.

**6.9. The dual of a flag,** $\operatorname{ev}\boldsymbol{P}$**.** Uniformly,

$$\operatorname{ev}, \widetilde{\operatorname{ev}} : \operatorname{ftype} f \longmapsto \operatorname{ftype} f^*.$$

By Theorem 5.14, the uniform cotype of saturated flags of type $\boldsymbol{P}$ is $\operatorname{ev}\boldsymbol{P}$ when $\boldsymbol{P}$ is a standard tableau on $1, \ldots, n$. This is trivially extended to multisaturated flags via $\operatorname{ev}\boldsymbol{P} = (\operatorname{sh}\Delta^{n-k}\boldsymbol{P})_{k=n-h}^{l}$, with degenerate possibilities given by using $\widetilde{\Delta}$ instead of $\Delta$.

*Tableau Rules.* Let $\boldsymbol{P}$ be a standard tableau on distinct entries. Repeatedly apply the operator $\Delta$ (for ev) or $\widetilde{\Delta}$ (for $\widetilde{\operatorname{ev}}$) to the subtableau of unstarred entries, and when an entry $k$ is deleted, place an entry $k^*$ in the cell that is concurrently deleted. Finally, a tableau with only starred entries is obtained, and each $k^*$ is replaced by $n + 1 - k$.

*Local Rules.* This does not have a representation in the form of two lines of partitions, but rather, has the form of a triangular array of partitions; see Section 5.2.

This can also be extended to flags that aren't from $\hat{0}$ to $\hat{1}$. The following map is uniform on flags in semi-primary lattices of type $\lambda$.

$$\operatorname{ev}^\lambda, \widetilde{\operatorname{ev}}^\lambda : \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}^*} f^*$$

*Tableau Rules.* Let $\boldsymbol{P}$ be a standard skew tableau of shape $\mu/\nu$ with $\mu \leq \lambda$. Let $r = |\nu|$ and $s = |\lambda| - |\mu|$. Fill $\nu$ with entries $l_1, l_2, \ldots, l_r$ by traversing its cells down the first column, then down the second column, and so on. Fill $\lambda/\mu$ with entries $(h+1)_1, (h+1)_2, \ldots, (h+1)_s$ in a similar fashion. Apply the ordinary ev or $\widetilde{\operatorname{ev}}$ as described above, but replace all $(h+1)_i^*$'s by $\bullet$'s and remove all cells with $l_i^*$'s.

This map is uniform: extend the flag $f = (f_l, \ldots, f_h)$ to the flag $(\hat{0}, f_l, \ldots, f_h, \hat{1})$ (filling $\nu$ with $l$'s and $\lambda/\mu$ with $(h+1)$'s), saturate it with a canonical saturation ($\mathcal{S}$), dualize it (apply ev), unsaturate it ($\mathcal{S}^{-1}$), and remove the leading $\hat{1}^*$ (replacing the $(h+1)_i$'s by $\bullet$'s) and the trailing $\hat{0}^*$ (removing the cells with $l_i$'s).

**Example 6.8.** Let $\lambda = (3, 3, 2)$ and $n = 7$. In semi-primary lattices of type $\lambda$, consider flags $f = (f_3, f_4, f_5)$ with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$, whose saturation to shape $\lambda$ is $\boldsymbol{P}^s$:

$$\boldsymbol{P} = \begin{array}{|c|c|c|}\hline \bullet & \bullet & 4 \\ \hline \bullet & 5 \\ \cline{1-2}\end{array} \qquad \boldsymbol{P}^s = \begin{array}{|c|c|c|}\hline 3_1 & 3_3 & 4 \\ \hline 3_2 & 5 \\ \cline{1-2} 6_1 & 6_2 \\ \cline{1-2}\end{array}\,.$$

This is a relabeling of the tableau used throughout Section 5.1. Do repeated applications of $\Delta$, keeping track of the vacated cells by replacing them with starred deleted entries:

$$
\begin{array}{ccc} 3_1 & 3_3 & 4 \\ 3_2 & 5 & \\ 6_1 & 6_2 & \end{array}
\qquad
\begin{array}{ccc} 3_2 & 3_3 & 4 \\ 5 & 6_2 & \\ 6_1 & 3_1^* & \end{array}
\qquad
\begin{array}{ccc} 3_3 & 4 & 3_2^* \\ 5 & 6_2 & \\ 6_1 & 3_1^* & \end{array}
\qquad
\begin{array}{ccc} 4 & 6_2 & 3_2^* \\ 5 & 3_3^* & \\ 6_1 & 3_1^* & \end{array}
$$

$$
\begin{array}{ccc} 5 & 6_2 & 3_2^* \\ 6_1 & 3_3^* & \\ 4^* & 3_1^* & \end{array}
\qquad
\begin{array}{ccc} 6_1 & 6_2 & 3_2^* \\ 5^* & 3_3^* & \\ 4^* & 3_1^* & \end{array}
\qquad
\begin{array}{ccc} 6_2 & 6_1^* & 3_2^* \\ 5^* & 3_3^* & \\ 4^* & 3_1^* & \end{array}
\qquad
\begin{array}{ccc} 6_2^* & 6_1^* & 3_2^* \\ 5^* & 3_3^* & \\ 4^* & 3_1^* & \end{array}
$$

Replace $6_i$'s by $\bullet$'s and remove $3_i$'s to obtain that uniformly, we expect

$$\operatorname{ftype}_{\hat{0}^*} f^* = \operatorname{ev}^\lambda \boldsymbol{P} = \begin{array}{cc} \bullet & \bullet \\ 5^* & \\ 4^* & \end{array} = \begin{array}{cc} \bullet & \bullet \\ 3 & \\ 4 & \end{array}.$$

## 7. The Robinson-Schensted correspondence

**7.1. The Robinson-Schensted correspondence.** Let $S_n$ be the group of permutations on $1, \ldots, n$. Robinson [**3**] and Schensted [**24**] created an algorithmic bijection between ordered pairs of standard tableaux of the same shape with entries $1, \ldots, n$, and $S_n$. See Sagan [**22**] for a comprehensive account. There have been many generalizations of it, including for repeated entries (Knuth [**15**]), skew tableaux (Sagan and Stanley [**23**]), bijections of a similar nature with other structures (Fomin [**4**]), and oscillating chains in Young's lattice (Roby [**21**]). The basic algorithm is as follows.

**Robinson-Schensted Row Insertion.** Let $\boldsymbol{P}$ be a standard tableau on distinct entries, and $k$ be a number not in $\boldsymbol{P}$. Insert $k$ in the first row by either replacing the smallest number on the row larger than $k$ with $k$, or by appending $k$ to the end of the row if it is larger than all entries on it. If a number was replaced, insert it into the second row in the same fashion. Continue this on successive rows until the process terminates by appending a number to a row. The final tableau is the result of row inserting $k$ into $\boldsymbol{P}$.

**Robinson-Schensted Algorithm.** Let $\sigma \in S_n$. Let $\boldsymbol{P}_0 = \emptyset$, and for $k = 1, \ldots, n$, form $\boldsymbol{P}_k$ by row inserting $\sigma(k)$ into $\boldsymbol{P}_{k-1}$. Define $\boldsymbol{P}(\sigma) = \boldsymbol{P}_n$, called the **insertion tableau**. Form a tableau $Q(\sigma)$ on $1, \ldots, n$, called the **recording tableau**, by placing $k$ at the cell where inserting $\sigma(k)$ into $\boldsymbol{P}_{k-1}$ terminates, so that $\boldsymbol{Q}(\sigma) = (\operatorname{sh} \boldsymbol{P}_0, \ldots, \operatorname{sh} \boldsymbol{P}_n)$.

The inverse map is given by a deletion process.

**Robinson-Schensted Deletion.** Let $\boldsymbol{P}$ be a standard tableaux on distinct entries, and $c$ be an inner corner of $\boldsymbol{P}$. The deletion step beginning at $c$ is as follows. Let $e$ be the entry at $c$. Remove cell $c$. If $e$ is on the top row, we are done. Otherwise, go up one row. Replace the largest entry smaller than $e$ with $e$ and let $e$ be the number that was replaced. If we're on the top row, we're done, and otherwise, go up another row, and replace the largest entry smaller than $e$ with $e$. Continue in this fashion until a number is removed from the top row. That number is said to be **deleted** from the tableau.

**Inverse Robinson-Schensted Algorithm.** Let $(\boldsymbol{P},\boldsymbol{Q})$ be a pair of standard tableaux of equal shape with entries $1,\ldots,n$. Let $c_k \in \mathbb{P}\times\mathbb{P}$ be the cell containing $k$ in $\boldsymbol{Q}$. Let $\boldsymbol{P}_n = \boldsymbol{P}$, and for $k = n,\ldots,1$, form $\boldsymbol{P}_{k-1}$ from $\boldsymbol{P}_k$ by a deletion step beginning at cell $c_k$, resulting in some entry $\sigma(k)$ being deleted from the tableau. Denote the permutation $\sigma$ obtained in this fashion by $\sigma_{\mathrm{RS}}(\boldsymbol{P},\boldsymbol{Q})$.

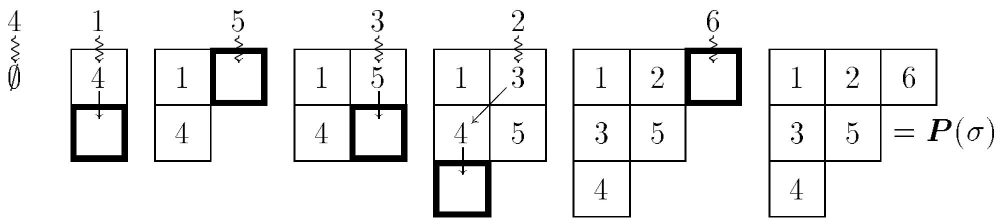
**Theorem 7.1 (Robinson-Schensted Correspondence).** *The maps*

$$\sigma \mapsto (\boldsymbol{P}(\sigma),\boldsymbol{Q}(\sigma)) \qquad \textit{and} \qquad (\boldsymbol{P},\boldsymbol{Q}) \mapsto \sigma_{\mathrm{RS}}(\boldsymbol{P},\boldsymbol{Q})$$

*are inverse bijections between $S_n$ and pairs of standard tableaux of the same shape on $1,\ldots,n$.*

See Robinson [**3**] and Schensted [**24**] for the original proof.

**Example 7.2.** The **one-line** form of a permutation $\sigma$ is the sequence of numbers $\sigma(1)\,\sigma(2)\cdots$. Let $\sigma = 4\ 1\ 5\ 3\ 2\ 6$. The sequence of tableaux $\boldsymbol{P}_0,\ldots,\boldsymbol{P}_6$ and the insertion paths are as follows. The outer corner in which the path will terminate is shown in bold.



$$\boldsymbol{Q}(\sigma) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 & \\ \cline{1-2} 5 & & \\ \cline{1-1}\end{array}$$

The inverse Robinson-Schensted algorithm is performed via deletion paths obtained by translating the arrows one tableau right and reversing their direction.

**7.2. Bruhat order.** Let $\sigma \in S_n$ and $A(\sigma) = (a_{ij})_{0\le i,j\le n}$ be the matrix with entries $a_{ij} = \#\{\, j' : j' \le j \text{ and } \sigma(j') \le i' \,\}$. Note that $\sigma(j) = \min\{\, i : a_{ij} > a_{i,j-1} \,\}$ and $\sigma(j) = i$ iff $a_{ij} > a_{i,j-1}$ but $a_{i-1,j} = a_{i-1,j-1}$.

The **Bruhat order** is a graded partial order on $S_n$ given by $\sigma \le \tau$ iff $A(\sigma) \ge A(\tau)$ entrywise. See [**29**, p. 528] and [**31**, p. 83] for discussion of the Bruhat order in this form, and [**17**, pp. 5–8] for an in-depth discussion of other descriptions and properties of the order.

**7.3. Relative positions of flags in modular lattices.** Throughout this section, $f = (f_0 \lessdot \cdots \lessdot f_n)$ and $f' = (f'_0 \lessdot \cdots \lessdot f'_n)$ are saturated flags from $\hat{0}$ to $\hat{1}$ in a modular lattice.

**Definition 7.3.** This **relative position** of flags $f$ and $f'$ is the permutation $\sigma(f, f')$ given by $\sigma(k) = \min\Big\{ i \,:\, f_i \wedge f'_k \gtrdot f_i \wedge f'_{k-1} \Big\}$ for $k = 1, \ldots, n$.

We can also replace $f_i \wedge f'_k \gtrdot f_i \wedge f'_{k-1}$ by $\rho(f_i \wedge f'_k) > \rho(f_i \wedge f'_{k-1})$ in the above. The relative position is completely determined by the ranks $\rho(f_i \wedge f'_k)$ for $0 \leq i, k \leq n$.

Another construction of $\sigma(f, f')$ is as follows. For each $0 \leq k \leq n$, consider $g^{(i)} = (f_i \wedge f'_0 \lesseqdot \cdots \lesseqdot f_i \wedge f'_n) = f' \wedge f_i$. The bottom element of $g^{(i)}$ is $\hat{0}$ and the top element is $f'_n \wedge f_i = f_i$, so of the weak cover relations, $i$ are strict. Clearly $g^{(i-1)} = g^{(i)} \wedge f_{i-1}$. Consider the $\lesseqdot$ relations in $g^{(i)}$ and in $g^{(i-1)}$. One $\lessdot$ in $g^{(i)}$ becomes $=$ in $g^{(i-1)}$, and all other $\lessdot$ or $=$ remain the same. If $f_i \wedge f'_{k-1} \lessdot f_i \wedge f'_k$ but $f_{i-1} \wedge f'_{k-1} = f_{i-1} \wedge f'_k$, define $\sigma(k) = i$. This associates a unique $k$ to each $i = n, \ldots, 1$, and since all relations are $\lessdot$ in $g^{(n)}$ and $=$ in $g^{(0)}$, there is for each $k$ a unique $i$ when the $k$th relation switches from $\lessdot$ to $=$. So $\sigma(f, f')$ is a permutation.

**Theorem 7.4 (Steinberg [29]).** *In the invariant subspace lattice over an infinite field, the generic relative position of saturated flags $(f, f')$ with* $\operatorname{ftype} f = \boldsymbol{P}$ *and* $\operatorname{ftype} f' = \boldsymbol{Q}$ *is $\sigma_{\mathrm{RS}}(\boldsymbol{P}, \boldsymbol{Q})$.*

See also van Leeuwen [**31**].

**Conjecture 7.5.** *Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be standard tableaux of shape $\lambda$. In $q$-regular semi-primary lattices of type $\lambda$, as $q \to \infty$, all but $O(q^{-1})$ pairs $(f, f')$ of flags with* $\operatorname{ftype} f = \boldsymbol{P}$ *and* $\operatorname{ftype} f' = \boldsymbol{Q}$ *have $\sigma(f, f') = \sigma_{\mathrm{RS}}(\boldsymbol{P}, \boldsymbol{Q})$.*

The tableaux $\boldsymbol{P} = (\lambda^{(0)}, \ldots, \lambda^{(n)})$ and $\boldsymbol{Q} = (\mu^{(0)}, \ldots, \mu^{(n)})$ are chains of partitions in Young's lattice, which is modular. Their relative position $\sigma(\boldsymbol{P}, \boldsymbol{Q})$ in Young's lattice is the permutation $\sigma(j) = i$ where the cell with $i$ in $\boldsymbol{P}$ has $j$ in $\boldsymbol{Q}$: comparing the tableaux representing the chains $\boldsymbol{P} \wedge \mu^{(j)}$ and $(\boldsymbol{P} \wedge \mu^{(j)}) \wedge \mu^{(j-1)} = \boldsymbol{P} \wedge \mu^{(j-1)}$, we see that the cell with $j$ in $\boldsymbol{Q}$ is deleted from the former and all other cells stay intact. The deleted entry is whatever was in that cell in $\boldsymbol{P}$.

**Conjecture 7.6.** *Let $(f, f')$ be a pair of saturated flags in a semi-primary lattice with* $\operatorname{ftype} f = \boldsymbol{P}$ *and* $\operatorname{ftype} f' = \boldsymbol{Q}$*. Then $\sigma(\boldsymbol{P}, \boldsymbol{Q}) \leq \sigma(f, f') \leq \sigma_{\mathrm{RS}}(\boldsymbol{P}, \boldsymbol{Q})$ holds in the Bruhat order.*

We will see later, in Corollary 9.20, that this holds in all semi-primary lattices whose type has two rows. For arbitrary types, The upper bound holds for invariant subspace lattices over infinite fields (see [**31**, Theorem 2.5.4]), and the lower bound is a consequence of the following.

**Theorem 7.7.** *Let $L$ and $M$ be modular lattices, and $\phi : L \mapsto M$ be a rank and order preserving map. Let $f$ and $f'$ be saturated flags in $L$ from $\hat{0}$ to $x$. Then $\sigma\Big(\phi(f), \phi(f')\Big) \leq \sigma(f, f')$.*

*Proof.* The map $\phi$ preserves order, but not necessarily meets and joins. We have $f_i \wedge f'_j \le f_i, f'_j$ so $\phi(f_i \wedge f'_j) \le \phi(f_i), \phi(f'_j)$, whence $\phi(f_i \wedge f'_j) \le \phi(f_i) \wedge \phi(f'_j)$. Take the ranks of both sides: $\rho(f_i \wedge f'_j) \le \rho\big(\phi(f_i) \wedge \phi(f'_j)\big)$. Thus, $\sigma(f, f') \ge \sigma\big(\phi(f), \phi(f')\big)$ in the Bruhat order. □

We consider an enumeration problem for pairs of flags of specified types in specified relative positions, similar to the enumeration problem for the interval type table of a flag considered in Section 5.5.

**Conjecture 7.8.** *Given standard tableaux $\boldsymbol{P}$ and $\boldsymbol{Q}$ of equal shape $\lambda$ on $1, \dots, n$, and a permutation $\sigma \in S_n$, the number of pairs $(f, f')$ of flags with* $\operatorname{ftype} f = \boldsymbol{P}$*,* $\operatorname{ftype} f' = \boldsymbol{Q}$*, and $\sigma(f, f') = \sigma$ is often a polynomial in $q$, depending only on $\boldsymbol{P}$, $\boldsymbol{Q}$, and $\sigma$, and on no further structure of the lattice. The polynomial is divisible by $M_\lambda(q)$. As a function of $\sigma$, its degree increases as $\sigma$ increases in the Bruhat order from $\sigma(\boldsymbol{P}, \boldsymbol{Q})$ to $\sigma_{\mathrm{RS}}(\boldsymbol{P}, \boldsymbol{Q})$.*

We consider three square arrays of partitions that each refine the notion of relative position. Their components $\lambda^{(ij)}$ for $0 \le i, j \le n$ are as follows.

| | $\lambda^{(ij)}$ | $\rho(f_i \wedge f'_j)$ |
|---|---|---|
| (1) | $\operatorname{type}(f_i \wedge f'_j)$ | $\lvert\lambda^{(ij)}\rvert$ |
| (2) | $\operatorname{type}[f_i, f_i \vee f'_j]$ | $j - \lvert\lambda^{(ij)}\rvert$ |
| (3) | $\operatorname{type}(f_i \vee f'_j)$ | $i + j - \lvert\lambda^{(ij)}\rvert$ |

The relative position of $(f, f')$ can be computed from any of these, because the values of $\rho(f_i \wedge f'_j)$ can be determined. In subsequent sections it is shown that given $\operatorname{ftype} f = \boldsymbol{P}$ and $\operatorname{ftype} f' = \boldsymbol{Q}$, computation of (1) is similar to Robinson-Schensted deletion; (2) is similar to a transposed deletion operating on columns instead of rows; and (3) is similar to insertion in skew tableaux.

**Conjecture 7.9.** *For all but $O(q^{-1})$ pairs $(f, f')$ with* $\operatorname{ftype} f = \boldsymbol{P}$ *and* $\operatorname{ftype} f' = \boldsymbol{Q}$*, in (1) the tableaux $\boldsymbol{P}_n, \dots, \boldsymbol{P}_0$ obtained by Robinson-Schensted deletion from $(\boldsymbol{P}, \boldsymbol{Q})$ are $\boldsymbol{P}_k = (\lambda^{(0k)}, \dots, \lambda^{(nk)})$, and in (2) the same, but with the transposed deletion process.*

We will later see that (3) does not have a generic value.

In the next section, we will consider enumeration of the number of pairs $(f, x)$ where $x \lessdot \hat{1}$ and the desired values of $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ and $\operatorname{ftype}_{\hat{0}}(f \wedge x)$ are given. Subsequent sections will consider similar problems for (2) and (3), and will encounter similar difficulties. Let $\boldsymbol{P}_{n-1}$ be the tableau obtained from a Robinson-Schensted deletion step in $\boldsymbol{P}$ beginning at cell $(\operatorname{sh} \boldsymbol{P}) / \operatorname{type} x$. The most common value of $\operatorname{ftype}_{\hat{0}}(f \wedge x)$ given $\boldsymbol{P}$ and $\operatorname{type} x$ is $\boldsymbol{P}_{n-1}$. To compute the relative position, we look at flags $f \wedge f'_k$. Since $f \wedge f'_{k-1} = (f \wedge f'_k) \wedge f'_{k-1}$, it would be tempting to repeat the process to obtain a series of Robinson-Schensted deletion steps. The uniformity condition in Theorem 5.12 that we use elsewhere to compose steps is *not* met here: $f \mapsto f \wedge x$ generically, but not uniformly, maps flags with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ to flags $g$ with $\operatorname{ftype}_{\hat{0}} g = \boldsymbol{P}_{n-1}$; topologically, $\{\, f \wedge x \,:\, \operatorname{ftype}_{\hat{0}}(f \wedge x) = \boldsymbol{P}_{n-1} \text{ and } \operatorname{ftype}_{\hat{0}} f = \boldsymbol{P} \,\}$ is not dense in $\{\, g \,:\, g_n = x \text{ and } \operatorname{ftype}_{\hat{0}} g = \boldsymbol{P}_{n-1} \,\}$, while enumeratively, the cardinality of the former set is a fraction $O(q^{-1})$ of the latter, rather than $1-O(q^{-1})$. Some other technique is required to continue. Steinberg [**29**] resolves this for invariant subspace lattices over infinite fields by using properties of the Zariski topology and a larger lattice (all pairs of saturated flags in the vector space $V$, without the condition of invariance under $N$) that concurrently embeds invariant subspace lattices of all types of order $\dim V$.

**7.4. Robinson-Schensted deletion, $\mathcal{D}_c\boldsymbol{P}$.** Let $\lambda/\mu = \operatorname{sh}\boldsymbol{P}$; the parameter $c$ is either $\emptyset$ or an inner corner of $\lambda$ not in $\mu$.

Pick $x \leq f_h$. The following map is generic but not uniform.

$$\begin{array}{ll} \mathcal{D}_c, \widetilde{\mathcal{D}}_c : \ \operatorname{ftype} f \longmapsto \operatorname{ftype}(f \wedge x) & c = \lambda/(\operatorname{type}[f_l \wedge x, f_h \wedge x]) \\ \mathcal{D}_c, \widetilde{\mathcal{D}}_c : \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}}(f \wedge x) & c = \lambda/\operatorname{type} x \end{array}$$

*Tableau Rules.* See Figure 8. Let $\boldsymbol{P}$ be a standard skew tableau on distinct entries, and $c$ be an inner corner of $\operatorname{osh}\boldsymbol{P}$ or $c = \emptyset$. If $c = \emptyset$ then $\widetilde{\mathcal{D}}_c\boldsymbol{P} = \boldsymbol{P}$. Otherwise, proceed as follows.

We insert an entry $e$ into a row by replacing its largest entry (possibly $\bullet$) smaller than $e$ with $e$; the number it replaces is **bumped** from the row.

- **($\mathcal{D}_c$):** Begin by letting $e$ be the entry at $c$ and removing cell $c$ from the tableau. Insert $e$ into the row above $c$, bumping out a number. Insert that number one row higher, bumping out another number. Continue this until either an entry is bumped out from the top row, or $\bullet$ is bumped out.
- **($\widetilde{\mathcal{D}}_c$):** Begin by letting $e$ be the entry at $c$ and removing cell $c$ from the tableau. Either choose any higher up row and insert $e$ on it (provided the column $e$ lands in continues to increase from top to bottom), or choose to stop. Take the bumped out number, and either choose a still higher row to insert it into, provided the column remains increasing, or choose to stop. Continue in this fashion until optionally stopping; bumping out $\bullet$; or bumping out an entry from the top row.

The final value bumped out is said to be **deleted** from the tableau.

*Local Rules.* If $c = \emptyset$, let $\mu^{(h)} := \lambda^{(h)}$, and if $c$ is a single cell, let $\mu^{(h)}$ be $\lambda^{(h)}$ with $c$ deleted. Then for $k = h, h-1, \ldots, l+1$, propagate as follows.

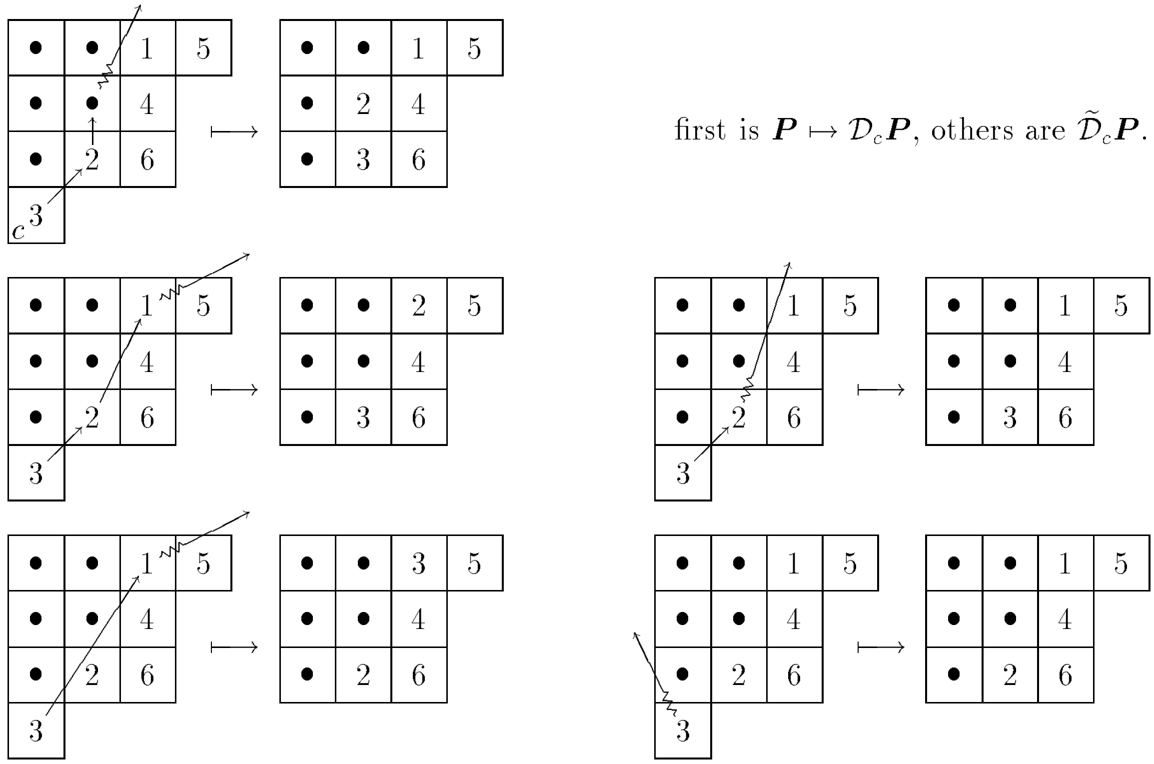
$$\begin{array}{c|c|c|c|c|l} \multicolumn{1}{c}{} & \multicolumn{1}{c}{k-1} & \multicolumn{1}{c}{k} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{h} & \text{column} \\ \cline{2-3}\cline{5-5} & \gamma & \delta & & \lambda^{(h)} & \boldsymbol{P} \\ \cline{2-3}\cline{5-5} \longleftarrow & (\alpha) & \beta & & (\lambda^{(h)} \ominus c) & \widetilde{\mathcal{D}}_c\boldsymbol{P} \\ \cline{2-3}\cline{5-5} \end{array}$$

**$\boldsymbol{\beta = \gamma \lessdot \delta}$ does not hold:** Let $\alpha := \beta \wedge \gamma$.

**$\boldsymbol{\beta = \gamma \lessdot \delta}$ holds:** Let $c := \delta/\beta$.

- **$\mathbf{row}(\boldsymbol{c}) = \mathbf{1}$:** Let $\alpha := \gamma$.
- **$\mathbf{row}(\boldsymbol{c}) > \mathbf{1}$ ($\boldsymbol{\mathcal{D}_c}$):** Delete the last cell of $\gamma$ on the row above $c$ to obtain $\alpha$.
- **$\mathbf{row}(\boldsymbol{c}) > \mathbf{1}$ ($\boldsymbol{\widetilde{\mathcal{D}}_c}$):** Choose either to let $\alpha := \gamma$, or to delete any inner corner of $\gamma$ whose row is less than $\operatorname{row}(c)$ to obtain $\alpha$.

The unique $k$ such that $\lambda^{(k)} > \lambda^{(k-1)}$ and $\mu^{(k)} = \mu^{(k-1)}$ is the entry deleted from the tableau.



first is $\boldsymbol{P} \mapsto \mathcal{D}_c\boldsymbol{P}$, others are $\widetilde{\mathcal{D}}_c\boldsymbol{P}$.

FIGURE 8. All possible degenerate skew Robinson-Schensted deletion games.

We will show that the tableau interpretation and the local rule interpretation of the games $\mathcal{D}_c$ and $\widetilde{\mathcal{D}}_c$ are the same, and that they describe $f \wedge x$.

**Proposition 7.10.** *The tableau rules and local rules given above for $\mathcal{D}_c$ and $\widetilde{\mathcal{D}}_c$ describe the same game.*

*Proof.* Let $\boldsymbol{P} = (\lambda^{(l)}, \ldots, \lambda^{(h)})$ and $\widetilde{\mathcal{D}}_c\boldsymbol{P} = (\mu^{(l)}, \ldots, \mu^{(h)})$ (as given by the tableau rules). Since each cell of $\boldsymbol{P}$ either vanishes, stays the same, or increases in $\widetilde{\mathcal{D}}_c\boldsymbol{P}$, we have $\mu^{(k)} \leq \lambda^{(k)}$ for $l \leq k \leq h$. If $e$ is the entry bumped out, then $\mu^{(k)} = \lambda^{(k)}$ for $l \leq k < e$ because no cells with entries less than $e$ are changed. Also, $\mu^{(k)} \lessdot \lambda^{(k)}$ for $e \leq k \leq h$ because the number of entries less than or equal to $k$ in $\boldsymbol{P}$ is exactly one more than in $\widetilde{\mathcal{D}}_c\boldsymbol{P}$ since $e$ was bumped out.

For $k = l, \ldots, h$, let $\boldsymbol{P}^{(k)} = (\lambda^{(l)} \wedge \mu^{(k)}, \ldots, \lambda^{(k)} \wedge \mu^{(k)}, \mu^{(k+1)}, \ldots, \mu^{(h)})$, which we picture as follows:

$$\begin{array}{lccccccccc}
 & \lambda^{(l)} \leq_{\boxed{l+1}} \lambda^{(l+1)} \leq_{\boxed{l+2}} \cdots \leq_{\boxed{k}} & \lambda^{(k)} & & & & \\
 & & \text{\rotatebox{90}{$\leq$}}\ \boxed{\phantom{x}} \to e \;= c & & & & \\
\boldsymbol{P}^{(k)} \;= & & \mu^{(k)} & \leq_{\boxed{k+1}} \mu^{(k+1)} \leq_{\boxed{k+2}} \cdots \leq_{\boxed{h}} \mu^{(h)} & & &
\end{array}$$

Reading left to right, the cells added in successive partitions are filled with the boxed entries, except that the entry $e \leq k$ placed in the square (if any) of $\lambda^{(k)}/\mu^{(k)}$ is removed, possibly to be filled later in the chain with an entry larger than $k$.

The tableau $\boldsymbol{P}^{(h)}$ is $\boldsymbol{P}$ with the initial cell $c$, containing an entry $e$, deleted. Now compare $\boldsymbol{P}^{(k)}$ and $\boldsymbol{P}^{(k-1)}$, successively for $k = h, h-1, \ldots, l+1$. The two chains differ only in the partitions

$$\begin{array}{cc} \lambda^{(k-1)} & \lambda^{(k)} \\ \mu^{(k-1)} & \mu^{(k)} \end{array}$$

Let $c_0 = \lambda^{(k)}/\mu^{(k)}$ and $c_1 = \lambda^{(k-1)}/\mu^{(k-1)}$. If these are the same, the tableaux $\boldsymbol{P}^{(k-1)}$ and $\boldsymbol{P}^{(k)}$ are equal. We will show that the local rules for $\widetilde{\mathcal{D}}_c$ mimic the changes in these tableaux when $c_0$ and $c_1$ are unequal. When $c_0$ and $c_1$ are different squares, we have

$$\boxed{\begin{array}{ccc} \lambda^{(k-1)} & \lessdot_{c_0} & \lambda^{(k)} \\ \dot{\vee}_{c_1} & & \dot{\vee}_{c_0} \\ \mu^{(k-1)} & \lessdot_{c_1} & \mu^{(k)} \end{array}}$$

Thus, the cell $c_0$ that was deleted in $\boldsymbol{P}^{(k)}$ to make way for a larger entry contained $k$. To form $\boldsymbol{P}^{(k-1)}$, bump the entry $e < k$ out of $c_1$ and place $k$ in $c_1$.

By the local rules, $c_1$ is any inner corner of $\lambda^{(k-1)}$ on an earlier row than $c_0$, and by the tableau rules, $k$ displaces the rightmost entry on an earlier row than $c_0$, provided the column remains increasing; these correspond to each other. In the generic game $\mathcal{D}_c$, this is restricted by $1 \leq \operatorname{row}(c_1) \lessdot \operatorname{row}(c_0)$.

When $c_0$ is a square but $c_1$ is $\emptyset$, the value $k$ is bumped from the tableau. In $\widetilde{\mathcal{D}}_c$, choosing $c_1 = \emptyset$ is permitted at any step, while in $\mathcal{D}_c$, it is only permitted when $\operatorname{row}(c_0) = 1$. $\square$

**Lemma 7.11.** *In a q-regular semi-primary lattice, the number of solutions to*

$$\text{type}\boxed{\begin{array}{ccc} (f_{k-1}) & \lessdot & f_k \\ \veebar & & \veebar \\ (g_{k-1}) & \lessdot & g_k \end{array}} = \boxed{\begin{array}{ccc} \gamma & \lessdot & \delta \\ \veebar & & \veebar \\ \alpha & \lessdot & \beta \end{array}} \qquad f_{k-1} \neq g_k$$

*(that is, $\alpha, \beta, \gamma, \delta$ are given partitions, $f_k \geq g_k$ are given elements of respective types $\delta \geq \beta$, and we seek elements $f_{k-1}, g_{k-1}$ to satisfy the remaining relations) depends on the partitions, as follows.*

(1) $(\beta \neq \gamma)$ $\boxed{\begin{array}{ccc} \gamma & \lessdot_{|r|} & \delta \\ \veebar & & \veebar \\ \alpha & \lessdot_{|r|} & \beta \end{array}}$ $\quad \langle \delta'_r \rangle - \left\langle \delta'_{r+1} \right\rangle$

(2) $(r \neq s)$ $\boxed{\begin{array}{ccc} \gamma & \lessdot_{|s|} & \delta \\ \dot{\vee}_{|r|} & & \dot{\vee}_{|s|} \\ \alpha & \lessdot_{|r|} & \beta \end{array}}$ $\quad \left(\langle \beta'_r \rangle - \left\langle \beta'_{r+1} \right\rangle\right)(q-1)$
*further, in this case, $r > s$, even when the lattice is not q-regular.*

(3) $\boxed{\begin{array}{ccc} \gamma & \lessdot_{|r|} & \delta \\ \dot{\vee}_{|r|} & & \dot{\vee}_{|r|} \\ \alpha & \lessdot_{|r|} & \beta \end{array}}$ $\quad \left(\langle \beta'_r \rangle - \left\langle \beta'_{r+1} \right\rangle\right) q$

*Proof.*

(1) Let $f_{k-1}$ be any lower cover of $f_k$ of type $\gamma$, in one of $\langle \delta'_r \rangle - \left\langle \delta'_{r+1} \right\rangle$ ways. Since $g_k \leq f_k$, also $f_{k-1} \wedge g_k \leq g_k$. This meet is an upper bound for $g_{k-1}$. We have $\beta \wedge \gamma \lessdot \beta$, so $\operatorname{type}(f_{k-1} \wedge g_k) \leq \operatorname{type} f_{k-1} \wedge \operatorname{type} g_k = \gamma \wedge \beta$. Thus $f_{k-1} \wedge g_k$ has rank at least one lower than $g_k$, but also it has rank at most one lower, so $g_{k-1} := f_{k-1} \wedge g_k$ in one way.