(2) In the remaining cases, $|\alpha| \lessdot |\beta| = |\gamma| \lessdot |\delta|$, and $f_{k-1} \wedge g_k = g_{k-1}$ and $f_{k-1} \vee g_k = f_k$, since $f_{k-1}, g_k$ are distinct elements in the rank two interval $[g_{k-1}, f_k]$.

For any $f_{k-1} \lessdot f_k$ of type $\gamma$, we have $f_k[s-1] \leq g_k, f_{k-1}$ by Theorem 3.8 as the types of $f_k, g_k, f_{k-1}$ agree in the first $s-1$ columns. Thus $f_k[s-1] \leq g_k \wedge f_{k-1} = g_{k-1}$, so $\alpha = \operatorname{type} g_{k-1}$ also agrees with $\beta$, $\gamma$, and $\delta$ in the first $s-1$ columns, whence $r \geq s$.

Pick any $g_{k-1} \lessdot g_k$ with $\operatorname{type} g_{k-1} = \alpha$, in one of $\langle \beta'_r \rangle - \left\langle \beta'_{r+1} \right\rangle$ ways.

Next we determine which elements of $(g_{k-1}, f_k)$ are suitable choices for $f_{k-1}$. We have $\operatorname{type}[g_{k-1}, f_k] = \boxminus$ by applying Corollary 4.72 to $\operatorname{ftype}_{\hat{0}}(g_{k-1}, g_k, f_k) = (\alpha, \beta, \delta)$ because the cell $\beta/\alpha$ is weakly right and strictly above $\delta/\beta$.

If $r > s$, there are two partitions between $\alpha$ and $\delta$. The element $g_{k-1} \vee f_k[s]$ has type $\alpha \vee \delta[s]$ (add the left square of $\delta/\alpha$ to $\alpha$) but the other $q$ elements (including $g_k$) have type $\gamma$, so there are $q-1$ choices of $f_{k-1}$ distinct from $g_k$.

(3) Continuing the analysis of the previous case, if $r = s$, then $\delta/\alpha$ is a vertical brick, so all $q+1$ elements in $(g_{k-1}, f_k)$ have type $\gamma$, and there are $q$ choices of $f_{k-1}$ (since it is distinct from $g_k$, which is also in this interval). □

**Theorem 7.12.** *Consider a $q$-regular semi-primary lattice of type $\lambda$. Let $\mu \lessdot \lambda$ and $c = \lambda/\mu$. Let $\boldsymbol{P}$ be a skew tableau of outer shape $\lambda$, and fix $\boldsymbol{Q} = \widetilde{\mathcal{D}}_c \boldsymbol{P}$.*

*Pick $x \lessdot \hat{1}$ of type $\mu$. The number of flags $f$ with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ and $\operatorname{ftype}_{\hat{0}}(f \wedge x) = \boldsymbol{Q}$ is a product of terms that depend on how entries move from $\boldsymbol{P}$ to $\boldsymbol{Q}$:*

$$\prod_{k \,\in\, \boldsymbol{Q}} \begin{cases} \langle \operatorname{row}(\boldsymbol{P}, k) \rangle - \langle \operatorname{row}_1(\boldsymbol{P}, k) \rangle & \textit{if } k \textit{ is in the same cell;} \\ \Big( \langle \operatorname{row}(\boldsymbol{Q}, k) \rangle - \langle \operatorname{row}_1(\boldsymbol{Q}, k) \rangle \Big) \cdot q & \textit{if } k \textit{ moves up one cell;} \\ \Big( \langle \operatorname{row}(\boldsymbol{Q}, k) \rangle - \langle \operatorname{row}_1(\boldsymbol{Q}, k) \rangle \Big) \cdot (q-1) & \textit{if } k \textit{ moves up and right.} \end{cases}$$

*This is a polynomial of degree*

$$n(\operatorname{sh} Q) + \textit{length of deletion path} - 1.$$

*It is maximized by having the deletion path contain a cell on every row above $c$, which happens uniquely in the generic game $\boldsymbol{Q} = \mathcal{D}_c \boldsymbol{P}$. The game is not uniform, however.*

*Conversely, in all semi-primary lattices, for any flag $f = (f_l \leq \cdots \leq f_h)$ and element $x \leq f_h$,*

$$\begin{aligned} \operatorname{ftype}(f \wedge x) &= \widetilde{\mathcal{D}}_c \operatorname{ftype} f && \textit{where } c = (\operatorname{type}[f_l, f_h])/(\operatorname{type}[f_l \wedge x, f_h \wedge x]), \\ \operatorname{ftype}_{\hat{0}}(f \wedge x) &= \widetilde{\mathcal{D}}_c \operatorname{ftype}_{\hat{0}} f && \textit{where } c = (\operatorname{type} f_h)/(\operatorname{type} x). \end{aligned}$$

The generic case in the invariant subspace lattice over an infinite field was first done by Steinberg [**29**, Lemma 1.2], and later by van Leeuwen [**31**, Lemma 2.5.1].

*Proof.* Since $g_k = f_k \wedge x$, we have $g_{k-1} = f_{k-1} \wedge x = f_{k-1} \wedge f_k \wedge x = f_{k-1} \wedge g_k$. Since $x \leq \hat{1}$, we have $g_k \leq f_k$, and $\mu^{(k)} \leq \lambda^{(k)}$. We will construct $f$ and $g$ with $g = f \wedge x$ using these local conditions.

Begin with $f_h := \hat{1}$ and $g_h := x$. For some $k = h, h-1, \ldots, l+1$, we have

$$\longleftarrow \operatorname{type} \boxed{\begin{array}{ccc} (f_{k-1}) & \leq & f_k \\ \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} \\ (g_{k-1}) & \leq & g_k \end{array}} = \boxed{\begin{array}{ccc} \gamma & \leq & \delta \\ \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} \\ \alpha & \leq & \beta \end{array}}$$

(that is, $f_k$ and $g_k$ and all the partitions are given, and we will choose $f_{k-1}$ and $g_{k-1}$ to satisfy this).

**$\boldsymbol{\gamma = \delta}$:** Then $f_{k-1} = f_k$; meet both sides with $x$ to obtain $g_{k-1} = g_k$, so that $\alpha = \beta$ also. Neither tableau has an entry $k$.

$$\boxed{\begin{array}{ccc} \gamma & = & \delta \\ \rotatebox[origin=c]{90}{$\leq$} & & \rotatebox[origin=c]{90}{$\leq$} \\ \alpha & = & \beta \end{array}} \qquad \text{1 choice}$$

**$\boldsymbol{\alpha = \beta}$ and $\boldsymbol{\gamma \lessdot \delta}$:** Then $\beta = \alpha \leq \beta \wedge \gamma \leq \gamma \lessdot \delta$. Since $\beta$ is a weak lower cover of $\delta$ and $\beta \leq \gamma \lessdot \delta$, in fact, $\alpha = \beta = \gamma$. So $f_{k-1} := g_{k-1} := g_k$ in one way, and $k$ is deleted on the deletion path from $\boldsymbol{P}$ to $\boldsymbol{Q}$.

$$\boxed{\begin{array}{ccc} \gamma & \lessdot & \delta \\ \| & & \rotatebox[origin=c]{90}{$\lessdot$} \\ \alpha & = & \beta \end{array}} \qquad \text{1 choice}$$

**$\boldsymbol{\alpha \lessdot \beta}$ and $\boldsymbol{\gamma \lessdot \delta}$:** If $\beta = \delta$ then $g_k = f_k$; meet both sides with $f_{k-1}$ to obtain $g_{k-1} = f_{k-1}$ also, so $\alpha = \gamma$. In view of this, the configurations of partitions in Lemma 7.11 are all the configurations that arise when $\alpha \lessdot \beta$ and $\gamma \lessdot \delta$. The possibilities are as follows.

If $\beta \neq \gamma$, then $k$ is in the same cell of $\boldsymbol{P}$ and $\boldsymbol{Q}$.

$$(\beta \neq \gamma) \quad \boxed{\begin{array}{ccc} \gamma & \lessdot_{|r|} & \delta \\ \rotatebox[origin=c]{90}{$\leq$} & & \rotatebox[origin=c]{90}{$\leq$} \\ \alpha & \lessdot_{|r|} & \beta \end{array}} \qquad \begin{array}{l} \langle \delta'_r \rangle - \left\langle \delta'_{r+1} \right\rangle \; = \\ \quad \langle \mathrm{row}(\boldsymbol{P}, k) \rangle - \langle \mathrm{row}_1(\boldsymbol{P}, k) \rangle \\ \text{choices} \end{array}$$

If $\beta = \gamma$, then $k$ is on the deletion path. It can move strictly up and right from $\boldsymbol{P}$ to $\boldsymbol{Q}$, bumping a smaller entry (as $\beta/\alpha$ is a cell in $\delta$, so that cell in $\boldsymbol{P}$ has a smaller entry than $k$):

$$(r > s) \quad \boxed{\begin{array}{ccc} \gamma & \lessdot_{|s|} & \delta \\ \rotatebox[origin=c]{90}{$\lessdot$}_{|r|} & & \rotatebox[origin=c]{90}{$\lessdot$}_{|s|} \\ \alpha & \lessdot_{|r|} & \beta \end{array}} \qquad \begin{array}{l} \left( \langle \beta'_r \rangle - \left\langle \beta'_{r+1} \right\rangle \right) (q-1) \; = \\ \quad \Big( \langle \mathrm{row}(\boldsymbol{Q}, k) \rangle - \langle \mathrm{row}_1(\boldsymbol{Q}, k) \rangle \Big) (q-1) \\ \text{choices;} \end{array}$$

or it can move up one cell from $\boldsymbol{P}$ to $\boldsymbol{Q}$:

$$\boxed{\begin{array}{ccc} \gamma & \lessdot_{|r|} & \delta \\ \rotatebox[origin=c]{90}{$\lessdot$}_{|r|} & & \rotatebox[origin=c]{90}{$\lessdot$}_{|r|} \\ \alpha & \lessdot_{|r|} & \beta \end{array}} \qquad \begin{array}{l} \left( \langle \beta'_r \rangle - \left\langle \beta'_{r+1} \right\rangle \right) q \; = \\ \quad \Big( \langle \mathrm{row}(\boldsymbol{Q}, k) \rangle - \langle \mathrm{row}_1(\boldsymbol{Q}, k) \rangle \Big) q \\ \text{choices.} \end{array}$$

Now multiply all the numbers of choices given over all $k \in \boldsymbol{P}$, that is, all $k$ with $\gamma \lessdot \delta$, to obtain the number of choices of $f$. The deleted entry $k$, if any, yields a factor of 1, so the product may be taken over all $k \in \boldsymbol{Q}$, to obtain the formula in the statement of the theorem. Each term of the product has degree $\mathrm{row}(\boldsymbol{Q}, k) - 1$, plus 1 if there is a factor of $q$ or $q - 1$ (that is, if $k$ is on the deletion path, but is not the entry bumped out). So the degree is the sum over all entries of $\boldsymbol{Q}$ of their row number minus 1 (totalling $n(\mathrm{sh}\, Q)$), plus the number of cells in $\boldsymbol{Q}$ on the deletion path, which is one less than the length of the deletion path since the deleted entry isn't counted in this length. The degree is uniquely maximized by the ordinary Robinson-Schensted deletion path, with each entry (but the smallest) on the path moving up one row, until it terminates at the top row or a $\bullet$. So $\mathcal{D}_c\boldsymbol{P}$ is the generic value of $\widetilde{\mathcal{D}}_c\boldsymbol{P}$.

It is not, however, uniform. Suppose $\beta$ and $\gamma$ are distinct lower covers of $\delta$ and $\beta/\alpha$ is strictly below and left of $\delta/\beta$. In terms of tableau, $k$ is larger than the entry deleted in the game $\mathcal{D}_c\boldsymbol{P}$, and is to the left of the cell on the deletion path on the row above $k$; thus, this configuration arises for some $k$ unless the deletion path moves the first entry in each row to the first entry of the preceding row (and deletes the smallest entry of the tableau). By applying Corollary 4.72 to $\operatorname{ftype}_{\hat{0}}(g_{k-1}, g_k, f_k) = (\alpha, \beta, \delta)$, we generically expect $\operatorname{type}[g_{k-1}, f_k] = \begin{array}{|c|c|}\hline \phantom{.} & \phantom{.} \\ \hline\end{array}$, but in fact, $g_{k-1}$ is the meet of two lower covers of $f_k$, so $\operatorname{type}[g_{k-1}, f_k] = \begin{array}{|c|}\hline \phantom{.} \\ \hline \phantom{.} \\ \hline\end{array}$. Only a fraction at most $O(q^{-1})$ of the lower covers of $g_k$ of type $\alpha$ may be chosen as $g_{k-1}$, so most flags $g$ from $\hat{0}$ to $x$ with $\operatorname{ftype} g = \boldsymbol{Q}$ cannot be represented as $f \wedge x$ with $\operatorname{ftype} f = \boldsymbol{P}$.

For the converse statement in the theorem, ignoring all the enumerative parts of the above proof shows that

$$\begin{aligned} &\mathcal{D}_c, \widetilde{\mathcal{D}}_c : \operatorname{ftype} f \longmapsto \operatorname{ftype}(f \wedge x) && \text{where } f_l \le x < f_h \text{ and } c = \lambda/(\operatorname{type}[f_l, x]) \\ &\mathcal{D}_c, \widetilde{\mathcal{D}}_c : \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}}(f \wedge x) && \text{where } x < f_h \text{ and } c = \lambda/\operatorname{type} x. \end{aligned}$$

If $x = f_h$, we have $f \wedge x = f$ and $c = \emptyset$, and indeed, $\widetilde{\mathcal{D}}_\emptyset$ does not change $\boldsymbol{P}$. If $f_l \not\le x < f_h$, then $f_l \vee x = f_h$, because $f_h$ is an upper bound of the two; there is nothing between $x$ and $f_h$; and the join is not $x$ since $f_l \not\le x$. Thus,

$$[f_l \wedge x, f_h \wedge x] = [f_l \wedge x, x] \cong [f_l, f_l \vee x] = [f_l, f_h],$$

so $c = \emptyset$ and $\operatorname{ftype}(f \wedge x) = \operatorname{ftype} f = \widetilde{\mathcal{D}}_\emptyset \operatorname{ftype} f$. $\square$

**7.5. Sample enumeration of the number of flags realizing a meet insersection type table.**

**Example 7.13.** In a $q$-regular semi-primary lattice of type $(3,1)$, we'll classify and enumerate all pairs of flags $(f, f')$ with $\operatorname{ftype} f = \boldsymbol{P}$ and $\operatorname{ftype} f' = \boldsymbol{Q}$ where

$$\boldsymbol{P} = \boldsymbol{Q} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}\,.$$

They will be classified according to their **meet type table** $\Lambda = (\lambda^{(ij)})_{0 \le i,j \le n}$ where $\lambda^{(ij)} = \operatorname{type}(f_i \wedge f'_j)$. We encode such a table by a sequence of tableaux $\boldsymbol{P}_n, \ldots, \boldsymbol{P}_0$ where $\boldsymbol{P}_j = (\lambda^{(0j)}, \ldots, \lambda^{(nj)})$. The transpose of $\Lambda = (\lambda^{(ij)})_{0 \le i,j \le n}$ is $\Lambda' = (\lambda^{(ji)})_{0 \le i,j \le n}$. Apply the degenerate Robinson-Schensted deletion algorithm to the pair $(\boldsymbol{P}, \boldsymbol{Q})$ by performing degenerate deletion steps $\widetilde{\mathcal{D}}_c$ on $\boldsymbol{P}$ with $c$ successively being the cell with $n, n-1, \ldots, 1$ in $\boldsymbol{Q}$, and forming a permutation $\sigma$ by listing the deleted entries in reverse order. The possibilities are in Figure 9. Let $N_k$ be the number of pairs $(f, f')$ of flags realizing meet type table $\Lambda_k$ in a $q$-regular semi-primary lattice.

(1) First we show that $\Lambda_7$ never occurs in any semi-primary lattice, so that $N_7 = 0$. By Theorem 4.76, $f_1 = \hat{1}[[\square]] = C\hat{1}$ because

$$\operatorname{ftype}_{\hat{0}} f = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} = \begin{array}{|c|}\hline 1 \\ \hline\end{array} + \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 4 \\ \hline\end{array}\,.$$

Also, $f'_1 = \hat{1}[[\square]]$ for the same reason, so $f_1 \wedge f'_1 = \hat{1}[[\square]]$ and $\boldsymbol{P}_1 = \begin{array}{|c|}\hline 1 \\ \hline\end{array}$. Thus $N_7 = 0$.

|  | $\sigma$ | $\boldsymbol{P}_4$ | $\boldsymbol{P}_3$ | $\boldsymbol{P}_2$ | $\boldsymbol{P}_1$ | $\boldsymbol{P}_0$ |
|---|---|---|---|---|---|---|
| $\Lambda_1 = \Lambda'_1$ | 1 4 3 2 | $\begin{matrix}1&2\\3&\\4&\end{matrix}$ | $\begin{matrix}1&3\\4&\end{matrix}$ | $\begin{matrix}1&4\end{matrix}$ | $\begin{matrix}1\end{matrix}$ | $\emptyset$ |
| $\Lambda_2 = \Lambda'_3$ | 1 3 4 2 | $\begin{matrix}1&2\\3&\\4&\end{matrix}$ | $\begin{matrix}1&3\\4&\end{matrix}$ | $\begin{matrix}1&3\end{matrix}$ | $\begin{matrix}1\end{matrix}$ | $\emptyset$ |
| $\Lambda_3 = \Lambda'_2$ | 1 4 2 3 | $\begin{matrix}1&2\\3&\\4&\end{matrix}$ | $\begin{matrix}1&2\\4&\end{matrix}$ | $\begin{matrix}1&4\end{matrix}$ | $\begin{matrix}1\end{matrix}$ | $\emptyset$ |
| $\Lambda_4 = \Lambda'_4$ | 1 2 4 3 | $\begin{matrix}1&2\\3&\\4&\end{matrix}$ | $\begin{matrix}1&2\\4&\end{matrix}$ | $\begin{matrix}1&2\end{matrix}$ | $\begin{matrix}1\end{matrix}$ | $\emptyset$ |
| $\Lambda_5 = \Lambda'_5$ | 1 3 2 4 | $\begin{matrix}1&2\\3&\\4&\end{matrix}$ | $\begin{matrix}1&2\\3&\end{matrix}$ | $\begin{matrix}1&3\end{matrix}$ | $\begin{matrix}1\end{matrix}$ | $\emptyset$ |
| $\Lambda_6 = \Lambda'_6$ | 1 2 3 4 | $\begin{matrix}1&2\\3&\\4&\end{matrix}$ | $\begin{matrix}1&2\\3&\end{matrix}$ | $\begin{matrix}1&2\end{matrix}$ | $\begin{matrix}1\end{matrix}$ | $\emptyset$ |
| $\Lambda_7 = \Lambda'_7$ | 3 4 1 2 | $\begin{matrix}1&2\\3&\\4&\end{matrix}$ | $\begin{matrix}1&4\\3&\end{matrix}$ | $\begin{matrix}3&4\end{matrix}$ | $\begin{matrix}3\end{matrix}$ | $\emptyset$ |
| $\Lambda_8 = \Lambda'_8$ | 1 4 3 2 | $\begin{matrix}1&2\\3&\\4&\end{matrix}$ | $\begin{matrix}1&4\\3&\end{matrix}$ | $\begin{matrix}1&4\end{matrix}$ | $\begin{matrix}1\end{matrix}$ | $\emptyset$ |

FIGURE 9. All possible degenerate Robinson-Schensted games.

(2) We evalutate $N_k$ for other $k$'s. Since $f_1 = f_1' = \hat{1}[[\square]]$, we have $\mathrm{type}[f_1, f_4] = \mathrm{type}[f_1', f_4'] = (1^3)$, so $\mathrm{type}[f_i, f_j] = \mathrm{type}[f_i', f_j'] = (1^2)$ for $(i,j) \in \{(1,3),(2,4)\}$. Thus, we could use an approach similar to that of Section 5.5: develop a system of equations in the $N_k$'s by replacing $f_2$ with some other atom in the interval $[f_1, f_3]$, and similarly replacing $f_3, f_2', f_3'$, to see how it could change the meet type table and whether a ratio of the frequencies of the possibilities can be determined. In larger examples, however, it is often not possible to determine from $\Lambda$ whether $\mathrm{type}[f_i, f_j] = \square\square$ or $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$ for $j - i = 2$.

We give here a more direct approach than a system of equations, but it, too, often fails in larger examples when types of intervals of length 2 cannot be determined.

Since $\Lambda_2 = \Lambda_3'$, there is a bijection between pairs of flags with meet type tables $\Lambda_3$ and $\Lambda_2$, namely $(f, f') \mapsto (f', f)$, so $N_2 = N_3$. We'll determine $N_3$. First expand $\Lambda_3$ as a square grid of partitions; see Figure 10. The tableau $\boldsymbol{P}$ is the chain going up the right-hand edge and the tableau $\boldsymbol{Q}$ is the chain going across the top edge. The games $\widetilde{\mathcal{D}}_c$ are played on columns from right to left, rather than on rows from top to bottom. This way, the orientation of the grid so obtained agrees with that in Roby [**21**].

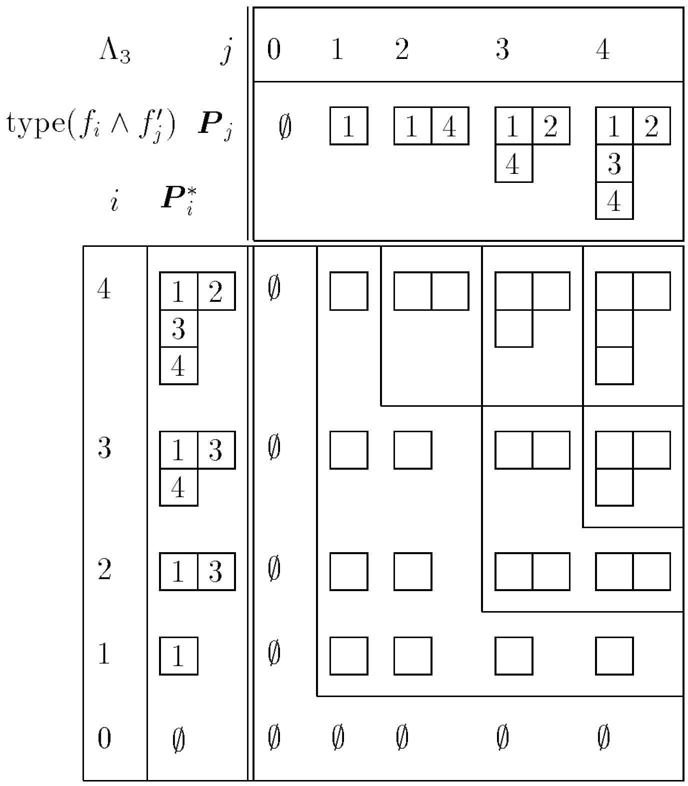



FIGURE 10. The graphical meet type table of $\Lambda_3$.

The grid is divided into regions of adjacent grid cells whose partitions are equal, because $f_i \wedge f_j'$ is constant on each such region: since $f_i \wedge f_j' \geq f_{i-1} \wedge f_j'$ always holds, if $\lambda^{(ij)} = \lambda^{(i-1,j)}$ then $f_i \wedge f_j' = f_{i-1} \wedge f_j'$, and similarly, if $\lambda^{(ij)} = \lambda^{(i,j-1)}$ then $f_i \wedge f_j' = f_i \wedge f_{j-1}'$. Form a graded graph (Figure 11) by rotating the grid an eighth of a revolution counterclockwise, collapsing each region to a node, and forming an edge between regions that share a horizontal or vertical boundary (not just a corner). (More generally, if we form such a graph for any pair of flags $(f, f')$ in a lower semimodular lattice by coalescing regions of constant $f_i \wedge f'_j$ into single nodes, the graph is the Hasse diagram of a lower semimodular sublattice.)

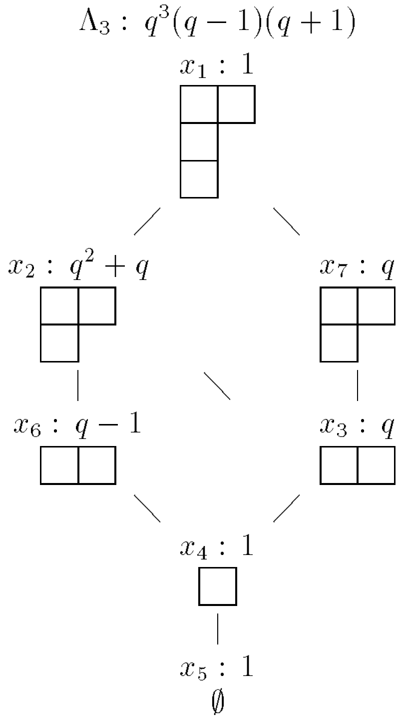



FIGURE 11. The types in the meet semilattice of $\Lambda_3$.

The values of $f_i \wedge f'_j$ on various regions are denoted $x_1, \ldots, x_7$, and we choose these values in the order given by the subscripts. The number of choices of each element is specified after the colon. First choose $x_1 = \hat{1}$ of type $(3,1)$ in one way. Choose $x_2 < x_1$ with type $x_2 = (2,1)$ in $\langle 3\rangle - \langle 1\rangle = q^2 + q$ ways, and similarly for $x_3, x_4, x_5$. The interval $[x_4, x_2]$ has type $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ as noted previously, but in larger examples, we wouldn't always be able to determine this type, and the procedure would break down now. Here we can continue: $x_6$ is an atom of $[x_4, x_2]$ of type $\square\square$ in the lattice, not equal to $x_3$. Of the $q+1$ atoms in the interval, one has type $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, namely $x_2[[\begin{smallmatrix}\square\\\square\end{smallmatrix}]]$, and the $q$ others, including $x_3$, have type $\square\square$, so there are $q-1$ choices of $x_6$. Similarly, the interval $[x_3, x_1]$ has type $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, and all atoms in it have type $(2,1)$, so there are $q$ choices of elements not equal to $x_2$. Thus, $N_3 = (q^2+q)q(q-1)q = q^3(q-1)(q+1)$.

(3) In general, how many solutions $x_4 \in (x_1, x_3)$ are there in a configuration

$$\begin{tikzcd}[column sep=small] & \text{type}\, x_1 = \delta \arrow[dl, dash] \arrow[dr, dash] & \\ \text{type}\, x_2 = \gamma \arrow[dr, dash] & & (\text{type}\, x_4 = \beta) \arrow[dl, dash] \\ & \text{type}\, x_3 = \alpha & \end{tikzcd}$$

where $x_1, x_2, x_3$ are given and $\mathrm{type}[x_3, x_1] = \boxminus$? There are $q+1$ elements in the interval $(x_3, x_1)$. If $\delta/\alpha$ is a brick, they all have the same type, and if it is not a brick, one has the type given by adding the left-hand square of $\delta/\alpha$ to $\alpha$ and the other $q$ have the type given by adding the right-hand square: if the columns of the squares of $\delta/\alpha$ are $r < s$ then $z = x_1[r] \vee x_3$ is the unique solution of $z \in (x_3, x_1)$ with $\mathrm{type}\, x_3 \lessdot_{|r|} \mathrm{type}\, z \lessdot_{|s|} \mathrm{type}\, x_1$. Altogether, the number of choices of $x_4$ in every circumstance is as follows.

$$\boxed{\begin{array}{ccc}\gamma & \lessdot_{|s|} & \delta \\ \dot\vee_{|r|} & & \dot\vee_{|r|} \\ \alpha & \lessdot_{|s|} & \beta\end{array}} \qquad \begin{cases} 1 & \text{if } r > s \text{, namely } x_1[s] \vee x_3; \\ q & \text{if } r < s \text{ since only } x_2 \text{ has type } \gamma; \\ q & \text{if } r = s \text{, since all } x \in (x_3, x_1) \\ & \quad \text{have equal type and } x_4 \neq x_2. \end{cases}$$

$$\boxed{\begin{array}{ccc}\gamma & \lessdot_{|s|} & \delta \\ \dot\vee_{|r|} & & \dot\vee_{|s|} \\ \alpha & \lessdot_{|r|} & \beta\end{array}} \qquad \begin{cases} 0 & \text{if } r < s \text{ since only} \\ & \quad x_2 = x_3[r] \vee x_1 \text{ has type } \gamma; \\ q-1 & \text{if } r > s \text{ since } x_4 \neq x_1 \\ & \quad \text{but they have equal type;} \\ q & \text{if } r = s. \end{cases}$$

By applying Corollary 4.72 to $\mathrm{ftype}_{\hat{0}}(x_3, x_2, x_1) = (\alpha \lessdot_{|r|} \gamma \lessdot_{|s|} \delta)$ in both diagrams, if $r \geq s$ then $\mathrm{type}[x_3, x_1] = \boxminus$ holds necessarily, while if $r < s$ it may or may not hold; in particular, in the first configuration with $r < s$ and $\delta/\alpha$ not a brick, it is necessary to specify that $\mathrm{type}[x_3, x_1] = \boxminus$ because without doing so, $\mathrm{type}[x_3, x_1] = \square\!\square$ is possible.

(4) In Figure 12, we solve for $N_1, N_4, N_5, N_6, N_8$ in a similar fashion. In each case we choose a flag $\hat{1} = x_1 > \cdots > x_5 = \hat{0}$, and then fill in intervals of type $\boxminus$. All length two intervals required to have type $\boxminus$ do, because $x_4 = \hat{1}[[\square]]$ and they are subintervals of $[x_4, \hat{1}]$. The highest degree polynomial is $N_1$, of degree 6, corresponding to the ordinary Robinson-Schensted algorithm on $(\boldsymbol{P}, \boldsymbol{Q})$.

**Example 7.14.** This method can break down.

(1) In a semi-primary lattice of type $\boxplus$, how many flags (not of maximal length) $f = (f_2 \lessdot f_3 \lessdot f_4 = \hat{1})$, $f' = (f'_2 \lessdot f'_3 \lessdot f'_4 = \hat{1})$ have the following meet meet type table?

$$\Lambda = (\lambda^{(ij)})_{2 \leq i,j \leq 4} = \begin{array}{c|ccc} \multicolumn{4}{c}{\mathrm{type}(f_i \wedge f'_j)} \\ {}_i\diagdown^j & 2 & 3 & 4 \\ \hline 4 & \square\!\square & \begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix} & \boxplus \\ 3 & \square & \boxminus & \begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix} \\ 2 & \emptyset & \square & \square\!\square \end{array}$$

Let $z_{ij} = f_i \wedge f'_j$, and try to choose the $z_{ij}$'s in a similar fashion to the previous example. In what order do we choose the $z_{ij}$'s? First, $z_{44} = \hat{1}$. Then choose $z_{43}$ or $z_{34}$; the type table is symmetric, so without loss of generality, choose $z_{43}$ in one of $\langle 2 \rangle = q + 1$ ways. The initial saturated chains

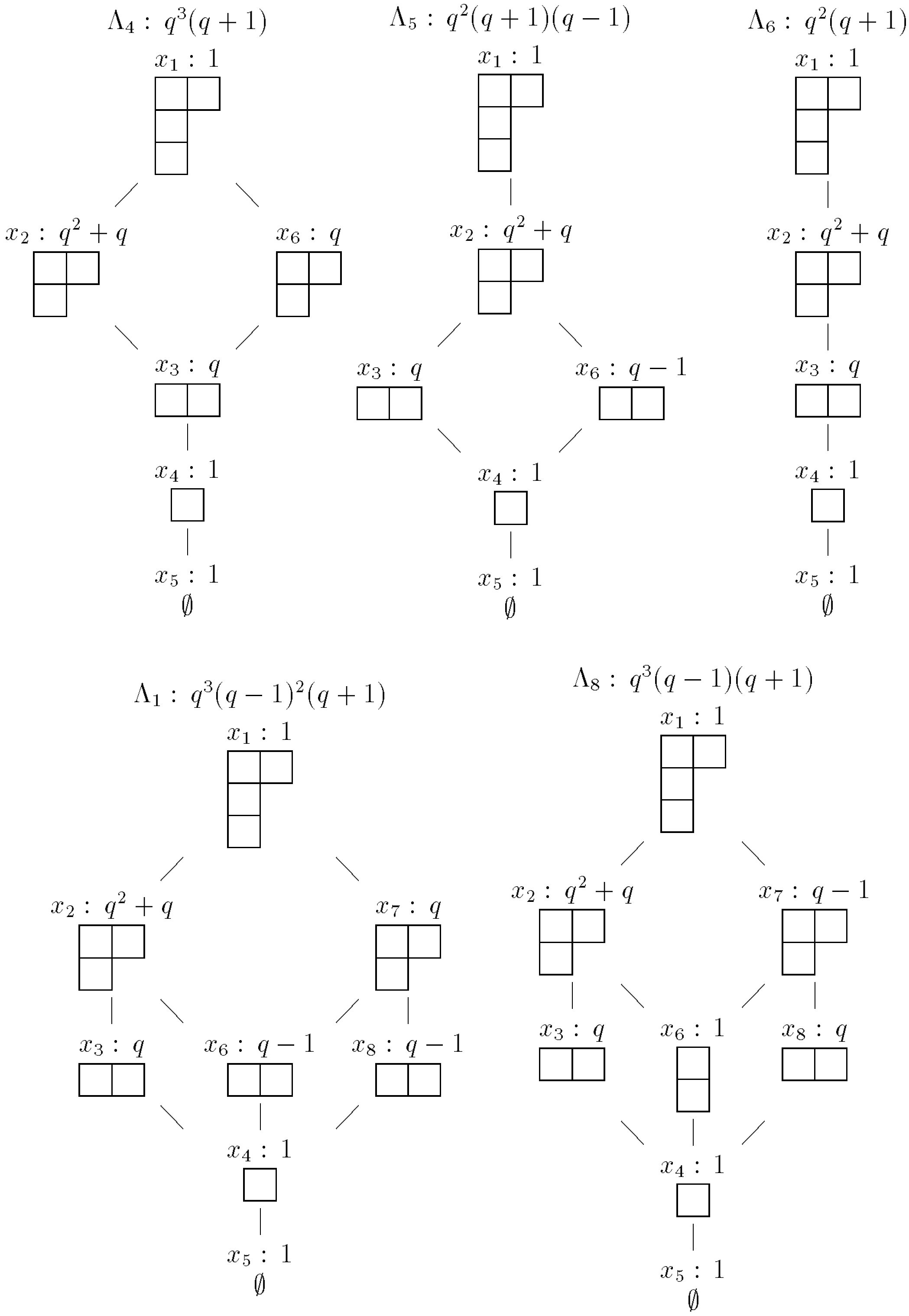



FIGURE 12. More meet semilattices.

we could form by continuing in this fashion are (a) $(z_{44}, z_{43}, z_{42}, z_{32}, z_{22})$; (b) $(z_{44}, z_{43}, z_{33}, z_{32}, z_{22})$; or (c) $(z_{44}, z_{43}, z_{33}, z_{23}, z_{22})$. Continuing with either (b) or (c), we will ultimately have $\operatorname{ftype}_{\hat{0}}(z_{32} \lessdot z_{33} \lessdot z_{43}) = \begin{smallmatrix} \bullet & 3 \\ 2 & \end{smallmatrix}$, so that we do not know whether $\operatorname{type}[z_{32}, z_{43}] = \square\!\square$ or $\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}$; we need to know this when selecting $z_{42}$. With (a) on the other hand, we ultimately have a flag $(z_{23} \lessdot z_{33} \lessdot z_{34})$ with the same problem when choosing $z_{24}$.

We can enumerate the pairs of flags with meet type table $\Lambda$ by another approach. The pair $(x_{24}, x_{42})$ is of two independent 2-cycles whose join is $\hat{1}$. There are $q^2+q$ join decompositions of $\hat{1}$ by Theorem 4.48. The interval $[x_{42}, \hat{1}]$ has type $\square\!\square$ and the only intermediate element is $x_{43} = x_{42} \vee A\hat{0} = x_{42} \vee x_{24}[1]$, while $x_{34} = x_{24} \vee x_{42}[1]$. Since variables on the upper and right boundaries are all expressed in terms of the same join decomposition of $\hat{1}$, their meets have the types shown in $\Lambda$. Thus, all $q^2 + q$ join decompositions of $\hat{1}$ yield pairs of flags whose meet type table is $\Lambda$, so there are $q^2 + q$ such $(f, f')$.

(2) Now take any partition $\lambda$, let $n = |\lambda|$, and form the grid $\Gamma = (\gamma^{(ij)})_{n+2 \le i,j \le n+4}$ where $\gamma^{(ij)} = \lambda \cup \lambda^{(i-n,j-n)}$. Try to find flags $f = (f_{n+2}, f_{n+3}, f_{n+4})$ and $f' = (f'_{n+2}, f'_{n+3}, f'_{n+4})$ with meet type table $\Gamma$. Let $z_{ij} = f_i \wedge f'_j$. The problems with sequences (a), (b), (c) above happen again, *mutatis mutandis.* If $\lambda'_1 > \lambda'_2$, then $\gamma^{(n+2,n+4)}/\gamma^{(n+2,n+2)}$ and $\gamma^{(n+4,n+2)}/\gamma^{(n+2,n+2)}$ are no longer horizontal bricks, so $\operatorname{type}[z_{n+2,n+2}, z_{n+2,n+4}]$ and $\operatorname{type}[z_{n+2,n+2}, z_{n+4,n+2}]$ no longer need to be $\square\!\square$; either or both could be $\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}$. The join decomposition approach cannot be applied. Further *ad hoc* methods could be developed to solve this problem. It would be preferable to find a systematic approach to enumerate the number of flags realizing a wide variety of meet type tables.

**7.6. Transposed Robinson-Schensted deletion, $\mathcal{D}^*_c\boldsymbol{P}$.** Let $\lambda/\mu = \operatorname{sh}\boldsymbol{P}$; the parameter $c$ is either $\emptyset$ or an inner corner of $\lambda$ not in $\mu$.

Pick $x \ge f_l$. The following map is generic but not uniform.

$$\begin{array}{lll} \mathcal{D}^*_c, \widetilde{\mathcal{D}^*_c}: \ \operatorname{ftype} f \longmapsto \operatorname{ftype}(f \vee x) & & x \gtrdot f_l \text{ and } c = \lambda/(\operatorname{type}[f_l \vee x, f_h \vee x]) \\ \mathcal{D}^*_c, \widetilde{\mathcal{D}^*_c}: \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_x(f \vee x) & & x \gtrdot \hat{0} \text{ and } c = \lambda/\operatorname{type}[x, f_h \vee x] \end{array}$$

The games $\mathcal{D}_c$ and $\mathcal{D}^*_c$ are the same except that the roles of rows and columns are reversed. The games $\widetilde{\mathcal{D}}_c$ and $\widetilde{\mathcal{D}^*_c}$ are not transposed versions of each other, however.

*Tableau Rules.* Let $\boldsymbol{P}$ be a standard skew tableau on distinct entries, and $c$ be an inner corner of $\operatorname{osh}\boldsymbol{P}$ or $c = \emptyset$. If $c = \emptyset$ then $\widetilde{\mathcal{D}}_c\boldsymbol{P} = \boldsymbol{P}$. Otherwise, proceed as follows.

**($\mathcal{D}^*_c$):** We insert an entry $e$ into a column by replacing its largest entry (possibly $\bullet$) smaller than $e$ with $e$; the number it replaces is **bumped** from the column.

**($\widetilde{\mathcal{D}^*_c}$):** We insert an entry $e$ into a column by placing $e$ where the largest entry (possibly $\bullet$) smaller than it is, then moving that entry up one cell, moving the entry there up one cell, and so on, until we either choose to stop, or we are forced to because moving up an entry would cause a row not to increase left to right. At least the cell where $e$ is placed changes, and possibly more do.

The final entry displaced that does not displace the one above it is **bumped** from the column.

Begin by letting $e$ be the entry at $c$ and removing cell $c$ from the tableau. Insert $e$ one column left of $c$. Insert the bumped entry one column left of that, bumping out another number. Continue this until either an entry is bumped out from the first column, or $\bullet$ is bumped out. The final entry bumped out is said to be **deleted** from the tableau.

The local rules are a cross between transposing the rules to $\widetilde{\mathcal{D}}_c$ on the one hand, and the rules to $\tilde{\jmath}^{c'}$ played in reverse on the other hand.

*Local Rules.* If $c = \emptyset$, let $\mu^{(h)} := \lambda^{(h)}$, and if $c$ is a single cell, let $\mu^{(h)}$ be $\lambda^{(h)}$ with $c$ deleted. Then for $k = h, h-1, \ldots, l+1$, propagate as follows.

$$
\begin{array}{c|c|c|c|c|l}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{k-1} & \multicolumn{1}{c}{k} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{h} & \text{column} \\
\cline{2-3}\cline{5-5}
 & \gamma & \delta & & \lambda^{(h)} & \boldsymbol{P} \\
\cline{2-3}\cline{5-5}
\longleftarrow & (\alpha) & \beta & & (\lambda^{(h)} \ominus c) & \widetilde{\mathcal{D}}^*_c \boldsymbol{P} \\
\cline{2-3}\cline{5-5}
\end{array}
$$

- **$\boldsymbol{\beta = \gamma \lessdot \delta}$ does not hold:** Let $\alpha := \beta \wedge \gamma$.
- **$\boldsymbol{\beta = \gamma \lessdot \delta}$ holds:** Let $c := \delta/\beta$.
  - **$\mathbf{col}(\boldsymbol{c}) = \mathbf{1}$:** Let $\alpha := \gamma$.
  - **$\mathbf{col}(\boldsymbol{c}) > \mathbf{1}$ ($\boldsymbol{\mathcal{D}_c}$):** Delete the bottom cell of $\gamma$ in the column left of $c$ to obtain $\alpha$.
  - **$\mathbf{col}(\boldsymbol{c}) > \mathbf{1}$ ($\boldsymbol{\widetilde{\mathcal{D}}_c}$):** Choose one of the following as $\alpha$:
    - delete the cell above $c$ in $\gamma$, provided there is one and doing so yields a partition;
    - delete the bottom cell of $\gamma$ in the column left of $c$; or
    - $\gamma$.

The tableau rules and local rules describe the same game. The proof is similar to Proposition 7.10, but the positions of consecutive cells on the path are different.

**Theorem 7.15.** *Consider a $q$-regular semi-primary lattice of type $\lambda$. Let $\mu \lessdot \lambda$ and $c = \lambda/\mu$. Let $\boldsymbol{P}$ be a skew tableau, and fix $\boldsymbol{Q} = \widetilde{\mathcal{D}}^*_c \boldsymbol{P}$. Let $e$ be the deleted entry, or $e = \mathrm{low}(\boldsymbol{P}) = \bullet$ if no entry is deleted.*

*Pick an atom $x$ of cotype $\mu$. There are*

$$
\prod_{k \in \boldsymbol{P}} \begin{cases} \langle \mathrm{row}(\boldsymbol{Q}, k)\rangle - \langle \mathrm{row}_1(\boldsymbol{P}, k)\rangle & \text{if } k > e \text{ and } \mathrm{col}(\boldsymbol{P}, k) = \mathrm{col}(\boldsymbol{Q}, k); \\ \langle \mathrm{row}(\boldsymbol{P}, k)\rangle - \langle \mathrm{row}_1(\boldsymbol{Q}, k)\rangle & \text{if } k > e \text{ and } \mathrm{col}(\boldsymbol{P}, k) > \mathrm{col}(\boldsymbol{Q}, k); \\ q^{\mathrm{row}(\boldsymbol{P},k)-1} & \text{if } k = e \\ \langle \mathrm{row}(\boldsymbol{P}, k)\rangle - \langle \mathrm{row}_1(\boldsymbol{P}, k)\rangle & \text{if } k < e, \end{cases}
$$

*flags $f$ with $\mathrm{ftype}_{\hat{0}} f = \boldsymbol{P}$ and $\mathrm{ftype}_x(f \vee x) = \boldsymbol{Q}$. This is a polynomial of degree*

$$n(\mathrm{sh}\, \boldsymbol{P}) - \#\text{ of entries on deletion path that move up one cell}$$

*It is maximized by having the deletion path contain a cell in every column weakly left of $c$, which happens uniquely in the generic (but not uniform) game $\boldsymbol{Q} = \mathcal{D}^*_c \boldsymbol{P}$.*

*Conversely, in all semi-primary lattices, for any flag* $f = (f_l \leq \cdots \leq f_h)$,

$$\begin{array}{rcll} \operatorname{ftype}(f \vee x) & = & \widetilde{\mathcal{D}}^*_c \operatorname{ftype} f & \textit{where } x \geq f_l \textit{ and} \\ & & & c = (\operatorname{type}[f_l, f_h])/(\operatorname{type}[f_l \vee x, f_h \vee x]) \\ \operatorname{ftype}_x(f \vee x) & = & \widetilde{\mathcal{D}}^*_c \operatorname{ftype}_{\hat{0}} f & \textit{where } x \geq \hat{0} \textit{ and} \\ & & & c = (\operatorname{type}[\hat{0}, f_h])/(\operatorname{type}[x, f_h \vee x]). \end{array}$$

The generic case in the invariant subspace lattice over an infinite field was done by van Leeuwen [**31**, Lemma 2.5.2].

*Proof.* We begin with $f_h := g_h := \hat{1}$. For some $k$ with $l < k \leq h$, after having chosen $f_h \geq f_{h-1} \geq \cdots \geq f_k$ we have

$$\longleftarrow \operatorname{type} \boxed{\begin{array}{cc} [\hat{0}, (f_{k-1})] & [\hat{0}, f_k] \\ \rotatebox{90}{$\geq$} & \rotatebox{90}{$\geq$} \\ [x, (g_{k-1})] & [x, g_k] \end{array}} = \boxed{\begin{array}{ccc} \gamma & \leq & \delta \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\ \alpha & \leq & \beta \end{array}}$$

We will choose $f_{k-1}$ to satisfy these types, in one of several ways. If $\gamma = \delta$ then $f_{k-1} := f_k$ so $g_{k-1} := g_k$ and $\alpha = \delta$. There is no entry $k$ in $\boldsymbol{P}$ or $\boldsymbol{Q}$.

$$\boxed{\begin{array}{ccc} \gamma & = & \delta \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\ \alpha & = & \beta \end{array}} \qquad \text{1 choice of } f_{k-1} \text{ and } g_{k-1}$$

Assume that $\gamma \lessdot \delta$.

**$x \leq f_{k-1}$:** We want to extend the partial flag $(\hat{0} \lessdot x \lessdot f_k)$ to $(\hat{0} \lessdot x \leq f_{k-1} \lessdot f_k)$, with specified types for each interval. By Theorem 5.4 and Corollary 5.5, we have the following possible relations among the partitions, and the following number of choices of $f_{k-1}$ in each case.

The entry $k$ can bump out the bottommost entry smaller than $k$ in the column just left of $k$:

$$\boxed{\begin{array}{ccc} \gamma & \lessdot_{|r+1|} & \delta \\ \rotatebox{90}{$\lessdot$}_{|r|} & & \rotatebox{90}{$\lessdot$}_{|r+1|} \\ \alpha & \lessdot_{|r|} & \beta \end{array}} \qquad \begin{array}{l} \left\langle \delta'_{r+1} \right\rangle - \left\langle \beta'_{r+1} \right\rangle = \\ \quad \langle \operatorname{row}(\boldsymbol{P}, k) \rangle - \langle \operatorname{row}_1(\boldsymbol{Q}, k) \rangle \\ \text{choices} \end{array}$$

or $k$ can stay in the same cell ($r \neq s$) or move up one cell ($r = s$):

$$\boxed{\begin{array}{ccc} \gamma & \lessdot_{|r|} & \delta \\ \rotatebox{90}{$\lessdot$}_{|s|} & & \rotatebox{90}{$\lessdot$}_{|s|} \\ \alpha & \lessdot_{|r|} & \beta \end{array}} \qquad \begin{array}{l} \langle \beta'_r \rangle - \left\langle \delta'_{r+1} \right\rangle = \\ \quad \langle \operatorname{row}(\boldsymbol{Q}, k) \rangle - \langle \operatorname{row}_1(\boldsymbol{P}, k) \rangle \\ \text{choices} \end{array}$$

**$x \not\leq f_{k-1}$ and $x \leq f_k$:** Then $g_{k-1} = f_{k-1} \vee x = f_k$ and $g_k = f_k \vee x = f_k$, so $\alpha = \beta \lessdot \delta$. Also

$$[\hat{0}, f_{k-1}] = [x \wedge f_{k-1}, f_{k-1}] \cong [x, x \vee f_{k-1}] = [x, f_k]$$

so $\alpha = \beta = \gamma \lessdot_{|r|} \delta$ for some $r$, and $k$ is the entry deleted on the deletion path.

Pick $f_{k-1}$ to be a coatom of $[\hat{0}, f_k]$ of type $\gamma$ not also in $[x, f_k]$. The coatoms of type $\gamma$ are the coatoms of $[A^{r-1}\hat{0} \wedge f_k, f_k]$ not also in $[A^r\hat{0} \wedge f_k, f_k]$. Since $x$ is an atom, $A\hat{0} \geq x$, so coatoms of both intervals are weakly larger than $x$

unless $r = 1$. Thus we must have $r = 1$, and $f_{k-1}$ is any coatom of $[\hat{0}, f_k]$ not also in $[x, f_k]$, in one of $\langle \delta'_1 \rangle - \langle \beta'_1 \rangle$ ways.

$$\boxed{\begin{array}{ccc} \gamma & \lessdot_{|1|} & \delta \\ \| & & \gtrdot_{|1|} \\ \alpha & = & \beta \end{array}} \qquad \begin{array}{l} \langle \delta'_1 \rangle - \langle \beta'_1 \rangle \;=\; q^{\mathrm{row}(\boldsymbol{P},k)-1} \\ \text{choices} \end{array}$$

$\boldsymbol{x \not\leq f_k}$**:** Then $[\hat{0}, f_k] \cong [x, f_k \vee x]$. We must have $\alpha = \gamma$ and $\beta = \delta$ for all $f_{k-1} \lessdot f_k$. Entry $k$ is smaller than the deleted entry, and does not move. Choose $f_{k-1}$ to be any coatom $f_{k-1}$ of $[\hat{0}, f_k]$ with type $\gamma$.

$$\boxed{\begin{array}{ccc} \gamma & \lessdot_{|r|} & \delta \\ \| & & \| \\ \alpha & \lessdot_{|r|} & \beta \end{array}} \qquad \begin{array}{l} \langle \delta'_r \rangle - \left\langle \delta'_{r+1} \right\rangle \;= \\ \quad \langle \mathrm{row}(\boldsymbol{P}, k) \rangle - \langle \mathrm{row}_1(\boldsymbol{P}, k) \rangle \\ \text{choices} \end{array}$$

Now multiply all the numbers of choices given over all $k \in \boldsymbol{P}$, that is, all $k$ with $\gamma \lessdot \delta$, to obtain the number of choices of $f$ stated in the theorem. When $k > e$ and $\mathrm{col}(\boldsymbol{P}, k) = \mathrm{col}(\boldsymbol{Q}, k)$, the number of choices of $f_{k-1}$ has degree $\mathrm{row}(\boldsymbol{Q}, k) - 1$, which equals $\mathrm{row}(\boldsymbol{P}, k) - 1$ when $k$ is in the same cell of $\boldsymbol{P}$ and $\boldsymbol{Q}$, and equals $\mathrm{row}(\boldsymbol{P}, k) - 2$ when $k$ moves up one cell. For all other $k \in \boldsymbol{P}$, the number of choices of $f_{k-1}$ has degree $\mathrm{row}(\boldsymbol{P}, k) - 1$. So the total degree of the number of choices of $f$ is the sum of $\mathrm{row}(\boldsymbol{P}, k) - 1$ over all $k \in \boldsymbol{P}$ (totalling $n(\mathrm{sh}\,\boldsymbol{P})$), minus the number of $k$'s which move up one cell from $\boldsymbol{P}$ to $\boldsymbol{Q}$. This degree is maximized by never moving up one cell and always choosing to move to the next column left, which happens uniquely in the generic game $\mathcal{D}^*_c \boldsymbol{P}$. See the note following this proof for further commentary on maximizing this degree.

The game $\mathcal{D}^*_c$ is not in general uniform, however, as we do not obtain most flags in $[x, \hat{1}]$ of type $\boldsymbol{Q}$ as possible images of $f \vee x$ over $f$'s of type $\boldsymbol{P}$. Suppose there is some $k$ with $\mathrm{row}(\boldsymbol{P}, k) < \mathrm{row}(\boldsymbol{Q}, k)$, which happens in all games except those whose deletion path simply shifts a row horizontally left one cell (possibly deleting the leftmost entry). The number of choices of $f_{k-1}$ has degree $\mathrm{row}(\boldsymbol{P}, k) - 1$, while the number of choices of a coatom in $[x, \hat{1}]$ of type $\mu^{(k-1)}$ has degree $\mathrm{row}(\boldsymbol{Q}, k) - 1$, which is larger. Thus, at most a fraction $O(q^{-1})$ of all flags in $[x, \hat{1}]$ of type $\boldsymbol{Q}$ are possible images of $f \vee x$ over all $f$ of type $\boldsymbol{P}$ in $\mathcal{L}$, for most games $\boldsymbol{P}$ and $\boldsymbol{Q}$.

For the converse statement in the theorem, ignoring the enumerative portions of the above proof yields that the valid configurations of partitions $\alpha$, $\beta$, $\gamma$, $\delta$ are precisely those given by the local rules for the game $\widetilde{\mathcal{D}}^*_c$. $\square$

*Note.* The formula in Proposition 5.7 for the number of flags $f$ with $\mathrm{ftype}\, f = \boldsymbol{P}$ and $\mathrm{ftype}\, \partial f = \boldsymbol{Q} = \widetilde{\Delta} \boldsymbol{P}$ is a special case of the formula in Theorem 7.15 for the number of flags $f$ such that $\mathrm{ftype}\, f = \boldsymbol{P}$ and $\mathrm{ftype}(f \vee x) = \boldsymbol{Q} = \widetilde{\mathcal{D}}^*_c \boldsymbol{P}$, yet the generic games $\Delta$ and $\mathcal{D}^*_c$ are different. This is because the boundary conditions of the deletion paths are different; in $\widetilde{\Delta}$, it must terminate at the top left corner, while in $\widetilde{\mathcal{D}}^*_c$, it must begin at $c$. In the case of $\widetilde{\Delta}$, we minimize the number of entries bumped up one cell by only allowing an entry to switch columns by moving horizontally one cell, yielding the generic game $\Delta$.

**7.7. Robinson-Schensted internal insertion, $\mathcal{I}_c \boldsymbol{P}$.** The game $\mathcal{I}_c \boldsymbol{P}$ below is the internal insertion game $R_c$ defined by Sagan and Stanley in [**23**, p. 164], and $\widetilde{\mathcal{I}}_c \boldsymbol{P}$ is a nondeterministic variant.

When $x \gtrdot f_l$, we have

$$\mathcal{I}_c, \widetilde{\mathcal{I}}_c : \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}}(f \vee x) \qquad c = \operatorname{type} x / \operatorname{type} f_l.$$

This is neither generic nor uniform. Unlike most other games considered, there usually is not a single value of $\operatorname{ftype}_{\hat{0}}(f \vee x)$ that occurs generically, that is, with asymptotic probability 1; there is some finite number $N$ of outcomes that each asymptotically occur with probability $1/N$, and all other outcomes asymptotically occur with probability 0. The concept of uniformity does not similarly generalize to this situation, however, and the game $\mathcal{I}_c$ is usually not one of the $N$ "generic" possibilities. See the note at the end of this section for further discussion.

*Tableau Rules.* Let $\boldsymbol{P}$ be a standard skew tableau of shape $\lambda/\mu$ on distinct entries, and $c$ be an outer corner of $\mu$ or $\emptyset$. If $c = \emptyset$ then $\widetilde{\mathcal{I}}_c \boldsymbol{P} = \boldsymbol{P}$. Otherwise, proceed as follows.

We insert an entry $e$ into a row by replacing its smallest entry larger than $e$ with $e$, **bumping out** that entry, or if all entries are smaller than $e$, by appending $e$ to the end of the row.

- **($\boldsymbol{\mathcal{I}_c}$):** Place $\bullet$ at $c$. If $c$ is an outer corner of $\lambda$ then stop, and otherwise, let $e$ be the entry that was at $c$. Insert $e$ one row lower; if $e$ is placed at an outer corner, stop, and otherwise, insert the bumped out number yet another row one lower, continuing until a number is placed at an outer corner.
- **($\boldsymbol{\widetilde{\mathcal{I}}_c}$):** Place $\bullet$ at $c$. If $c$ is an outer corner of $\lambda$ then stop, and otherwise, let $e$ be the entry that was at $c$. Choose either to stop, or choose any lower row, and insert $e$ on it, provided that the column $e$ lands in continues to increase from top to bottom, and if $e$ is appended to the end of the row, it is at an outer corner. If indeed $e$ is appended, stop. Otherwise, either choose to stop, or choose a lower row, and try to insert the bumped out number in the same fashion. Continue this until either optionally stopping, or until forced to stop by placing a number at an outer corner.

If we terminate by placing $e$ at an outer corner, this is an insertion path with no entry deleted. If we terminate $\widetilde{\mathcal{I}}_c$ by optionally stopping, the resulting tableau has the same outer shape as $\boldsymbol{P}$, and the final entry bumped out is said to be **deleted** from the tableau.

*Local Rules.* Let $\lambda := \operatorname{osh} \boldsymbol{P}$. Place $\lambda^{(l)} \oplus c$ in column $l$. Then for $k = l+1, l+2, \ldots, h$, propagate as follows.

$$\begin{array}{r|c|c|c|c|l}
\text{column} & \multicolumn{1}{c}{l} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{k-1} & \multicolumn{1}{c}{k} & \\
\cline{2-2}\cline{4-5}
\boldsymbol{P} & \lambda^{(l)} & & \gamma & \delta & \\
\cline{2-2}\cline{4-5}
\widetilde{\mathcal{I}}_c \boldsymbol{P} & (\lambda^{(l)} \oplus c) & & \alpha & (\beta) & \longrightarrow \\
\cline{2-2}\cline{4-5}
\end{array}$$

- **$\boldsymbol{\gamma \lessdot \alpha = \delta}$ is false:** Let $\beta := \alpha \vee \delta$.
- **$\boldsymbol{\gamma \lessdot_{|r|} \alpha = \delta}$ ($\boldsymbol{\mathcal{I}_c}$):** Let $\beta \gtrdot_{|s|} \alpha$ with $s$ maximal subject to $s \leq r$ and $\alpha$ has an outer corner in column $s$.
- **$\boldsymbol{\gamma \lessdot_{|r|} \alpha = \delta}$ ($\boldsymbol{\widetilde{\mathcal{I}}_c}$):** Choose any $\beta \gtrdot \alpha$ with $\operatorname{col}(\beta/\alpha) \leq r$, or choose to let $\beta := \alpha$.

**Theorem 7.16.** *For any flag $f$ and $x > f_l$, we have* $\operatorname{ftype}(f \vee x) = \widetilde{\mathcal{I}}_c(\operatorname{ftype} f)$, *where* $c = \operatorname{type} x / \operatorname{type} f_l$.

*Fix $c$, $\boldsymbol{P}$, and $\boldsymbol{Q}$ with $\boldsymbol{Q} = \widetilde{\mathcal{I}}_c \boldsymbol{P}$, and let $e$ be the deleted entry, or $e = \infty$ if no entry is deleted. In a $q$-regular semi-primary lattice of type* $\operatorname{osh} \boldsymbol{Q}$, *the number of pairs $(x, f)$ where $f$ is a flag and $x > f_l$, with* $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ *and* $\operatorname{ftype}_{\hat{0}}(f \vee x) = \boldsymbol{Q}$, *is a product of factors depending on how entries $k$ move from $\boldsymbol{P}$ to $\boldsymbol{Q}$.*

$$\prod_{k \,\in\, \boldsymbol{P}} \begin{cases} \langle \operatorname{row}(\boldsymbol{P}, k)\rangle - \langle \operatorname{row}_1(\boldsymbol{P}, k)\rangle & \text{if } k \geq e\text{; else, assume } k < e\text{:} \\ \langle \operatorname{row}(\boldsymbol{Q}, k)\rangle - \langle \operatorname{row}_1(\boldsymbol{Q}, k)\rangle & \text{if } k \text{ stays in the same cell;} \\ \Big(\langle \operatorname{row}(\boldsymbol{P}, k)\rangle - \langle \operatorname{row}_1(\boldsymbol{P}, k)\rangle\Big) q & \text{if } k \text{ moves down one cell;} \\ \Big(\langle \operatorname{row}(\boldsymbol{P}, k)\rangle - \langle \operatorname{row}_1(\boldsymbol{P}, k)\rangle\Big)(q-1) & \text{if } k \text{ moves strictly left and down.} \end{cases}$$

$$\times g^{\operatorname{osh} \boldsymbol{Q}}_{\operatorname{osh} \boldsymbol{P}, (1)}(q)$$

*This is a polynomial of degree*

$$\begin{cases} n(\operatorname{sh} \boldsymbol{P}) + \text{length of insertion/deletion path} - 2 & \text{if } e < \infty; \\ n(\operatorname{osh} \boldsymbol{Q}) - n(\operatorname{osh} \boldsymbol{P}) + n(\operatorname{sh} P) + \text{length of insertion path} - 1 & \text{if } e = \infty. \end{cases}$$

*For specified values of $\boldsymbol{Q}$ and $\lambda \lessdot \operatorname{sh} \boldsymbol{Q}$, this is maximized over all $(c, \boldsymbol{P})$ where* $\operatorname{osh} \boldsymbol{P} = \lambda$ *and $\boldsymbol{Q} = \widetilde{\mathcal{I}}_c \boldsymbol{P}$, by a unique pair for which $\boldsymbol{Q} = \mathcal{I}_c \boldsymbol{P}$.*

*Proof.* Suppose $x > f_l$ and $g = f \vee x$. Since $f_k \leq f_k \vee x$, we have $\lambda^{(k)} \leq \mu^{(k)}$ for all $k$. Whenever $f_{k-1} = f_k$, we also have $g_{k-1} = g_k$, while when $f_{k-1} \lessdot f_k$, we have $g_{k-1} \leq g_k$. At most one $k$ satisfies $f_{k-1} \lessdot f_k$ and $g_{k-1} = g_k$: for such a $k$ we have $f_{k'} = g_{k'}$ for all $k' \geq k$, so that $f_{k'-1} \lessdot f_{k'} \Rightarrow g_{k'-1} \lessdot g_{k'}$ for $k' \geq k$. If there is such a $k$, denote it by $e$, and then $\boldsymbol{Q}$ is obtained from $\boldsymbol{P}$ by inserting $\bullet$ and deleting $e$ along some path; if there is no such $k$, let $e = \infty$, and then $\boldsymbol{Q}$ is obtained from $\boldsymbol{P}$ by inserting $\bullet$ along some path.

Now we solve for flags $f$ and elements $x > f_l$ with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ and $\operatorname{ftype}_{\hat{0}}(f \vee x) = \boldsymbol{Q}$ for some tableaux $\boldsymbol{P}$ and $\boldsymbol{Q}$. If $f$ and $g$ are flags and $x$ is an element with $g = f \vee x$, then for each $k$ we have $g_k = f_k \vee x$, so

$$g_k = g_{k-1} \vee g_k = g_{k-1} \vee f_k \vee x = (g_{k-1} \vee x) \vee f_k = g_{k-1} \vee f_k.$$

This relation allows us to select successive pairs $(f_k, g_k)$ by looking at a local part of the flags. While the local rules for the game $\boldsymbol{Q} = \widetilde{\mathcal{I}}_c \boldsymbol{P}$ were described in the order $k = l, l+1, \ldots, h$ to satisfy the initial condition given by $c$, the selection process proceeds in the order $k = h, h-1, \ldots, l+1$.

If $e < \infty$, let $f_h := g_h = \hat{1}$, and otherwise, let $g_h := \hat{1}$ and $f_h$ be a coatom with $\operatorname{type} f_h = \operatorname{osh} \boldsymbol{P}$, in one of $g^{\operatorname{osh} \boldsymbol{Q}}_{\operatorname{osh} \boldsymbol{P}, (1)}(q)$ ways.

For some $k = h, h-1, \ldots, l+1$ we have

$$\longleftarrow \operatorname{type} \boxed{\begin{matrix} (f_{k-1}) & \leq & f_k \\ \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} \\ (g_{k-1}) & \leq & g_k \end{matrix}} = \boxed{\begin{matrix} \gamma & \leq & \delta \\ \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} \\ \alpha & \leq & \beta \end{matrix}}$$

(that is, $f_k$ and $g_k$ and all the partitions are given, and we will choose $f_{k-1}$ and $g_{k-1}$ to satisfy this).

**$\gamma = \delta$:** Then $f_{k-1} := f_k$, and $g_k = f_k \vee g_{k-1} = f_{k-1} \vee g_{k-1} = g_{k-1}$ so $g_{k-1} := f_{k-1}$ and $\alpha = \beta$. Neither tableau has an entry $k$.

$$\boxed{\begin{array}{ccc} \gamma & = & \delta \\ \rotatebox{-90}{$\leq$} & & \rotatebox{-90}{$\leq$} \\ \alpha & = & \beta \end{array}} \qquad 1 \text{ choice}$$

**$\alpha = \beta$ and $\gamma \lessdot \delta$:** The only way to resolve the weak cover relations is $\gamma \lessdot \delta = \alpha = \beta$. So $g_{k-1} := g_k$, and $f_{k-1}$ is any lower cover of $f_k$ of type $\gamma$. The entry $k$ is deleted on the path from $\boldsymbol{P}$ to $\boldsymbol{Q}$.

$$\boxed{\begin{array}{ccc} \gamma & \lessdot & \delta \\ \rotatebox{-90}{$\lessdot$} & & \parallel \\ \alpha & = & \beta \end{array}} \qquad \begin{array}{l} \langle \delta'_r \rangle - \left\langle \delta'_{r+1} \right\rangle = \\ \quad \langle \mathrm{row}(\boldsymbol{P}, k) \rangle - \langle \mathrm{row}_1(\boldsymbol{P}, k) \rangle \\ \text{choices} \end{array}$$

**$\alpha \lessdot \beta$ and $\gamma \lessdot \delta$:** If $\alpha = \gamma$ then $g_{k-1} = f_{k-1}$; join both sides with $f_k$ to obtain $g_k = f_k$ also, so $\beta = \delta$. In view of this, the configurations of partitions in Lemma 7.11 (but turned upside down, so the roles of $f$ and $g$ are reversed, and the roles of $\alpha$ and $\beta$ are swapped with $\gamma$ and $\delta$) are all the configurations that arise when $\alpha \lessdot \beta$ and $\gamma \lessdot \delta$. The possibilities are as follows.

If $\alpha \neq \delta$, then $k$ is in the same cell of $\boldsymbol{P}$ and $\boldsymbol{Q}$.

$$(\alpha \neq \delta) \quad \boxed{\begin{array}{ccc} \gamma & \lessdot_{|r|} & \delta \\ \rotatebox{-90}{$\leq$} & & \rotatebox{-90}{$\leq$} \\ \alpha & \lessdot_{|r|} & \beta \end{array}} \qquad \begin{array}{l} \langle \beta'_r \rangle - \left\langle \beta'_{r+1} \right\rangle = \\ \quad \langle \mathrm{row}(\boldsymbol{Q}, k) \rangle - \langle \mathrm{row}_1(\boldsymbol{Q}, k) \rangle \\ \text{choices} \end{array}$$

The vertical relations are equality when $k$ is larger than the deleted entry, and in this case, the count is also $\langle \mathrm{row}(\boldsymbol{P}, k) \rangle - \langle \mathrm{row}_1(\boldsymbol{P}, k) \rangle$.

If $\alpha = \delta$, then $k$ is on the deletion path. It can move down and left from $\boldsymbol{P}$ to $\boldsymbol{Q}$, bumping a larger entry (as all entries at most $k$ in $\boldsymbol{P}$ are located in the cells of $\delta$, and $\beta/\alpha$ is an outer corner of $\delta$):

$$(r > s) \quad \boxed{\begin{array}{ccc} \gamma & \lessdot_{|r|} & \delta \\ \rotatebox{-90}{$\lessdot$}_{|r|} & & \rotatebox{-90}{$\lessdot$}_{|s|} \\ \alpha & \lessdot_{|s|} & \beta \end{array}} \qquad \begin{array}{l} \Big(\langle \delta'_r \rangle - \left\langle \delta'_{r+1} \right\rangle\Big)(q-1) = \\ \quad \Big(\langle \mathrm{row}(\boldsymbol{P}, k) \rangle - \langle \mathrm{row}_1(\boldsymbol{P}, k) \rangle\Big)(q-1) \\ \text{choices;} \end{array}$$

or it can move down one cell from $\boldsymbol{P}$ to $\boldsymbol{Q}$, bumping a larger entry:

$$\boxed{\begin{array}{ccc} \gamma & \lessdot_{|r|} & \delta \\ \rotatebox{-90}{$\lessdot$}_{|r|} & & \rotatebox{-90}{$\lessdot$}_{|r|} \\ \alpha & \lessdot_{|r|} & \beta \end{array}} \qquad \begin{array}{l} \Big(\langle \delta'_r \rangle - \left\langle \delta'_{r+1} \right\rangle\Big)q = \\ \quad \Big\langle (\langle \mathrm{row}(\boldsymbol{P}, k) \rangle - \langle \mathrm{row}_1(\boldsymbol{P}, k) \rangle\Big)q \\ \text{choices.} \end{array}$$

Finally we set $x := g_l$, and then $g = f \vee x$.

Multiply the numbers of choices together to obtain the polynomial stated in the theorem. The factors $\langle \mathrm{row}(\boldsymbol{P}, k) \rangle - \langle \mathrm{row}_1(\boldsymbol{P}, k) \rangle$ have degree $\mathrm{row}(\boldsymbol{P}, k) - 1$, as does the factor $\langle \mathrm{row}(\boldsymbol{Q}, k) \rangle - \langle \mathrm{row}_1(\boldsymbol{Q}, k) \rangle$ when it occurs, as it occurs when $k$ is in the same cell of $\boldsymbol{P}$ and $\boldsymbol{Q}$. The factors $q$ and $q-1$ have degree 1, and occur when $k \in \boldsymbol{P}$ is on the insertion/deletion path, but isn't the deleted entry; the total number of such $k$'s is two less than the length of the insertion/deletion path if an entry is deleted ($e < \infty$), or one less if none is ($e = \infty$). When no entry is deleted, there is also the Hall polynomial, of degree $\mathrm{row}(\mu^{(h)}/\lambda^{(h)}) - 1$. The total degree is as stated in the theorem.

When there is a deleted entry, the total degree equals

$$n(\operatorname{sh}\boldsymbol{Q}) + (\operatorname{row}(c) - 1) + \text{length of insertion path} - 1.$$

For fixed $\boldsymbol{Q}$ and fixed $\lambda \lessdot \operatorname{sh}\boldsymbol{Q}$, consider $(c, \boldsymbol{P})$ as stated in the theorem. Given the first cell $c$ and the last cell $c' = (\operatorname{osh}\boldsymbol{Q})/\lambda$ on the insertion path, the length of the insertion path is at most $\operatorname{row}(c') - \operatorname{row}(c) + 1$, uniquely obtained by inserting $\bullet$ at $c$ and then bumping entries to successive rows without skipping any rows (this might not be possible, given $c$ and $c'$). So an upper bound for the degree is $n(\operatorname{sh}\boldsymbol{Q}) + \operatorname{row}(c') - 1$. This is obtained only by $\boldsymbol{P} = \mathcal{D}_{c'}\boldsymbol{Q}$, with $c$ set to the cell of the deleted entry in this game. $\square$

*Note.* For all other games yet and still to be considered, we sought the value of $\boldsymbol{Q}$ that, for a specified $\boldsymbol{P}$, maximized the number of pairs of flags $(f, g)$ satisfying the game. The maximization in this theorem is *backwards*, because that turns out to be the condition required to get the classically defined internal insertion game.

Let's consider the ordinary maximization question. Given $\boldsymbol{P}$ and $c$, what game $\boldsymbol{Q} = \widetilde{\mathcal{I}}_c\boldsymbol{P}$ maximizes the degree of the number of choices of $(f, x)$? In a given lattice $\mathcal{L}$, the shape of $\boldsymbol{Q}$ is predetermined as $\operatorname{type}\mathcal{L}$, and the degree is maximized by maximizing the length of the insertion/deletion path. We consider the case when there is a deleted entry, and when there is no deleted entry.

Let

$$\boldsymbol{P} = \begin{array}{|c|c|c|}\hline \bullet & \bullet & 1 \\ \hline \bullet & 3 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}$$

and $c = (1, 3)$ be the cell with 1. The possible values of $\boldsymbol{Q} = \widetilde{\mathcal{I}}_c\boldsymbol{P}$ are listed below, along with the factor for each $k$; the Hall polynomial; and the product of all factors, equal to the total number of pairs $(f, x)$ with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ and $\operatorname{ftype}_{\hat{0}}(f \vee x) = \boldsymbol{Q}$ in a lattice whose type is $\operatorname{osh}\boldsymbol{Q}$.

| | | | | | |
|---|---|---|---|---|---|
| $\widetilde{\mathcal{I}}_c\boldsymbol{P}$ | $\begin{array}{\|c\|c\|c\|}\hline \bullet & \bullet & \bullet \\ \hline \bullet & 3 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}$ | $\begin{array}{\|c\|c\|c\|}\hline \bullet & \bullet & \bullet \\ \hline \bullet & 1 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}$ | $\begin{array}{\|c\|c\|c\|}\hline \bullet & \bullet & \bullet \\ \hline \bullet & 3 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$ | $\begin{array}{\|c\|c\|c\|}\hline \bullet & \bullet & \bullet \\ \hline \bullet & 1 \\ \cline{1-2} 2 & 3 \\ \cline{1-2}\end{array}$ $\mathcal{I}_c\boldsymbol{P}$ | $\begin{array}{\|c\|c\|c\|}\hline \bullet & \bullet & \bullet \\ \hline \bullet & 1 \\ \cline{1-2} 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$ |
| $k = 1$ | $1$ | $1 \cdot (q-1)$ | $1 \cdot (q-1)$ | $1 \cdot (q-1)$ | $1 \cdot (q-1)$ |
| 2 | $q^2$ | $q^2$ | $q^2$ | $q^2$ | $q^2$ |
| 3 | $q$ | $q$ | $q$ | $q \cdot q$ | $q \cdot (q-1)$ |
| Hall | $1$ | $1$ | $1$ | $q^2 + q$ | $q^3 + q^2$ |
| total | $q^3$ | $q^3(q-1)$ | $q^3(q-1)$ | $q^5(q-1)(q+1)$ | $q^5(q-1)^2(q+1)$ |

Of the three games $\widetilde{\mathcal{I}}_c\boldsymbol{P}$ with a deleted entry, corresponding to $f_h := \hat{1}$, two achieve the maximum degree 4, so there is not a unique generic value. A generic value can be thought of as a value that occurs with probability $1 - O(q^{-1})$ as $q \to \infty$. As $q \to \infty$, these two values each occur with probability $\frac{1}{2} + O(q^{-1})$, and the third with probability $o(q^{-1})$; it might be interesting to do an analysis generalizing the notion of genericity to multiple outcomes with well defined asymptotically positive probabilities.

In the games without a deleted entry, so that $f_h \lessdot \hat{1}$ and $x \not\leq f_h$, the value of $\widetilde{\mathcal{I}}_c\boldsymbol{P}$ achieving the maximum degree is *not* $\mathcal{I}_c\boldsymbol{P}$.

## 8. The leftward and rightward vertical strip games

We introduce new tableau games similar in spirit to evacuation and Robinson-Schensted that can be expressed in terms of tableaux, or in terms of grids of partitions. They are all uniform.

$$\begin{aligned} \mathcal{L}, \widetilde{\mathcal{L}} &: \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}} Af \\ \mathcal{CL}, \mathcal{C}\widetilde{\mathcal{L}} &: \ \operatorname{ftype} f \longmapsto \operatorname{ftype} Af \\ \mathcal{R}, \widetilde{\mathcal{R}} &: \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}} Cf \end{aligned}$$

Note that for any flag $f$, the intervals $[f_k, A_{f_h} f_k]$ and $[C_{f_l} f_k, f_k]$ are elementary, so that $\operatorname{type}(A_{f_h} f_k)/\operatorname{type} f_k$ and $\operatorname{type} f_k/\operatorname{type}(C_{f_l} f_k)$ are vertical strips by Proposition 4.67. In the games $j^c$ and $\Delta$, we formed a deletion path by creating a hole $\star$ into which an entry would slide. In the vertical strip games, we instead form vertical strips of holes $\star$ into which entries slide. In the leftward vertical strip game, a vertical strip is formed that moves left while entries move into it, and in the rightward vertical strip game, the directions are reversed.

### 8.1. The leftward vertical strip game, $\mathcal{L}\boldsymbol{P}$.

$$\mathcal{L}, \widetilde{\mathcal{L}} : \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}} Af$$

*Tableau Rules.* We form a sequence of tableau $\boldsymbol{P}^{(h)}, \ldots, \boldsymbol{P}^{(l)}$. Set $\boldsymbol{P}^{(h)} := \boldsymbol{P}$. To transform $\boldsymbol{P}^{(k)}$ to $\boldsymbol{P}^{(k-1)}$ for $k = h, \ldots, l+1$, if there is no $k$ in $\boldsymbol{P}^{(k)}$, let $\boldsymbol{P}^{(k-1)} := \boldsymbol{P}^{(k)}$, and otherwise do the following.

- **($\boldsymbol{\mathcal{L}}$):** Slide $k$ to the bottom leftmost $\star$ in a column strictly right of $k$, provided there is such a $\star$; if there is not, replace $k$ by $\star$.
- **($\widetilde{\boldsymbol{\mathcal{L}}}$):** If the cell right of $k$ has $\star$, slide $k$ into it; otherwise, choose either to slide $k$ into the bottommost $\star$ in any column right of $k$, or to replace $k$ by $\star$ without putting a new $k$ anywhere.

Finally, replace all $\star$'s with $\bullet$'s in $\boldsymbol{P}^{(l)}$ to obtain $\widetilde{\mathcal{L}}\boldsymbol{P}$.

*Local Rules.* Let $\mu^{(h)} := \lambda^{(h)}$, and then for $k = h, h-1, \ldots, l+1$, propagate as follows.

$$\begin{array}{cc|c|c|c|l} & k-1 & k & & h & \text{column} \\ \cline{2-3}\cline{5-5} \longleftarrow & \alpha & \beta & & \lambda^{(h)} & \boldsymbol{P} \\ \cline{2-3}\cline{5-5} & (\gamma) & \delta & & (\lambda^{(h)}) & \widetilde{\mathcal{L}}\boldsymbol{P} \\ \cline{2-3}\cline{5-5} \end{array}$$

- **$\boldsymbol{\beta = \alpha}$:** Let $\gamma := \delta$.
- **$\boldsymbol{\beta \gtrdot_{|s|} \alpha}$ ($\boldsymbol{\mathcal{L}}$):** If $\delta/\beta$ has no cells in columns $s+1, s+2, \ldots$, let $\gamma := \beta$, and otherwise, choose minimum $r > s$ so that $\delta/\beta$ has a cell in column $r$, and set $\gamma \lessdot_{|r|} \delta$.
- **$\boldsymbol{\beta \gtrdot_{|s|} \alpha}$ ($\widetilde{\boldsymbol{\mathcal{L}}}$):** If $\delta/\alpha$ has a horizontal brick, remove the right square of the brick from $\delta$ to obtain $\gamma$. Otherwise, $\delta/\alpha$ is a vertical strip. Either choose an $r > s$ such that column $r$ of $\delta/\alpha$ is nonempty, and set $\gamma \lessdot_{|r|} \delta$, or choose to let $\gamma := \delta$.

Note that these rules ensure $\delta/\beta$ and $\gamma/\alpha$ are vertical strips, and that $\alpha \leq \beta \leq \gamma \leq \delta$.

**Theorem 8.1.** *The local rules and the tableau rules for $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ describe the same game.*

*Proof.* Apply the local rules to $\boldsymbol{P}$ to determine a possible value of $\widetilde{\mathcal{L}}\boldsymbol{P}$. Define the tableau $\boldsymbol{P}^{(k)}$ (with $l \leq k \leq h$) as the following chain of partitions, with the entry $e$ to fill the squares added between two partitions shown between them as $\boxed{e}$:

$$
\begin{array}{ll}
 & \mu^{(l)} \lessdot_{\boxed{l+1}} \mu^{(l+1)} \lessdot_{\boxed{l+2}} \cdots \lessdot_{\boxed{k}} \mu^{(k)} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\quad \lessdot_{\boxed{\star}} \\
\boldsymbol{P}^{(k)} = & \qquad\qquad\qquad\qquad\qquad\qquad\quad \lambda^{(k)} \lessdot_{\boxed{k+1}} \cdots \lessdot_{\boxed{h}} \lambda^{(h)}
\end{array}
$$

Since $\mu^{(h)} = \lambda^{(h)}$, we have $\boldsymbol{P}^{(h)} = \boldsymbol{P}$. We will show that the tableau rules describe the transformation from $\boldsymbol{P}^{(k)}$ to $\boldsymbol{P}^{(k-1)}$.

When $\beta = \alpha$, there is no $k$ in $\boldsymbol{P}^{(k)}$, and both sets of rules yield $\boldsymbol{P}^{(k-1)} := \boldsymbol{P}^{(k)}$.

There is a horizontal brick in $\delta/\alpha$ exactly when $\star$ is in the cell right of $k$ in $\boldsymbol{P}^{(k)}$; moving $k$ right one cell is the same as removing the right square of the brick from $\delta$ to obtain $\gamma$.

When $\delta/\alpha$ is a vertical strip and $\beta \gtrdot_{|s|} \alpha$, letting $\gamma := \delta$ is the same as replacing $k$ with $\star$, and letting $\gamma \lessdot_{|r|} \delta$ for some $r > s$ is the same as sliding $k$ to the bottommost $\star$ in column $r$. A partition $\gamma$ with $\alpha \leq \gamma \lessdot_{|r|} \delta$ exists provided $r > s \geq 1$ and column $r$ of $\delta/\alpha$ is nonempty.

Finally, given $\boldsymbol{P}^{(l)}$, replacing the $\star$'s by $\bullet$'s yields the tableau $(\lambda^{(l)}, \ldots, \lambda^{(h)})$. $\square$

**Example 8.2.** We will compute all possible values of $\widetilde{\mathcal{L}}\boldsymbol{P}$ for

$$
\boldsymbol{P} = \boldsymbol{P}^{(8)} = \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & 7 \\ \cline{1-2} 3 & 8 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}
$$

We begin with $k = 8$. There is no $\star$ in the tableau, so there is nowhere to slide 8, so 8 is replaced by $\star$.

$$
\boldsymbol{P}^{(7)} = \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & 7 \\ \cline{1-2} 3 & \star \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}
$$

There is no $\star$ in any column right of 7, so 7 is replaced by $\star$, and similarly, the same then happens with 6.

$$
\boldsymbol{P}^{(6)} = \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & \star \\ \cline{1-2} 3 & \star \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}
\qquad\qquad
\boldsymbol{P}^{(5)} = \begin{array}{|c|c|c|} \hline 1 & 4 & \star \\ \hline 2 & \star \\ \cline{1-2} 3 & \star \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}
$$

Both columns right of 5 contain $\star$. For $\widetilde{\mathcal{L}}\boldsymbol{P}$, there are choices as to how to proceed, which we'll examine later. To compute $\mathcal{L}\boldsymbol{P}$, slide 5 to the lower $\star$ of column 2, obtaining

$$\boldsymbol{P}^{(4)} = \begin{ytableau} 1 & 4 & \star \\ 2 & \star \\ 3 & 5 \\ \star \end{ytableau}$$

Next, the cell just right of 4 has $\star$, so slide 4 there.

$$\boldsymbol{P}^{(3)} = \begin{ytableau} 1 & \star & 4 \\ 2 & \star \\ 3 & 5 \\ \star \end{ytableau}$$

The column right of 3 has two $\star$'s; slide 3 to the lower one.

$$\boldsymbol{P}^{(2)} = \begin{ytableau} 1 & \star & 4 \\ 2 & 3 \\ \star & 5 \\ \star \end{ytableau}$$

The column right of 2 has a $\star$, where we slide 2.

$$\boldsymbol{P}^{(1)} = \begin{ytableau} 1 & 2 & 4 \\ \star & 3 \\ \star & 5 \\ \star \end{ytableau}$$

There are no $\star$'s in columns right of 1, so 1 is replaced by $\star$.

$$\boldsymbol{P}^{(0)} = \begin{ytableau} \star & 2 & 4 \\ \star & 3 \\ \star & 5 \\ \star \end{ytableau}$$

Replace the $\star$'s by $\bullet$'s to obtain

$$\boldsymbol{\mathcal{L}P} = \begin{ytableau} \bullet & 2 & 4 \\ \bullet & 3 \\ \bullet & 5 \\ \bullet \end{ytableau}$$

There are several places we could have made degenerate choices. The first was at $k = 5$; to compute $\boldsymbol{\mathcal{L}P}$, we put 5 in the lower $\star$ of column 2, but to compute $\widetilde{\boldsymbol{\mathcal{L}}}\boldsymbol{P}$, we can either do that, slide 5 to the $\star$ in column 3, or replace 5 by $\star$ without sliding 5 somewhere else. In the second case, we have

$$\boldsymbol{P}^{(4)} = \begin{ytableau} 1 & 4 & 5 \\ 2 & \star \\ 3 & \star \\ \star \end{ytableau}$$

There is no $\star$ in columns right of 4, so 4 is replaced by $\star$.

$$\boldsymbol{P}^{(3)} = \begin{array}{lll} 1 & \star & 5 \\ 2 & \star & \\ 3 & \star & \\ \star & & \end{array}$$

Each of 3, 2, and 1 in turn have $\star$ just to their right, so we are successively forced to have

$$\boldsymbol{P}^{(2)} = \begin{array}{lll} 1 & \star & 5 \\ 2 & \star & \\ \star & 3 & \\ \star & & \end{array} \qquad \boldsymbol{P}^{(1)} = \begin{array}{lll} 1 & \star & 5 \\ \star & 2 & \\ \star & 3 & \\ \star & & \end{array} \qquad \boldsymbol{P}^{(0)} = \begin{array}{lll} \star & 1 & 5 \\ \star & 2 & \\ \star & 3 & \\ \star & & \end{array}$$

Finally, replace the $\star$'s by $\bullet$'s to obtain another possible value of $\widetilde{\mathcal{L}}\boldsymbol{P}$.

We list all possible choices in Figure 13.

**Example 8.3.** Now consider the skew tableau

$$\boldsymbol{P} = \begin{array}{lll} \bullet & \bullet & 6 \\ \bullet & 7 & \\ \bullet & 8 & \\ 5 & & \end{array}$$

The movements of $k = 8, 7, 6$ are similar to the previous example, since the entries larger than 4 are in the same positions in both tableaux.

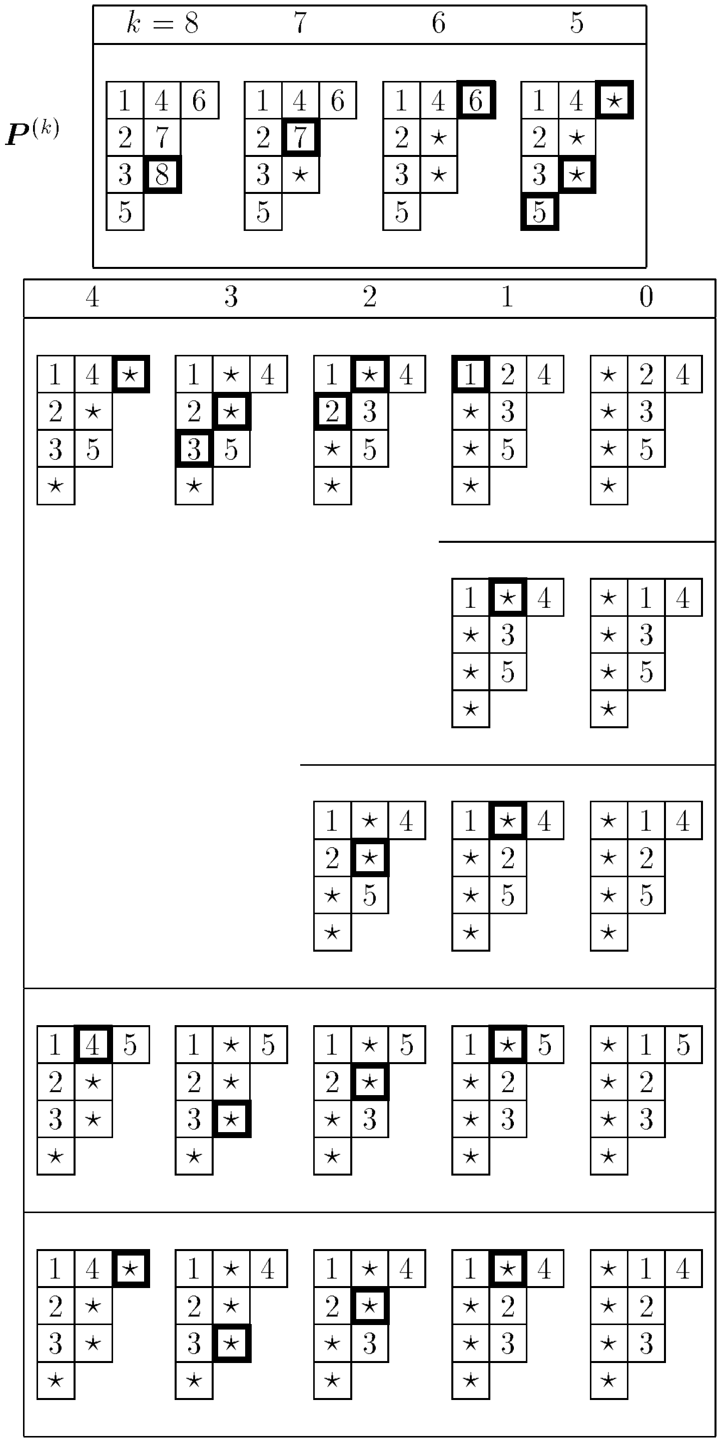
$$\begin{array}{ccc} \boldsymbol{P}^{(7)} & \boldsymbol{P}^{(6)} & \boldsymbol{P}^{(5)} \\ \begin{array}{lll} \bullet & \bullet & 6 \\ \bullet & 7 & \\ \bullet & \star & \\ 5 & & \end{array} & \begin{array}{lll} \bullet & \bullet & 6 \\ \bullet & \star & \\ \bullet & \star & \\ 5 & & \end{array} & \begin{array}{lll} \bullet & \bullet & \star \\ \bullet & \star & \\ \bullet & \star & \\ 5 & & \end{array} \end{array}$$

For $\boldsymbol{P}^{(4)}$, we have three choices again.

$$\begin{array}{lll} \bullet & \bullet & \star \\ \bullet & \star & \\ \bullet & 5 & \\ \star & & \end{array} \qquad \begin{array}{lll} \bullet & \bullet & 5 \\ \bullet & \star & \\ \bullet & \star & \\ \star & & \end{array} \qquad \begin{array}{lll} \bullet & \bullet & \star \\ \bullet & \star & \\ \bullet & \star & \\ \star & & \end{array}$$

Replace all $\star$'s by $\bullet$'s to obtain the possible values of $\widetilde{\mathcal{L}}\boldsymbol{P}$, the first of which is $\mathcal{L}\boldsymbol{P}$.

$$\begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & 5 & \\ \bullet & & \end{array} \qquad \begin{array}{lll} \bullet & \bullet & 5 \\ \bullet & \bullet & \\ \bullet & \bullet & \\ \bullet & & \end{array} \qquad \begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & \bullet & \\ \bullet & & \end{array}$$



A missing tableau abbreviates the tableau above it. The possible transformations of $\boldsymbol{P}^{(k)}$ to $\boldsymbol{P}^{(k-1)}$ are indicated: either slide $k$ to a $\boxed{\star}$ or change $\boxed{k}$ to $\boxed{\star}$. Finally, replace $\star$'s by $\bullet$'s in $\boldsymbol{P}^{(0)}$ to obtain $\widetilde{\mathcal{L}}\boldsymbol{P}$. The first game is $\mathcal{L}\boldsymbol{P}$.

FIGURE 13. All possible degenerate leftward vertical strip games.

## 8.2. The rightward vertical strip game, $\mathcal{R}\boldsymbol{P}$.

$$\mathcal{R}, \widetilde{\mathcal{R}} : \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}} Cf$$

*Tableau Rules.* Let $\boldsymbol{P}$ be a standard skew tableau on distinct entries. We form a sequence of tableau $\boldsymbol{P}^{(l)}, \dots, \boldsymbol{P}^{(h)}$. Set $\boldsymbol{P}^{(l)} := \boldsymbol{P}$. To transform $\boldsymbol{P}^{(k-1)}$ to $\boldsymbol{P}^{(k)}$ for $k = l+1, \dots, h$, if there is no $k$ in $\boldsymbol{P}^{(k-1)}$, let $\boldsymbol{P}^{(k)} := \boldsymbol{P}^{(k-1)}$, and otherwise do the following.

- **($\boldsymbol{\mathcal{R}}$):** Slide $k$ to the $\star$ that's upper rightmost in the columns strictly left of $k$, provided there is such a $\star$; if there is not, replace $k$ by $\star$.
- **($\widetilde{\boldsymbol{\mathcal{R}}}$):** If the cell left of $k$ has $\star$, slide $k$ into it; otherwise, choose either to slide $k$ into the topmost $\star$ in any column left of $k$, or to replace $k$ by $\star$ without putting a new $k$ anywhere.

Finally, delete all the cells with $\star$ from $\boldsymbol{P}^{(h)}$ to obtain $\widetilde{\mathcal{R}}\boldsymbol{P}$.

See Figure 14 for an example.

*Local Rules.* Let $\mu^{(l)} := \lambda^{(l)}$, and then for $k = l+1, l+2, \dots, h$, propagate as follows.

$$\begin{array}{r|c|c|c|c|l}
\text{column} & \multicolumn{1}{c}{l} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{k-1} & \multicolumn{1}{c}{k} & \\
\cline{2-2}\cline{4-5}
\boldsymbol{P} & \lambda^{(l)} & & \gamma & \delta & \\
\cline{2-2}\cline{4-5}
\widetilde{\mathcal{R}}\boldsymbol{P} & (\lambda^{(l)}) & & \alpha & (\beta) & \longrightarrow \\
\cline{2-2}\cline{4-5}
\end{array}$$

- **$\boldsymbol{\delta = \gamma}$ or $\boldsymbol{\delta \gtrdot_{|1|} \gamma}$:** Let $\beta := \alpha$.
- **$\boldsymbol{\delta \gtrdot_{|r|} \gamma}$ for some $\boldsymbol{r > 1}$ ($\boldsymbol{\mathcal{R}}$):** If $\delta/\alpha$ has no cells in columns $1, \dots, r-1$, let $\beta := \alpha$, and otherwise, choose maximum $s < r$ such that $\delta/\alpha$ has a cell in column $s$, and set $\beta \gtrdot_{|s|} \alpha$.
- **$\boldsymbol{\delta \gtrdot_{|r|} \gamma}$ for some $\boldsymbol{r > 1}$ ($\widetilde{\boldsymbol{\mathcal{R}}}$):** If $\delta/\alpha$ has a horizontal brick, add the left square of the brick to $\alpha$ to obtain $\beta$. Otherwise, $\delta/\alpha$ is a vertical strip. Either choose an $s < r$ such that column $s$ of $\delta/\alpha$ is nonempty and set $\beta \gtrdot_{|s|} \alpha$, or choose to let $\beta := \alpha$.

Note that these rules ensure $\delta/\beta$ and $\gamma/\alpha$ are vertical strips, and that $\alpha \leq \beta \leq \gamma \leq \delta$.

**Theorem 8.4.** *The local rules and the tableau rules for $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ describe the same game.*

*Proof.* Apply the local rules to $\boldsymbol{P}$ to determine a possible value of $\widetilde{\mathcal{R}}\boldsymbol{P}$. Define the tableau $\boldsymbol{P}^{(k)}$ (with $l \leq k \leq h$) as the following chain of partitions, with the entry $e$ to fill the squares added between two partitions shown between them as $\boxed{e}$:

$$\begin{array}{lll}
\boldsymbol{P}^{(k)} \;= & & \lambda^{(k)} \lessdot_{\boxed{k+1}} \cdots \lessdot_{\boxed{h}} \lambda^{(h)} \\
& & \gtrdot_{\boxed{\star}} \\
\mu^{(l)} \lessdot_{\boxed{l+1}} \mu^{(l+1)} \lessdot_{\boxed{l+2}} \cdots \lessdot_{\boxed{k}} \mu^{(k)} & &
\end{array}$$

Since $\mu^{(l)} = \lambda^{(l)}$, we have $\boldsymbol{P}^{(l)} = \boldsymbol{P}$. We will show that the tableau rules describe the transformation from $\boldsymbol{P}^{(k-1)}$ to $\boldsymbol{P}^{(k)}$.

When $\delta = \gamma$, there is no $k$ in $\boldsymbol{P}^{(k-1)}$, and both sets of rules yield $\boldsymbol{P}^{(k)} := \boldsymbol{P}^{(k-1)}$.

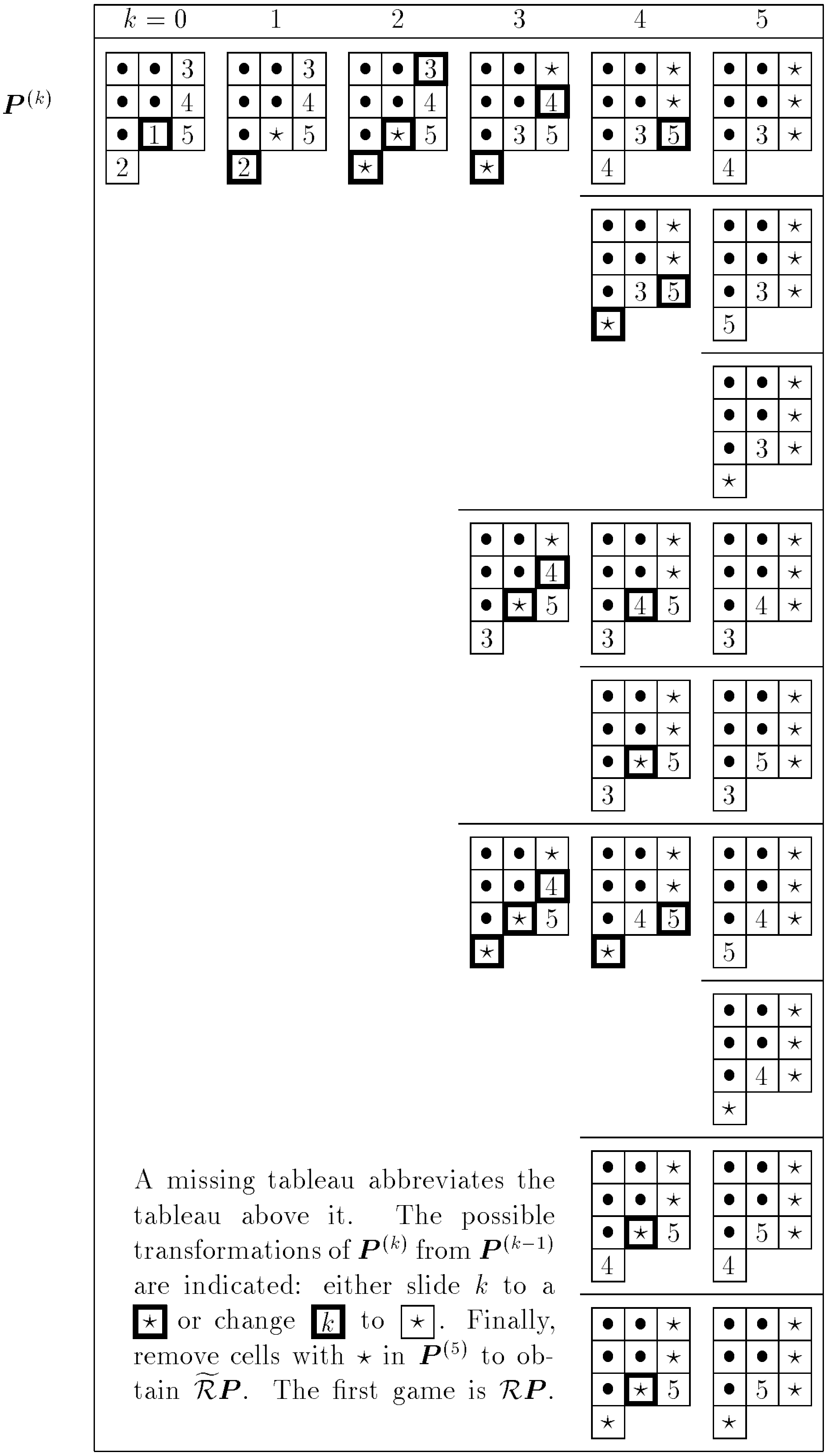



FIGURE 14. All possible degenerate rightward vertical strip games.

There is a horizontal brick in $\delta/\alpha$ exactly when $\star$ is in the cell left of $k$ in $\boldsymbol{P}^{(k-1)}$; moving $k$ left one cell is the same as adding the left square of the brick to $\alpha$ to obtain $\beta$.

When $\delta/\alpha$ is a vertical strip and $\delta >_{|r|} \gamma$, letting $\beta := \alpha$ is the same as replacing $k$ with $\star$, and letting $\beta >_{|s|} \alpha$ for some $s < r$ is the same as sliding $k$ to the topmost $\star$ in column $s$. A partition $\beta$ with $\gamma \geq \beta >_{|s|} \alpha$ exists provided $r > s \geq 1$ and column $s$ of $\gamma/\alpha$ is nonempty.

Finally, given $\boldsymbol{P}^{(h)}$, deleting the cells with $\star$ yields the tableau $(\mu^{(l)}, \ldots, \mu^{(h)})$. $\square$

Next we prove that $\operatorname{ftype}_{\hat{0}} Af = \widetilde{\mathcal{L}}(\operatorname{ftype} f)$ and $\operatorname{ftype}_{\hat{0}} Cf = \widetilde{\mathcal{R}}(\operatorname{ftype} f)$, and show that $\mathcal{L}$ and $\mathcal{R}$ are the uniform values.

**Definition 8.5.** The notation $x \leq_{\mathrm{e}} y$ means $x \leq y$ and $[x, y]$ is an elementary interval.

**Definition 8.6.** For a vertical strip $\lambda/\mu$ and integer $t \geq 0$, define

$$\begin{aligned}
\operatorname{vlen}(\lambda, \mu, t) &\stackrel{\text{def}}{=} |\lambda| - |\lambda[t] \vee \mu| \;=\; \#\{\, i : \lambda_i > \mu_i \geq t \,\} \\
&= \;\#\text{ of cells in } \lambda/\mu \text{ in columns } t+1, t+2, \ldots \\
\operatorname{vlen}'(\lambda, \mu, t) &\stackrel{\text{def}}{=} |\mu| - |\mathcal{C}\lambda \vee \mu[t]| \;=\; \#\{\, i : \lambda_i = \mu_i > t \,\} \\
&= \;\#\text{ of rows of } \lambda \text{ of length greater than } t \text{ without cells of } \lambda/\mu
\end{aligned}$$

Evaluate the right-hand expressions for vlen and vlen$'$ by summing the contribution of each row in the middle expressions. In vlen, the contribution of row $i$ is $\lambda_i - \max\{\min\{\lambda_i, t\}, \mu_i\}$. Since $\lambda_i \geq \mu_i$ for all $i$, this contribution is 1 when both $\lambda_i > t$ and $\lambda_i > \mu_i$, and is 0 otherwise. Similarly, the contribution of row $i$ in vlen$'$ is $\mu_i - \min\{\max\{\lambda_i - 1, 0\}, \mu_i, t\}$, which is 0 when either $\lambda_i > \mu_i$ or $\mu_i \leq t$, and is 1 otherwise. Also note that $\operatorname{vlen}(\lambda, \mu, t) + \operatorname{vlen}'(\lambda, \mu, t) = \lambda'_{t+1}$ because the sum is the number of parts of $\lambda$ of length at least $t+1$ that equal or cover the corresponding part of $\mu$, and all parts do because $\lambda/\mu$ is a vertical strip.

Let $\boldsymbol{P}$ be a skew tableau whose entries are distinct integers, and a vertical strip of $\star$'s. Let $k$ be an integer in $\boldsymbol{P}$ and $t = 0$ or 1. Let $r = \operatorname{col}(\boldsymbol{P}, k)$. Define

$$\begin{aligned}
\operatorname{vlen}_t(\boldsymbol{P}, k) &\stackrel{\text{def}}{=} \#\text{ of } \star\text{'s in } \boldsymbol{P} \text{ at least } t \text{ columns right of } k \\
\operatorname{vlen}'_t(\boldsymbol{P}, k) &\stackrel{\text{def}}{=} \operatorname{row}_t(\boldsymbol{P}, k) - \operatorname{vlen}_t(\boldsymbol{P}, k)
\end{aligned}$$

When $t$ is omitted, $t = 0$. If $\mu$ is the shape of the subtableau of $\boldsymbol{P}$ whose entries are $\bullet, l+1, \ldots, k$, and $\lambda$ is the shape of the subtableau whose entries are $\bullet, l+1, \ldots, k, \star$, and $r = \operatorname{col}(\boldsymbol{P}, k)$, then $\operatorname{vlen}_t(\boldsymbol{P}, k) = \operatorname{vlen}(\lambda, \mu, r+t-1)$ and $\operatorname{vlen}'_t(\boldsymbol{P}, k) = \operatorname{vlen}'(\lambda, \mu, r+t-1)$.

**Lemma 8.7.** *In a $q$-regular semi-primary lattice, the number $T(\alpha, \beta, \gamma, \delta)$ of solutions $(f_{k-1}, g_{k-1})$ to*

$$\operatorname{type} \boxed{\begin{array}{ccc} (f_{k-1}) & \leq & f_k \\ & \diagdown & \\ \rotatebox{90}{$\leq_{\mathrm{e}}$} & \rotatebox{90}{$\leq_{\mathrm{e}}$} & \rotatebox{90}{$\leq_{\mathrm{e}}$} \\ & & \diagdown \\ (g_{k-1}) & \leq & g_k \end{array}} = \boxed{\begin{array}{ccc} \gamma & \leq & \delta \\ \diagdown & & \\ \rotatebox{90}{$\leq_{\mathrm{v}}$} & \rotatebox{90}{$\leq_{\mathrm{v}}$} & \rotatebox{90}{$\leq_{\mathrm{v}}$} \\ & \diagdown & \\ \alpha & \leq & \beta \end{array}}$$

*in various situations is as follows.*

$$(1)\quad \begin{array}{|ccccc|}\hline \gamma & & = & & \delta \\ & \diagdown & & & \\ \leq_\vee & & \leq_\vee & & \leq_\vee \\ & & & \diagdown & \\ \alpha & & = & & \beta \\ \hline \end{array} \qquad \textit{1 choice}$$

$$(2)\quad \begin{array}{|ccccc|}\hline \gamma & & \lessdot_{|r|} & & \delta \\ & \diagdown & & & \\ \leq_\vee & & \leq_\vee & & \leq_\vee \\ & & & \diagdown & \\ \alpha & & = & & \beta \\ \hline \end{array} \qquad \langle \mathrm{vlen}(\delta,\beta,r-1)\rangle - \langle \mathrm{vlen}(\delta,\beta,r)\rangle \ \textit{choices}$$

$$(3)\quad \begin{array}{|ccccc|}\hline \gamma & & = & & \delta \\ & \diagdown & & & \\ \leq_\vee & & \leq_\vee & & \leq_\vee \\ & & & \diagdown & \\ \alpha & & \lessdot_{|s|} & & \beta \\ \hline \end{array} \qquad \langle \mathrm{vlen}'(\gamma,\beta,s-1)\rangle - \langle \mathrm{vlen}'(\gamma,\beta,s)\rangle \ \textit{choices}$$

$$(4)\quad \begin{array}{|ccccc|}\hline \gamma & & \lessdot_{|r|} & & \delta \\ & \diagdown & & & \\ \leq_\vee & & \leq_\vee & & \leq_{\mathrm{e}} \\ & & & \diagdown & \\ \alpha & & \lessdot_{|s|} & & \beta \\ \hline \end{array} \qquad \textit{where } g_{k-1} \not\leq_{\mathrm{e}} f_k$$

*We have $r > s$ for all solutions of this last case, even in semi-primary lattices that are not q-regular. For ones that are, choose $f_{k-1}$ in one of*

$$\langle \mathrm{vlen}(\delta,\beta,r-1)\rangle - \langle \mathrm{vlen}(\delta,\beta,r)\rangle$$

*ways, and then $g_{k-1}$ in one of the following number of ways;*

$$\begin{cases} \langle \mathrm{vlen}'(\gamma,\beta,s-1)\rangle - \langle \mathrm{vlen}'(\gamma,\beta,s)\rangle & \textit{if } \delta/\alpha \textit{ has a horizontal brick;} \\ q^{\mathrm{vlen}'(\delta,\beta,s-1)} & \textit{if } \delta \geq_\vee \alpha \textit{ and } r-1 = s; \\ q^{\mathrm{vlen}'(\delta,\beta,s-1)} - q^{\mathrm{vlen}'(\delta,\beta,s)} & \textit{if } \delta \geq_\vee \alpha \textit{ and } r-1 > s. \end{cases}$$

*In cases (1), (2), (4) we have $C_{g_{k-1}} f_k = g_k$, and in cases (1), (3), (4) we have $A_{f_k} g_{k-1} = f_{k-1}$.*

*Proof.* Consider a solution $f_{k-1}$ and $g_{k-1}$ to one of these configurations. In (1) and (2), we have $g_{k-1} = g_k$, so $[g_{k-1}, f_k]$ is elementary; thus, $C_{g_{k-1}} f_k = g_{k-1} = g_k$. In (1) and (3), $f_{k-1} = f_k$ so $[g_{k-1}, f_k]$ is elementary, so $A_{f_k} g_{k-1} = f_k = f_{k-1}$. In (4), the interval $[g_k, f_k]$ is elementary, so $g_k \geq C f_k$. We have $C_{g_{k-1}} f_k = C f_k \vee g_{k-1}$, which is at least $g_{k-1}$ and at most $g_k$ (because $C_{g_{k-1}} f_k \leq C_{g_k} f_k = g_k$). It's not $g_{k-1}$ because $[g_{k-1}, f_k]$ is given as not elementary, whence $C_{g_{k-1}} f_k = g_k$. Similarly in (4), $A_{f_k} g_{k-1} = f_{k-1}$.

Now we enumerate the solutions to each case.

(1) Since $\gamma = \delta$ and $\alpha = \beta$, we have $f_{k-1} := f_k$ and $g_{k-1} := g_k$, in one way.

(2) Set $g_{k-1} := g_k$ in one way.

Since $\gamma \lessdot_{|r|} \delta$, we have that $f_{k-1} \geq f_k[r-1]$ but $f_{k-1} \not\geq f_k[r]$. Also, $f_{k-1}$ is a coatom of $[g_k, f_k]$. Thus, $f_{k-1}$ is any coatom of $I_1 = \Big[f_k[r-1] \vee g_k, f_k\Big]$

not also in $I_2 = \left[f_k[r] \vee g_k, f_k\right]$. These are both elementary intervals, as they are contained in the elementary interval $[g_k, f_k]$, so altogether there are $\langle\rho(I_1)\rangle - \langle\rho(I_2)\rangle$ choices of $f_{k-1}$. We must find the ranks of $I_1$ and $I_2$.

Since $[g_k, f_k]$ is elementary, there is a hereditary decomposition of $g_k$ and $f_k$ by Theorem 4.69, and so by Theorem 4.62, $\operatorname{type}(f_k[r-1] \vee g_k) = \delta[r-1] \vee \beta$ and $\operatorname{type}(f_k[r] \vee g_k) = \delta[r] \vee \beta$. Thus $\rho(I_1) = \operatorname{vlen}(\delta, \beta, r-1)$ and $\rho(I_2) = \operatorname{vlen}(\delta, \beta, r)$.

(3) Set $f_{k-1} := f_k$ in one way.

We will compute

$$N(g_k, f_{k-1}, s) = \#\left\{ g_{k-1} \lessdot g_k \,:\, g_{k-1} \leq_{\mathrm{e}} f_{k-1} \text{ and } \operatorname{type} g_{k-1} \lessdot_{|s|} \operatorname{type} g_k \right\}.$$

Since $g_{k-1}$ is a lower cover of $g_k$ with type $\alpha \lessdot_{|s|} \beta$, we have $g_{k-1} \geq g_k[s-1]$ but $g_{k-1} \not\geq g_k[s]$. Since $[g_{k-1}, f_{k-1}]$ is elementary, $g_{k-1} \geq C f_{k-1}$.

Thus, $g_{k-1}$ is any coatom of $I_3 = \left[C f_{k-1} \vee g_k[s-1], g_k\right]$ not also in $I_4 = \left[C f_{k-1} \vee g_k[s], g_k\right]$. These intervals are both elementary, because they are contained in the elementary interval $[C f_{k-1}, f_{k-1}]$, so there are $\langle\rho(I_3)\rangle - \langle\rho(I_4)\rangle$ choices of $g_{k-1}$. We must find the ranks of the intervals.

Since $[g_k, f_{k-1}]$ is elementary, there is a hereditary decomposition of $g_k$ and $f_k$, so we may compute types as follows: $\operatorname{type}(C f_{k-1} \vee g_k[s-1]) = \mathcal{C}\gamma \vee \beta[s-1]$ and $\operatorname{type}(C f_{k-1} \vee g_k[s]) = \mathcal{C}\gamma \vee \beta[s]$. Thus $\rho(I_3) = \operatorname{vlen}'(\gamma, \beta, s-1)$ and $\rho(I_4) = \operatorname{vlen}'(\gamma, \beta, s)$, so $N(g_k, f_{k-1}, s) = \langle \operatorname{vlen}'(\gamma, \beta, s-1)\rangle - \langle \operatorname{vlen}'(\gamma, \beta, s)\rangle$.

(4) Given a solution $f_{k-1}$ and $g_{k-1}$, the values of $r$ and $s$ are

$$\begin{aligned} r &= 1 + \max\left\{ t \,:\, A^t\hat{0} \wedge f_{k-1} = A^t\hat{0} \wedge f_k \right\} \\ s &= 1 + \max\left\{ t \,:\, A^t\hat{0} \wedge g_{k-1} = A^t\hat{0} \wedge g_k \right\}. \end{aligned}$$

If $A^t\hat{0} \wedge g_{k-1} = A^t\hat{0} \wedge g_k$, apply $A_{f_k}$ to both sides to obtain $A^{t+1}\hat{0} \wedge A g_{k-1} \wedge f_k = A^{t+1}\hat{0} \wedge A g_k \wedge f_k$, which simplifies to $A^{t+1}\hat{0} \wedge f_{k-1} = A^{t+1}\hat{0} \wedge f_k$. Thus, $r > s$.

Select $f_{k-1}$ as in case (2) in one of $\langle \operatorname{vlen}(\delta, \beta, r-1)\rangle - \langle \operatorname{vlen}(\delta, \beta, r)\rangle$ ways.

If $\delta/\alpha$ has a horizontal brick, then $[g_{k-1}, f_k]$ is not elementary for all $g_{k-1} \lessdot g_k$ of type $\alpha$, by Proposition 4.67. So choose $g_{k-1}$ in the same fashion as in case (3).

Now assume $\delta/\alpha$ is a vertical strip.

If $[g_{k-1}, f_k]$ is elementary, then so is the subinterval $[g_{k-1}, f_{k-1}]$. Thus, the number of choices of $g_{k-1}$ is

$$\begin{aligned} N(g_k, f_{k-1}, s) - N(g_k, f_k, s) = {} & \Big(\langle \operatorname{vlen}'(\gamma, \beta, s-1)\rangle - \langle \operatorname{vlen}'(\gamma, \beta, s)\rangle\Big) \\ & - \Big(\langle \operatorname{vlen}'(\delta, \beta, s-1)\rangle - \langle \operatorname{vlen}'(\delta, \beta, s)\rangle\Big). \end{aligned}$$

For some $j$ we have $\delta/\gamma = (j, r)$, and then $\delta_j \gtrdot \gamma_j = \beta_j = \alpha_j = r - 1 \geq s$. Thus, $\operatorname{vlen}'(\gamma, \beta, s-1) \gtrdot \operatorname{vlen}'(\delta, \beta, s-1)$, while $\operatorname{vlen}'(\beta, \gamma, s) \geq \operatorname{vlen}'(\delta, \beta, s)$ with equality when $r - 1 = s$ and covering when $r - 1 > s$. Note that $\langle M+1\rangle - \langle M\rangle = q^M$ for all $M$, so when $r-1 = s$ there are $q^{\operatorname{vlen}'(\delta,\beta,s-1)}$ choices of $g_{k-1}$, and when $r - 1 > s$ there are $q^{\operatorname{vlen}'(\delta,\beta,s-1)} - q^{\operatorname{vlen}'(\delta,\beta,s)}$ choices. $\square$

**Definition 8.8.** Let $\boldsymbol{P} = (\lambda^{(l)}, \ldots, \lambda^{(h)})$ and $\boldsymbol{Q} = (\mu^{(l)}, \ldots, \mu^{(h)})$ be skew tableau such that $\lambda^{(k)} \gtrdot_\vee \mu^{(k)}$ for $l < k \leq h$. Let $\boldsymbol{P}^{(k)}$ be a skew tableau of shape $\lambda^{(h)}/\mu^{(l)}$ in which entries $e \leq k$ are in the same cells as in $\boldsymbol{Q}$; entries $e > k$ are in the same cells as in $\boldsymbol{P}$; and the vertical strip $\lambda^{(k)}/\mu^{(k)}$ is filled with $\star$. Define

$$
\begin{aligned}
E(\boldsymbol{P}, \boldsymbol{Q}; q) &\overset{\text{def}}{=} \prod_{k \,\in\, \boldsymbol{P}} \left( \left\langle \mathrm{vlen}(\boldsymbol{P}^{(k-1)}, k) + 1 \right\rangle - \left\langle \mathrm{vlen}_1(\boldsymbol{P}^{(k-1)}, k) \right\rangle \right) \\
&\times \prod_{\substack{k \,\in\, \boldsymbol{Q} \\ k \,\notin\, \boldsymbol{P}}} \left( \left\langle \mathrm{vlen}(\boldsymbol{P}^{(k)}, k) \right\rangle - \left\langle \mathrm{vlen}_1(\boldsymbol{P}^{(k)}, k) \right\rangle \right) \\
&\times \prod_{\substack{k \,\in\, \boldsymbol{Q} \\ k \,\in\, \boldsymbol{P}}} \begin{cases} \left( \left\langle \mathrm{vlen}'(\boldsymbol{P}^{(k)}, k) + 1 \right\rangle - \right. & \text{if } k \text{ slides} \\ \qquad \left. \left\langle \mathrm{vlen}'_1(\boldsymbol{P}^{(k)}, k) + 1 \right\rangle \right) & \quad \text{horizontally one cell;} \\ q^{\mathrm{vlen}'(\boldsymbol{P}^{(k)}, k)} & \quad \text{one column to a different row;} \\ q^{\mathrm{vlen}'(\boldsymbol{P}^{(k)}, k)} - q^{\mathrm{vlen}'_1(\boldsymbol{P}^{(k)}, k)} & \quad \text{more than one column.} \end{cases}
\end{aligned}
$$

This is the product of $T(\mu^{(k-1)}, \mu^{(k)}, \lambda^{(k-1)}, \lambda^{(k)})$ over all $k$. In configurations (2) and (4), we have $\mathrm{vlen}(\delta, \beta, r-1) = \mathrm{vlen}(\boldsymbol{P}^{(k-1)}, k) + 1$ and $\mathrm{vlen}(\delta, \beta, r) = \mathrm{vlen}_1(\boldsymbol{P}^{(k-1)}, k)$ because $\star$'s form the strip $\gamma/\alpha$ in $\boldsymbol{P}^{(k-1)}$, which differs from $\delta/\beta$ only in one cell in column $r = \mathrm{col}(\boldsymbol{P}^{(k-1)}, k)$. In (4), we have $\mathrm{vlen}'(\delta, \beta, s-1) = \mathrm{vlen}'(\boldsymbol{P}^{(k)}, k)$ because the vertical strip of $\star$'s in the tableau is also $\delta/\beta$, and the columns we examine are the same. The value in (3) is similar. Finally in (4), one cell of $\delta/\beta$ in column $r$ is deleted to form the strip $\gamma/\beta$, and since $r > s$, this deleted cell is counted by both $\mathrm{vlen}'(\gamma, \beta, s-1) = \mathrm{vlen}'(\boldsymbol{P}^{(k)}, k) + 1$ and $\mathrm{vlen}'(\gamma, \beta, s) = \mathrm{vlen}'_1(\boldsymbol{P}^{(k)}, k) + 1$.

**Theorem 8.9.** *For all multisaturated flags $f$ in a semi-primary lattice,* $\mathrm{ftype}_{\hat{0}}\, Af = \widetilde{\mathcal{L}}(\mathrm{ftype}_{\hat{0}}\, f)$ *and* $\mathrm{ftype}_{\hat{0}}\, Cf = \widetilde{\mathcal{R}}(\mathrm{ftype}_{\hat{0}}\, f)$.

*Now consider a $q$-regular semi-primary lattice of type $\lambda$. Let $\boldsymbol{P}$ be a skew tableau of outer shape $\lambda$.*

*Fix $\boldsymbol{Q} = \widetilde{\mathcal{R}}\boldsymbol{P}$ of outer shape $\mu$. The number of flags $f$ for which* $\mathrm{ftype}_{\hat{0}}\, f = \boldsymbol{P}$ *and* $\mathrm{ftype}_{\hat{0}}\, Cf = \boldsymbol{Q}$ *is $g^{\lambda}_{\mu,(m)'}(q) \cdot E(\boldsymbol{P}, \boldsymbol{Q}; q)$, where $m = |\lambda| - |\mu|$. Flags $f$ with* $\mathrm{ftype}_{\hat{0}}\, f = \boldsymbol{P}$ *are uniformly mapped by $C$ to flags $g$ from $f_l$ to $C_{f_l} f_h$ with* $\mathrm{ftype}_{\hat{0}}\, g = \mathcal{R}\boldsymbol{P}$.

*Fix $\boldsymbol{Q} = \widetilde{\mathcal{L}}\boldsymbol{P}$. The number of flags $g$ for which* $\mathrm{ftype}_{\hat{0}}\, f = \boldsymbol{P}$ *and* $\mathrm{ftype}_{\hat{0}}\, Af = \boldsymbol{Q}$ *is $E(\boldsymbol{Q}, \boldsymbol{P}; q)$. Flags $f$ with* $\mathrm{ftype}_{\hat{0}}\, f = \boldsymbol{P}$ *are uniformly mapped by $A$ to flags $g$ from $A_{f_h} f_l$ to $f_h$ with* $\mathrm{ftype}_{\hat{0}}\, g = \mathcal{L}\boldsymbol{P}$.

*Proof.* If $f$ and $g$ are flags with $g = Cf$, then we have $g_l = C_{f_l} f_l = g_l$ and $g_k = C_{f_l} f_k = Cf_k \vee f_l = Cf_k \vee Cf_{k-1} \vee f_l = Cf_k \vee g_{k-1}$. The intervals $[g_{k-1}, f_{k-1}]$ and $[g_k, f_k]$ are elementary, so by Proposition 4.67, $\mathrm{type}\, g_{k-1} \leq_\vee \mathrm{type}\, f_{k-1}$ and $\mathrm{type}\, g_k \leq_\vee \mathrm{type}\, f_k$. Since $f_{k-1} \leq f_k$, we have $Cf_k \leq f_{k-1}$. Also $g_{k-1} \leq f_{k-1}$, so $g_k \leq f_{k-1}$. Thus $\mathrm{type}\, g_k \leq \mathrm{type}\, f_{k-1}$. Also, $[g_k, f_{k-1}]$ is a subinterval of the elementary interval $[g_k, f_k]$, so it is elementary, and $\mathrm{type}\, g_k \leq_\vee \mathrm{type}\, f_{k-1}$.

Consider a semi-primary lattice of type $\lambda$. Let $\boldsymbol{P} = (\lambda^{(l)}, \ldots, \lambda^{(h)})$ be a standard tableau of outer shape $\lambda$, and fix $\boldsymbol{Q} = \widetilde{\mathcal{R}}\boldsymbol{P} = (\mu^{(l)}, \ldots, \mu^{(h)})$. To find all pairs of flags $(f, g)$ with $g = Cf$ and $\mathrm{ftype}_{\hat{0}}\, f = \boldsymbol{P}$ and $\mathrm{ftype}_{\hat{0}}\, Cf = \boldsymbol{Q}$, set $f_h := \hat{1}$ in one way, and choose $g_h \leq_{\mathrm{e}} f_h$ so that $\mathrm{type}\, g_h = \mu^{(h)}$ in one of $g^{\lambda^{(h)}}_{\mu^{(h)},(m(h))'}(q)$ ways (where $m(k) = |\lambda(k)| - \left|\mu^{(k)}\right|$). Then for $k = h, h-1, \ldots, l+1$, successively find pairs

$(f_{k-1}, g_{k-1})$ of weak lower covers of $f_k$ and $g_k$ with specified types $\lambda^{(k-1)}$ and $\mu^{(k-1)}$ such that $g_k = Cf_k \vee g_{k-1}$. Cases (1), (2), and (4) of Lemma 8.7 give all possibilities: if $\lambda^{(k-1)} = \lambda^{(k)}$ then $f_{k-1} = f_k$ so applying $C$ maintains equality, and we are in case (1), not (3). Suppose $\lambda^{(k-1)} \lessdot \lambda^{(k)}$. If $\mu^{(k-1)} = \mu^{(k)}$, we are in case (2). If $\mu^{(k-1)} \lessdot \mu^{(k)}$, then for any solution we find, we have $g_k = Cf_k \vee g_{k-1} > g_{k-1}$ so $g_{k-1} \not\geq Cf_k$, whence $g_{k-1} \not\leq_{\mathrm{e}} f_k$. Thus, we are in case (4).

In a $q$-regular semi-primary lattice, multiply the numbers of choices together to obtain

$$g^{\lambda^{(h)}}_{\mu^{(h)},(m(h))'}(q) \cdot E(\boldsymbol{P}, \boldsymbol{Q}; q)$$

flags $f$ with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ and $\operatorname{ftype}_{\hat{0}} Cf = \boldsymbol{Q}$. Denote this product $p(\boldsymbol{P}, \boldsymbol{Q}; q)$. This product is a monic polynomial in $q$: the Hall polynomial is monic, because its leading coefficient is the number of ways to fill a Littlewood-Richardson tableau of specified skew shape with just 1's, which can only be done in one way, and $E(\boldsymbol{P}, \boldsymbol{Q}; q)$ is a product of monic polynomials.

We show that $\mathcal{R}$ is the generic game by inducting on $\operatorname{high}(\boldsymbol{P}) - \operatorname{low}(\boldsymbol{P})$. Note that $d\mathcal{R}\boldsymbol{P} = \mathcal{R}d\boldsymbol{P}$, and by the inductive hypothesis, $\mathcal{R}d\boldsymbol{P}$ is the generic value of $\widetilde{\mathcal{R}}d\boldsymbol{P}$. Thus, $p(d\boldsymbol{P}, d\mathcal{R}\boldsymbol{P}; q)$ is a monic polynomial whose degree is the same as that of $F^{d\boldsymbol{P}}(q)$. If we show that the degrees of $p(\boldsymbol{P}, \mathcal{R}\boldsymbol{P}; q)$ and $F^{\boldsymbol{P}}(q)$ are the same, then all but $O(q^{-1})$ flags $f$ with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ have $\operatorname{ftype}_{\hat{0}} Cf = \mathcal{R}\boldsymbol{P}$, so $\mathcal{R}$ gives the generic value.

Consider the quotient

$$\frac{p(\boldsymbol{P}, \mathcal{R}\boldsymbol{P}; q)}{p(d\boldsymbol{P}, d\mathcal{R}\boldsymbol{P}; q)} = \frac{g^{\lambda^{(h)}}_{\mu^{(h)},(m(h))'(q)}}{g^{\lambda^{(h-1)}}_{\mu^{(h-1)},(m(h-1))'}(q)} \cdot T(\mu^{(h-1)}, \mu^{(h)}, \lambda^{(h-1)}, \lambda^{(h)})$$

We must show the degree of the ratio is $n(\operatorname{osh} \boldsymbol{P} / \operatorname{osh} d\boldsymbol{P}) = n(\delta/\gamma)$. We evaluate the difference in degrees of the Hall polynomials, and the degree of $T$, for cases (1), (2), and (4) of the preceding lemma. Note that the Hall polynomial $g^{\lambda}_{\mu,\nu}(q)$ has degree $n(\lambda) - n(\mu) - n(\nu)$, and in this case, $\nu$ is always a single column.

In (1), the ratio equals 1 and also $F^{\boldsymbol{P}}(q)/F^{d\boldsymbol{P}}(q) = 1$, so $\mathcal{R}$ is generic.

In (2), the ratio of Hall polynomials has degree $\delta'_r - |\delta/\gamma|$, and the factor $T$ has degree $\operatorname{vlen}(\delta, \beta, r-1) - 1$. When configuration (2) arises in $\mathcal{R}$, the vertical strip $\delta/\beta$ is wholly contained in columns $r, r+1, \ldots$. Thus, $\operatorname{vlen}(\delta, \beta, r-1) = |\delta/\beta|$, so the total ratio has degree $\delta'_r - 1 = n(\delta/\gamma)$, as required.

In (4), the ratio of Hall polynomials has degree $\delta'_r - \beta'_s$, and the factor $T$ has degree $\operatorname{vlen}(\delta, \beta, r-1) - 1 + \operatorname{vlen}'(\delta, \beta, s-1)$. When configuration (4) arises in $\mathcal{R}$ (and $\mathcal{L}$), the vertical strip $\delta/\beta$ has no cells in columns $s+1, s+2, \ldots, r-1$, so rows $\beta'_s + 1, \beta'_s + 2, \ldots, \delta'_s$ have length at least $s-1$ and no cells of the vertical strip. Thus, $\operatorname{vlen}'(\delta, \beta, r-1) = \operatorname{vlen}'(\delta, \beta, s-1) - (\delta'_s - \beta'_s)$, so the factor $T$ has degree

$$(\operatorname{vlen}'(\delta, \beta, s-1) + \operatorname{vlen}(\delta, \beta, s-1)) - 1 - \delta'_s + \beta'_s = \delta'_s - 1 - \delta'_s + \beta'_s = \beta'_s - 1.$$

The degree of the total ratio is $(\delta'_r - \beta'_s) + (\beta'_s - 1) = \delta'_r - 1 = n(\delta/\gamma)$, as required.

Now we show that $\mathcal{R}$ is uniform: we show that all but $O(q^{-1})$ of the flags $g$ with $\operatorname{ftype}_{\hat{0}} g = \mathcal{R}\boldsymbol{P}$ and $g_h \leq_{\mathrm{e}} \hat{1}$ have $q^{n(\operatorname{sh}\boldsymbol{P})}(1 + O(q^{-1}))$ inverses $f$ with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ and $g = Cf$. Begin with any $g_h \leq_{\mathrm{e}} \hat{1}$ with $\operatorname{type} g_h = \mu^{(h)}$. Suppose we have chosen $g_h, \ldots, g_k$ and $f_h, \ldots, f_k$, and we want to choose $g_{k-1}$ and $f_{k-1}$. In cases (1) and

(2), we set $g_{k-1} := g_k$ and choose any of $T(\alpha, \beta, \gamma, \delta)$ values for $f_{k-1}$; this count is independent of $g$. In case (4), we saw that the degree of $T$ in the generic game $\mathcal{R}$ is $\beta'_s - 1$, which is the same as the degree of the number of lower covers of $g_k$ of type $\alpha$. Thus, all but a fraction $O(q^{-1})$ of partial flags $g$ can be extended so that the inverse $f$ has an extension. For each extension of $g$, the extension of $f$ is unique: $f_{k-1} := A_{f_k} g_{k-1}$. Thus, by case (4), only a fraction $O(q^{-1})$ of $g$ with $\operatorname{ftype}_{\hat{0}} g = \mathcal{R}\boldsymbol{P}$ are not invertible (cases (1) and (2) impose no restrictions), and by case (2), there are $q^{n(\operatorname{sh}\boldsymbol{P})}(1 + O(q^{-1}))$ inverses of each (cases (1) and (4) yield unique choices of $f_{k-1}$ while (2) contributes many choices).

The proof for the game $\widetilde{\mathcal{L}}$ is similar. The roles of $f$ and $g$ are reversed from the lemma; we denote the configurations (1′), etc., to indicate that all $f$'s and $g$'s in the lemma should be swapped. We set $(\alpha, \beta, \gamma, \delta) = (\lambda^{(k-1)}, \lambda^{(k)}, \mu^{(k-1)}, \mu^{(k)})$ so that the partitions agree with those in the lemma. For pairs of flags $(f, g)$ with $g = Af$, we have $f_h = g_h$, and $g_k = A_{f_h} f_k = Af_k \wedge f_h = Af_k \wedge g_h$, so $g_{k-1} = Af_{k-1} \wedge g_h = Af_{k-1} \wedge (Af_k \wedge g_h) = Af_{k-1} \wedge g_k$. So a local part of $(f, g)$ has form (1′), (3′), or (4′), whence $\operatorname{ftype}_{\hat{0}} Af = \widetilde{\mathcal{L}}(\operatorname{ftype}_{\hat{0}} f)$.

We compute the number of flags $f$ with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ and $\operatorname{ftype}_{\hat{0}} Af = \boldsymbol{Q}$ as follows. The flag $g$ will equal $Af$ during the construction. Let $f_h := g_h := \hat{1}$ in one way. Then for each $k = h, h-1, \ldots, l+1$, choose $g_{k-1}$ and $f_{k-1}$ in one of $T(\lambda^{(k-1)}, \lambda^{(k)}, \mu^{(k-1)}, \mu^{(k)})$ ways; multiply all the values of $T$ together to obtain the total number $E(\boldsymbol{Q}, \boldsymbol{P}; q)$.

We show that $\mathcal{L}$ is the generic value of $\widetilde{\mathcal{L}}$. We must show for each $k$ that the degree of the polynomial $T$ equals $n(\beta/\alpha)$, the degree of the total number of weak lower covers of $f_k$ of type $\alpha$. In (1′) there is one choice of $(f_{k-1}, g_{k-1})$ and one weak lower cover of $f_k$ of type $\alpha$. In (4′), the generic value of the number of choices of $(f_{k-1}, g_{k-1})$ has degree $\beta'_s - 1$, by the same proof used for $\mathcal{R}$ above; this is also the degree of the number of lower covers of $f_k$ of type $\alpha$. When configuration (3′) arises in the game $\mathcal{L}$, the vertical strip $\delta/\beta$ is wholly contained in columns $1, \ldots, s$, so the degree of $T$ is $N_2(\gamma, \beta, s-1) - 1 = \beta'_s - 1$, which again is the degree of the number of lower covers of $f_k$ of type $\alpha$. Thus, the generic value of $\widetilde{\mathcal{L}}$ is $\mathcal{L}$.

Now we show that the generic value is uniform. Let $\boldsymbol{P}$ be a tableau of ordinary shape. Since $Af = (C(f^*))^*$ for all flags $f$, the generic value of $\operatorname{ftype}((C(f^*))^*)$ is a composition $\operatorname{ev}(\mathcal{C}(\operatorname{ev}(\boldsymbol{P})))$ of uniform games. Thus, $\mathcal{L}\boldsymbol{P} = \operatorname{ev}(\mathcal{C}(\operatorname{ev}(\boldsymbol{P})))$, and this is uniform. When $\boldsymbol{P}$ is skew with inner shape of size $N$, fill in the inner shape of $\boldsymbol{P}$ with a standard tableau on $l-N+1, l-N+2, \ldots, l$ to obtain a tableau $\boldsymbol{R}$ of ordinary shape. We have $\boldsymbol{P} = \partial^N \boldsymbol{R}$. Also, $\mathcal{L}\boldsymbol{P} = \partial^N \mathcal{L}\boldsymbol{R}$, and this composition of uniform games is uniform. □

**Corollary 8.10.** *For all multisaturated flags $f$ in a semi-primary lattice,* $\operatorname{ftype} Af = \mathcal{C}\widetilde{\mathcal{L}}(\operatorname{ftype} f)$, *and uniformly,* $\operatorname{ftype} Af = \mathcal{C}\mathcal{L}(\operatorname{ftype} f)$.

### 8.3. Relation of vertical strip games to evacuation and *jeu de taquin*.

**Theorem 8.11.** *Let $\boldsymbol{P}$ be a standard tableau of shape $\lambda \vdash n$ on entries $1, \ldots, n$. Form a tableau $\boldsymbol{Q}$ as follows: for each $r > 0$, take the entries that vanish in the game $\mathcal{L}^{r-1}\boldsymbol{P} \to \mathcal{L}^r\boldsymbol{P}$, complement them by subtracting each from $n+1$, and place the complements in column $r$ of $\boldsymbol{Q}$ in increasing order from top to bottom. Then $\boldsymbol{Q} = \operatorname{ev} \boldsymbol{P}$.*

*Proof.* We can compute the generic cotype of flags $f$ with ftype $f = \boldsymbol{P}$ in two ways. On the one hand, by Theorem 5.14, it is $\operatorname{ev}\boldsymbol{P}$. Alternately, the differences in successive ranks of $f_k, Af_k, A^2 f_k, \ldots,$ are the parts of Atype$[f_k, f_n]$. We evaluate the generic values of these ranks over all $k$ by the playing games $\boldsymbol{P}, \mathcal{L}\boldsymbol{P}, \mathcal{L}^2\boldsymbol{P}, \ldots$. If $k$ is deleted in the transition $\mathcal{L}^{r-1}\boldsymbol{P} \to \mathcal{L}^r\boldsymbol{P}$, then generically, $A^{r-1}f_k > A^{r-1}f_{k-1}$ and $A^r f_k = A^{r-1}f_{k-1}$, so the cotypes of $f_{k-1}$ and $f_k$ differ in column $r$. Thus, the complement $n+1-k$ to $k-1$ is in column $r$ of the cotype tableau. This determines the columns of all entries in the cotype tableau, and sorting the entries in each column determines their positions. $\square$

**Example 8.12.**

$$\boldsymbol{P} = \begin{array}{|c|c|c|}\hline 1&4&6\\ \hline 2&5\\ \cline{1-2} 3&7\\ \cline{1-2} 8\\ \cline{1-1}\end{array} \qquad \mathcal{L}\boldsymbol{P} = \begin{array}{|c|c|c|}\hline \bullet&2&4\\ \hline \bullet&3\\ \cline{1-2} \bullet&5\\ \cline{1-2} \bullet\\ \cline{1-1}\end{array} \qquad \mathcal{L}^2\boldsymbol{P} = \begin{array}{|c|c|c|}\hline \bullet&\bullet&3\\ \hline \bullet&\bullet\\ \cline{1-2} \bullet&\bullet\\ \cline{1-2} \bullet\\ \cline{1-1}\end{array} \qquad \mathcal{L}^3\boldsymbol{P} = \begin{array}{|c|c|c|}\hline \bullet&\bullet&\bullet\\ \hline \bullet&\bullet\\ \cline{1-2} \bullet&\bullet\\ \cline{1-2} \bullet\\ \cline{1-1}\end{array}$$

The entries that disappear in the first step are 1,6,7,8; in the second step, 2,4,5; and in the third step, 3. Subtracting them from 9, the first column of $\operatorname{ev}\boldsymbol{P}$ has 8,3,2,1; the second, 7,5,4; and the third, 6. So

$$\operatorname{ev}\boldsymbol{P} = \begin{array}{|c|c|c|}\hline 1&4&6\\ \hline 2&5\\ \cline{1-2} 3&7\\ \cline{1-2} 8\\ \cline{1-1}\end{array}\ .$$

**Theorem 8.13.** *Let $\boldsymbol{P}$ be a skew tableau on distinct entries. Form a Young tableau $\boldsymbol{Q}$ whose entries in the $r$th column are the entries that vanish in the game $\mathcal{R}^{r-1}\boldsymbol{P} \to \mathcal{R}^r\boldsymbol{P}$. Then $\boldsymbol{Q} = j(\boldsymbol{P})$.*

*Proof.* We can compute the generic value of ftype $f$ over flags with ftype$_{\hat{0}}\, f = \boldsymbol{P}$ in two ways. On the one hand, by Theorem 6.3, it is $j(\boldsymbol{P})$. Alternately, the differences in successive ranks of $f_k, C_{f_l}f_k, C_{f_l}^2 f_k, \ldots,$ are the parts of Ctype$[f_l, f_k]$. We evaluate the generic values of these ranks over all $k$ by the playing games $\boldsymbol{P}, \mathcal{R}\boldsymbol{P}, \mathcal{R}^2\boldsymbol{P}, \ldots$. If $k$ is deleted in the transition $\mathcal{R}^{r-1}\boldsymbol{P} \to \mathcal{R}^r\boldsymbol{P}$, then generically, $C_{f_l}^{r-1}f_k > C_{f_l}^{r-1}f_{k-1}$ and $C_{f_l}^r f_k = C_{f_l}^{r-1}f_{k-1}$, so type$[f_l, f_{k-1}]$ and type$[f_l, f_k]$ differ in column $r$. Thus, $k$ is in column $r$ of ftype $f$. This determines the columns of all entries in ftype $f$, and sorting the entries in each column determines their positions. $\square$

**Example 8.14.**

$$\boldsymbol{P} = \begin{array}{|c|c|c|}\hline \bullet&\bullet&3\\ \hline \bullet&\bullet&4\\ \hline \bullet&1&5\\ \hline 2\\ \cline{1-1} 6\\ \cline{1-1}\end{array} \qquad \mathcal{R}\boldsymbol{P} = \begin{array}{|c|c|}\hline \bullet&\bullet\\ \hline \bullet&\bullet\\ \hline \bullet&3\\ \hline 4\\ \cline{1-1}\end{array} \qquad \mathcal{R}^2\boldsymbol{P} = \begin{array}{|c|c|}\hline \bullet&\bullet\\ \hline \bullet&\bullet\\ \hline \bullet\\ \cline{1-1}\end{array} \qquad j(\boldsymbol{P}) = \begin{array}{|c|c|}\hline 1&3\\ \hline 2&4\\ \hline 5\\ \cline{1-1} 6\\ \cline{1-1}\end{array}$$

**8.4.** ***A*-statistics of flags.** Let $f = (f_0 \lessdot \cdots \lessdot f_n)$ be a saturated flag in a modular lattice and $\vec{r} = (r_0, \ldots, r_n)$ be a sequence of nonnegative integers. Define

$$\begin{aligned} A^{\vec{r}} f &\stackrel{\text{def}}{=} A^{r_0} f_0 \wedge A^{r_1} f_1 \wedge \cdots \wedge A^{r_n} f_n \\ \rho^{\vec{r}}(f) &\stackrel{\text{def}}{=} \rho(A^{\vec{r}} f) \end{aligned}$$

If some $A^{r_i} f_i$ with $i < n$ is omitted from the expression for $A^{\vec{r}} f$, an equivalent expression containing it can be formed by inserting $A^{r_n + n - i} f_i$ because $A^{n-i} f_i \geq f_n$. If the lattice has finite length and the term $A^{r_n} f_n$ is missing, we may insert $A^{\operatorname{arank} \mathcal{L}} f_n$ because that quantity equals $\hat{1}$.

**Definition 8.15.** For an arbitrary modular lattice, $\vec{r} \in \mathbb{N}^{n+1}$ is **reduced** when $r_0 \geq r_1 \geq \cdots r_n$. For a modular lattice of finite height, we further impose the conditions $r_n = 0$ and $r_0 \leq \operatorname{arank} \mathcal{L}$.

Suppose $\vec{r}$ is not reduced. If $r_i < r_{i+1}$ then $A^{r_i} f_i \wedge A^{r_{i+1}} f_{i+1} = A^{r_i} f_i = A^{r_i} f_i \wedge A^{r_i} f_{i+1}$, so we can replace $r_{i+1}$ by $r_i$. If $r_i - r_{i+1} > 1$, then since $A f_i \geq f_{i+1}$, we see that $A^{r_i} f_i \wedge A^{r_{i+1}} f_{i+1} = A^{r_{i+1}} f_{i+1} = A^{r_{i+1}+1} f_i \wedge A^{r_{i+1}} f_{i+1}$. Substitutions of these forms always lower exponents, so repeated substitutions of these forms eventually terminate in a reduced sequence $\vec{r}^*$ with $A^{\vec{r}} f = A^{\vec{r}^*} f$ for all flags $f$. Explicitly,

$$r_i^* \stackrel{\text{def}}{=} \min_{k=i}^{n} \left( k - i + \min_{j=0}^{k} r_j \right).$$

If we are only considering maximum length saturated flags with $f_0 = \hat{0}$ and $f_1 = \hat{1}$, then $A^{r_n} \hat{1} = \hat{1}$ for all $r_n$, so we can take $r_n$ to be 0, while $A^{r_0} \hat{0} = \hat{1}$ iff $r_0 \geq \operatorname{arank} \mathcal{L}$, so we can take $r_0 \leq \operatorname{arank} \mathcal{L}$.

Now we restrict our attention to semi-primary lattices. The reduced sequences for a semi-primary lattice of type $\lambda \vdash n$ have the form $0 = r_n \leq r_{n-1} \leq \cdots \leq r_0 \leq \lambda_1$, and there are $\sum_{k=0}^{\lambda_1} \binom{n}{k}$ of them. The quantities $\rho^{\vec{r}}(f)$ are interesting because they generalize the notions of Littlewood-Richardson sequences and interval type tables, and an extension of it generalizes relative positions. It generalizes the notion of Littlewood-Richardson sequences, since $\mathrm{ALR}_{[f_i, f_k]}(f_j) = ((\operatorname{type}[f_i, A^r_{f_k} f_j])')_{r \geq 0}$ and in turn,

$$\operatorname{type}[f_i, A^r_{f_k} f_j] = (\rho(A^{s-1} f_i \wedge A^r f_j \wedge f_k, A^s f_i \wedge A^r f_j \wedge f_k))_{s \geq 1},$$

which is derivable from such quantities. It also generalizes the notion of interval type tables, since

$$\operatorname{type}[f_i, f_j] = \rho(A^{s-1} f_i \wedge f_j, A^s f_i \wedge f_j)$$

is derivable from such quantities.

**Definition 8.16.**

$$\begin{aligned} R_i(\vec{r}) &\stackrel{\text{def}}{=} (r_0, \ldots, r_{i-1}, r_i + 1, r_{i+1}, \ldots, r_n) \\ R(\vec{r}) &\stackrel{\text{def}}{=} (r_0 + 1, \ldots, r_n + 1) \\ L_i(\vec{r}) &\stackrel{\text{def}}{=} (r_0, \ldots, r_{i-1}, r_i - 1, r_{i+1}, \ldots, r_n) \end{aligned}$$

**Proposition 8.17.** *The following are necessary conditions for a specification of $\rho^{\vec{r}}$'s to be realizable in a semi-primary lattice. For all $\vec{r} \in \mathbb{N}^{n+1}$,*

(1) $\rho^{\vec{r}} = \rho^{\vec{r}^*}$.
(2) $\rho^{\vec{r}} \leq \rho^{R_i(\vec{r})}$ *when* $r_{i-1} > r_i = r_{i+1}$, *for* $0 < i < n$;
(3) $\rho^{R(R_i(\vec{r}))} - \rho^{R(\vec{r})} \leq \rho^{R_i(\vec{r})} - \rho^{\vec{r}}$, *for* $0 \leq i \leq n$;
(4) $\rho^{R_i(\vec{r})} - \rho^{\vec{r}} \geq \rho^{R_iR_j(\vec{r})} - \rho^{R_j(\vec{r})}$, *for* $0 \leq i, j \leq n$ *and* $i \neq j$.

*Conditions (2)–(4) only need to be checked for reduced* $\vec{r}$.

*Proof.*

(1) This was shown when $\vec{r}^*$ was constructed.

(2) $A^{\vec{r}} f$ and $A^{R_i(\vec{r})} f$ differ by replacing $A^{r_i} f_i$ in the former with $A^{r_i+1} f_i$ in the latter. Let $r = r_{i-1}$. We have $A^r f_{i-1} \leq A^r f_i$ and $A^r f_{i-1} \geq A^{r-1} f_i$. Therefore,

$$\begin{aligned} A^r f_{i-1} \wedge A^{r-1} f_i \wedge A^{r-1} f_{i+1} &= A^{r-1} f_i = A^r f_{i-1} \wedge A^r f_i \wedge A^{r-1} f_i \\ &\leq A^r f_{i-1} \wedge A^{r-1} f_i \wedge A^{r-1} f_{i+1}. \end{aligned}$$

The initial expression is three terms of $A^{R_i(\vec{r})} f$, and the final expression is the replacement terms in $A^{\vec{r}} f$. Separately meet these with $A^{R_i(\vec{r})} f$ to obtain $A^{\vec{r}} f \leq A^{R_i(\vec{r})} f$.

(3)–(4) By Theorem 4.19, the lattice polynomials $At$ and $t \wedge A^{r_j} f_j$ in $t$ shrink interval lengths. Apply the former to the interval $[A^{\vec{r}} f, A^{R_i(\vec{r})} f]$ to obtain a weakly shorter interval $[A^{R(\vec{r})} f, A^{RR_i(\vec{r})}]$. Apply the latter to the interval $[A^{R_iR_j(\vec{r})} f, A^{R_j(\vec{r})} f]$ to obtain a weakly shorter interval $[A^{R_i(\vec{r})} f, A^{\vec{r}} f]$. ☐

**Definition 8.18.** For a flag $f = (f_l, \ldots, f_h)$ and $l \leq k \leq h$, let

$$A_k f \stackrel{\text{def}}{=} Ad^{h-k} = (A_{f_k} f_l, \ldots, A_{f_k} f_k).$$

For a tableau $\boldsymbol{P}$ and $\operatorname{low}(\boldsymbol{P}) \leq k \leq \operatorname{high}(\boldsymbol{P}) = h$, define $\widetilde{\mathcal{L}}_k \boldsymbol{P} = \widetilde{\mathcal{L}} d^{h-k} \boldsymbol{P}$, and similarly define $\mathcal{L}_k$.

**Proposition 8.19.** *Let* $\vec{r}$ *be a reduced sequence. Let* $s_i = \min\{\, j : r_j = i \,\}$ *for* $i = 0, \ldots, k$ *where* $k = r_0$. *Then* $\rho^{\vec{r}}(f) = \left|\widetilde{\mathcal{L}}_{s_k} \cdots \widetilde{\mathcal{L}}_{s_1} \widetilde{\mathcal{L}}_{s_0} \boldsymbol{P}\right|$ *for some possible outcome of the games* $\widetilde{\mathcal{L}}$.

*Proof.* Consider

$$\begin{aligned} A_{s_0} f &= (A f_j \wedge f_{s_0})_{0 \leq j \leq s_0} \\ &\phantom{=}\ \text{ending in } A f_{s_0} \wedge f_{s_0} = f_{s_0}; \\ A_{s_1} A_{s_0} f &= (A(A f_j \wedge f_{s_0}) \wedge (A f_{s_1} \wedge f_{s_0}))_{0 \leq j \leq s_1} \\ &= (A^2 f_j \wedge A f_{s_1} \wedge f_{s_0})_{0 \leq j \leq s_1} \\ &\phantom{=}\ \text{ending in } A^2 f_{s_1} \wedge A f_{s_1} \wedge f_{s_0} = A f_{s_1} \wedge f_{s_0} \\ &\ \vdots \\ A_{s_m} \cdots A_{s_0} f &= (A^m f_j \wedge A^{m-1} f_{s_{m-1}} \wedge \cdots \wedge f_{s_0})_{0 \leq j \leq f_{s_m}} \\ &\phantom{=}\ \text{ending in } A^m f_{s_m} \wedge \cdots \wedge f_{s_0} \\ &\ \vdots \\ A_{s_k} \cdots A_{s_0} f &= (A^k f_0 \wedge A^{k-1} f_{s_{k-1}} \wedge \cdots \wedge f_{s_0}) \\ &\phantom{=}\ \text{because } s_k = 0. \end{aligned}$$

The possible types are obtained by replacing each $A_{s_i}$ with $\widetilde{\mathcal{L}}_{s_i}$. Ultimately, a tableau with one partition $(\mu)$ is obtained, and and the required rank is $|\mu|$. $\square$

**Conjecture 8.20.** *Let $\boldsymbol{P}$ be a standard tableau. Let $\rho_g^{\vec{r}}$ be the generic ranks for flags of type $\boldsymbol{P}$, obtained by replacing $\widetilde{\mathcal{L}}$ with $\mathcal{L}$ in Proposition* 8.19. *Then $\rho_g^{\vec{r}} \le \rho^{\vec{r}}(f)$ for all $\vec{r}$ and flags $f$.*

Applied to interval type tables, the following corollary to the conjecture empirically holds for all the interval type tables in Appendix A.

**Definition 8.21.** Let $\lambda$ and $\mu$ be partitions. Then $\lambda$ **(row) dominates** $\mu$, written $\lambda \trianglerighteq \mu$, iff $\lambda_1 + \cdots + \lambda_k \ge \mu_1 + \cdots + \mu_k$ for all $k > 0$. Also, $\lambda$ column dominates $\mu$, written $\lambda \trianglerighteq' \mu$, when $\lambda' \trianglerighteq \mu'$. These differ from the usual definition of dominance in which $|\lambda| = |\mu|$ is a requirement; in the usual definition, $\lambda \trianglerighteq \mu$ iff $\mu \trianglerighteq' \lambda$, but this does not hold in the extended definition.

Let $\boldsymbol{P} = (\lambda^{(l)}, \dots, \lambda^{(h)})$ and $\boldsymbol{Q} = (\mu^{(l)}, \dots, \mu^{(h)})$ be tableaux. Then $\boldsymbol{P} \trianglerighteq \boldsymbol{Q}$ iff $\lambda^{(k)} \trianglerighteq \mu^{(k)}$ for $k = l, \dots, h$, and similarly for $\trianglerighteq'$.

**Corollary 8.22.** *Let $f = (f_0, \dots, f_n)$ be a flag with* $\operatorname{ftype} f = \boldsymbol{P}$. *Then* $\operatorname{ftype} \partial^i f \trianglerighteq' \Delta^i \boldsymbol{P}$ *for $0 \le i \le n$.*

*Proof, given the conjecture.* Since $\operatorname{ftype} \partial^i f = (\operatorname{type}[f_i, f_j])_{j=i}^n$, the number of squares in the first $k$ columns of the $j$th partition is $\rho(f_i, A^k f_i \wedge f_j) = \rho(A^k f_i \wedge f_j) - i$. If the conjecture holds, the generic value of $\rho(A^k f_i \wedge f_j)$ is weakly lower than the one this $f$ realizes. The generic value is the number of squares in the first $k$ columns of the $j$th partition in $\Delta^i \boldsymbol{P}$, so the realized value dominates the generic value. $\square$

**Example 8.23.** Note that $\widetilde{\Delta}^i \boldsymbol{P} \trianglerighteq' \Delta^i \boldsymbol{P}$ does not always hold, Consider

$$\boldsymbol{P} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} \qquad \Delta \boldsymbol{P} = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \quad \Delta^2 \boldsymbol{P} = \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline\end{array}$$

$$\widetilde{\Delta} \boldsymbol{P} = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \widetilde{\Delta}^2 \boldsymbol{P} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline\end{array}$$

The shapes of $\widetilde{\Delta}^2 \boldsymbol{P}$ and $\Delta^2 \boldsymbol{P}$ are $\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}$ and $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$, and $\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} \not\trianglerighteq' \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$. However, no flag with $\operatorname{ftype} f = \boldsymbol{P}$ has $\operatorname{ftype} \partial^2 f = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline\end{array}$ because

$$\boldsymbol{P} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} + \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline\end{array}$$

so by Theorem 4.76, $\operatorname{ftype} \partial^2 f = \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline\end{array}$.

**8.5. Counting the number of flags with given $A$-statistics.** The method developed in Section 5.5 for enumerating the number of flags realizing an interval type table generalizes to counting the number of flags realizing some specification of the $\rho^{\vec{r}}$'s. Let $\Lambda = (\rho^{\vec{r}})_{\vec{r}}$ be a collection of numbers satisfying Proposition 8.17, where the indicies run over reduced sequences $\vec{r}$. The **$A$-statistics** of a flag $f$ are $\Lambda(f) = (\rho^{\vec{r}}(f))_{\vec{r}}$.

Fix $k$ with $0 < k < n$. Consider two $k$-adjacent flags $f, f'$. Let $\vec{r}$ be a reduced sequence. If $r_k > r_{k+1}$ then

$$A^{r_k} f_k \wedge A^{r_{k+1}} f_{k+1} = A^{r_{k+1}} f_{k+1} = A^{r_k} f'_k \wedge A^{r_{k+1}} f'_{k+1}$$

so $A^{\vec{r}}f = A^{\vec{r}}f'$ and $\rho^{\vec{r}}(f) = \rho^{\vec{r}}(f')$. If $r_{k-1} = r_k$ then

$$A^{r_{k-1}}f_{k-1} \wedge A^{r_k}f_k = A^{r_{k-1}}f_{k-1} = A^{r_{k-1}}f'_{k-1} \wedge A^{r_k}f'_k.$$

so again, $A^{\vec{r}}f = A^{\vec{r}}f'$ and $\rho^{\vec{r}}(f) = \rho^{\vec{r}}(f')$. Thus, we say $\vec{r}$ is $k$-**free** when $r_{k-1} = r_k$ or $r_k > r_{k+1}$, and is $k$-**dependent** when $r_{k-1} > r_k = r_{k+1}$. When $\vec{r}$ is $k$-free, we may substitute for $f_k$ any atom of $[f_{k-1}, f_{k+1}]$ without changing the value of $A^{\vec{r}}f$. In particular, when $\vec{r}$ is reduced and $k$-dependent, both $R_k(\vec{r})$ and $(L_{k-1}(\vec{r}))^*$ are $k$-free, so $A^{R_k(\vec{r})}f$ and $A^{L_{k-1}(\vec{r})}f$ do not depend on $f_k$.

Consider $k$-dependent $\vec{r}$. We have $A^{L_{k-1}(\vec{r})}f \lessdot A^{\vec{r}}f \lessdot A^{R_k(\vec{r})}f$: the first cover is obtained by meeting $A^{r_k}f_{k-1} \lessdot A^{r_k}f_k$ with $A^{\vec{r}}f$, and the second by Proposition 8.17. The left and right expressions have no dependence on $f_k$, so they are the same if we substitute $f'$ for $f$. If $\rho^{L_{k-1}(\vec{r})} = \rho^{R_k(\vec{r})}$ then both weak covers are equality, so $\rho^{\vec{r}}(f) = \rho^{\vec{r}}(f') = \rho^{L_{k-1}(\vec{r})}$. If $\rho^{R_k(\vec{r})} - \rho^{L_{k-1}}(\vec{r}) = 2$ then $\rho^{\vec{r}}(f) = \rho^{\vec{r}}(f') > \rho^{L_{k-1}(\vec{r})}$. We say that $\Lambda$ is $\vec{r}$-**specializable** when $\rho^{R_k(\vec{r})} - \rho^{L_{k-1}}(\vec{r}) = 1$, that it is $\vec{r}$-**special** when $\rho^{\vec{r}} = \rho^{R_k(\vec{r})}$, and that it is $\vec{r}$-**generic** when $\rho^{\vec{r}} = \rho^{L_{k-1}(\vec{r})}$. A flag is $\vec{r}$-specializable, $\vec{r}$-special, or $\vec{r}$-generic when $\Lambda(f)$ is.

Let $\rho^{\vec{r}}$ be specializable. If $f \neq f'$ are $k$-adjacent, they cannot both be $\rho^{\vec{r}}$-special, for suppose they are. Then $A^{\vec{r}}f = A^{\vec{r}}f' = A^{R_k(\vec{r})}f$, but

$$\begin{aligned} A^{\vec{r}}f \wedge A^{\vec{r}}f' &= A^{R_k(\vec{r})}f \wedge A^{r_k}f_k \wedge A^{r_k}f'_k \\ &= A^{R_k(\vec{r})}f \wedge A^{r_k}(f_k \wedge f'_k) \\ &= A^{R_k(\vec{r})}f \wedge A^{r_k}f_{k-1} \\ &= A^{L_{k-1}(\vec{r})}f \\ &\lessdot A^{R_k(\vec{r})}f, \end{aligned}$$

a contradiction. So when an adjacency class is $\vec{r}$-specializable for a particular $\vec{r}$, at most one flag in it is $\vec{r}$-special.

In fact, exactly one flag is. Let $\vec{r}$ be reduced and $k$-dependent, and $f$ be $\vec{r}$-specializable. Let $f'$ be the $k$-adjacent flag with $f'_k = f_{k-1} \vee C^{r_k}A^{R_k(\vec{r})}f$. We must verify that $f'_k$ is an atom of $[f_{k-1}, f_{k+1}]$ and that $\rho(A^{\vec{r}}f') = \rho(A^{R_k(\vec{r})}f')$. Clearly $f'_k \geq f_{k-1}$. Let $r = r_{k+1} = r_k < r_{k-1}$. Then $C^{r_k}A^{R_k(\vec{r})}f \leq C^r A^r f_{k+1} \leq f_{k+1}$ so $f'_k \leq f_{k-1} \vee f_{k+1} = f_{k+1}$. Thus $f_{k-1} \leq f'_k \leq f_{k+1}$. Since

$$\begin{aligned} \rho(f'_k) &= \rho(f_{k-1}) + \rho(A^{R_k(\vec{r})}f) - \rho(A^r f_{k-1} \wedge A^{R_k(\vec{r})}f) \\ &= \rho(f_{k-1}) + \Big(\rho(A^{R_k(\vec{r})}f) - \rho(A^{L_{k-1}(\vec{r})}f)\Big) \\ &= \rho(f_{k-1}) + 1, \end{aligned}$$

$f_{k-1}$ is an atom of $[f_{k-1}, f_{k+1}]$. Finally, let

$$\begin{aligned} y &= A^{R_k(\vec{r})}f' \\ z &= A^{\vec{r}}f' = y \wedge A^{r_k}f'_k = y \wedge A^{r_k}(f_{k-1} \vee C^{r_k}y). \end{aligned}$$

Then $y \leq A^{r_k}C^{r_k}y \leq A^{r_k}(f_{k-1} \vee C^{r_k}y)$ so $z = y$. Thus, $f'$ is $\vec{r}$-special.

We have built up a generalization of the framework designed for interval type tables in Section 5.5. A similar technique could be applied to enumerate the number $N_\Lambda$ of flags realizing each $A$-statistic $\Lambda$. List all $\Lambda$'s satisfying Proposition 8.17 and split them into $k$-adjacency classes over all $k$. If a class lacks an $\vec{r}$-generic or an $\vec{r}$-special $\Lambda$ for some specializable $\vec{r}$, delete all members of that class from the list. A class

$\{\Lambda_0, \ldots, \Lambda_m\}$ in which $\Lambda_0$ is $\vec{r}$-generic for all specializable $\vec{r}$, and with exactly one $\Lambda_i$ that's $\vec{r}$-special for each specializable $\vec{r}$, yields the equation $N_{\Lambda_0}/(q+1-m) = N_{\Lambda_1} = \cdots = N_{\Lambda_i}$. It would be interesting to pursue this further. If the equations turn out to be solvable, it would provide a refinement of the polynomials that enumerate how many flags realize specified interval type tables.

The theory of hereditary flags in Section 5.6 computed the $A$-statistics of hereditary flags, and then only used the portion of those statistics necessary to construct the interval type table. For hereditary flags of composition tableau type $\boldsymbol{P}$, the $A$-statistics are given by substituting the operator $\mathcal{A}$ of that section for $\widetilde{\mathcal{L}}$ in Proposition 8.19. Let $\Lambda(\boldsymbol{P})$ be those statistics. In Theorem 5.52, it is shown that $f_{\mathrm{itype}\,\boldsymbol{P}}(q) = q^{2n(\lambda)-\mathrm{dom}(\boldsymbol{P})} M_\lambda(q^{-1})$; in fact, this also is the number of flags with $A$-statistics $\Lambda(\boldsymbol{P})$.

**8.6. Further statistics and future problems.** Consider the $C$-statistics of a flag, given as

$$\rho_{\vec{r}}(f) = \rho(C^{r_0} f_0 \vee \cdots \vee C^{r_n} f_n)$$

where $0 = r_0 \leq r_1 \leq \cdots \leq r_n \leq \lambda_1$ are the reduced sequences in semi-primary lattices of type $\lambda$. Let $C_m(f) = C\partial^{m-\mathrm{low}(f)}(f) = (C_{f_m} f_m, \ldots, C_{f_m} f_h)$ and $\mathcal{R}_m(\boldsymbol{P}) = \mathcal{R}\partial^{m-\mathrm{low}(\boldsymbol{P})}(\boldsymbol{P})$. Let $s_i = \max\{\, j : r_j = i \,\}$ for $i = 0, \ldots, k$, where $k = r_n$. Then for flags of type $\boldsymbol{P}$, we have $\rho_{\vec{r}}(f) = |\mathcal{R}_{s_k} \cdots \mathcal{R}_{s_0}(\boldsymbol{P})|$, analogously to Proposition 8.19. The enumeration theory is similar to that of the $A$-statistics.

We can also consider multiple flags, Combinations of the games $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{D}}_c$ can be used to compute possible values of $\rho(A^{\vec{r}} f \wedge A^{\vec{s}} g \wedge \cdots)$. We illustrate for two and three flags.

First define the operator $A_k^{\#}(f) = (A_{f_k} f_l, A_{f_k} f_{l+1}, \ldots, A_{f_k} f_k, f_{k+1}, \ldots, f_h)$. If $f$ is a multisaturated chain, so is $A_k^{\#}(f)$, because $A_{f_k}$ maintains weak covers in the $l$ through $k$th terms; $A_{f_k} f_k = f_k \leq f_{k+1}$; and the $k+1$ through $h$th terms already form a multisaturated chain. Also, for $\boldsymbol{P} = (\lambda^{(l)}, \ldots, \lambda^{(h)})$ and $\widetilde{\mathcal{L}}_k \boldsymbol{P} = (\mu^{(l)}, \ldots, \mu^{(k)})$, define $\widetilde{\mathcal{L}}_k^{\#} \boldsymbol{P} = (\mu^{(l)}, \ldots, \mu^{(k)}, \lambda^{(k+1)}, \ldots, \lambda^{(h)})$. In terms of tableau, we apply $\widetilde{\mathcal{L}}$ to the subtableau of $\boldsymbol{P}$ whose entries are at most $k$, and leave all the entries larger than $k$ in place. Similarly define $\mathcal{L}_k^{\#}$. Uniformly we have

$$\mathcal{L}_k^{\#}, \widetilde{\mathcal{L}}_k^{\#} : \mathrm{ftype}_{\hat{0}}\, f \longmapsto \mathrm{ftype}_{\hat{0}}\, A_k^{\#}(f).$$

Let $\mathrm{ftype}_{\hat{0}}\, f = \boldsymbol{P}$ and $\mathrm{ftype}_{\hat{0}}\, g = \boldsymbol{Q}$. Let $\vec{r}$ and $\vec{s}$ be reduced sequences and set $r_i' = \min\{\, j : r_j = i \,\}$ and $s_i' = \min\{\, j : s_j = i \,\}$. Let

$$\begin{aligned} f^{\#} &= (A^{r_i} f_i \wedge A^{r_{i+1}} f_{i+1} \wedge \cdots \wedge A^{r_n} f_n) \\ &= \cdots A_{r_1'}^{\#} A_{r_0'}^{\#}(f), \\ \boldsymbol{P}^{\#} &= \cdots \widetilde{\mathcal{L}}_{r_1'}^{\#} \widetilde{\mathcal{L}}_{r_0'}^{\#}(\boldsymbol{P}). \end{aligned}$$

Then $\mathrm{ftype}_{\hat{0}}\, f^{\#} = \boldsymbol{P}^{\#}$, with the uniform type found by replacing $\widetilde{\mathcal{L}}_{r_i'}^{\#}$ by $\mathcal{L}_{r_i'}^{\#}$. Similarly define $g^{\#}$ and $\boldsymbol{Q}^{\#}$.

Next, perform a skew Robinson-Schensted algorithm on $(\boldsymbol{P}^{\#}, \boldsymbol{Q}^{\#})$. Let $k_1 < \cdots < k_m$ be the entries of $\boldsymbol{Q}^{\#}$ and $c_1, \ldots, c_m$ be the cells of $\boldsymbol{Q}^{\#}$ with $k_1, \ldots, k_m$. Then

$\mathrm{ftype}_{\hat{0}}(f^{\#} \wedge A^{\vec{s}}g) = \widetilde{\mathcal{D}}_{c_1} \cdots \widetilde{\mathcal{D}}_{c_m}(\boldsymbol{P}^{\#})$, so the inner shape of this skew tableau is the value of $\mathrm{type}(A^{\vec{r}}f \wedge A^{\vec{s}}g)$.

For three flags $f, g, h$ of types $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}$, and reduced sequences $\vec{r}, \vec{s}, \vec{t}$, define $f^{\#}$, $g^{\#}$, $h^{\#}$ and $\boldsymbol{P}^{\#}, \boldsymbol{Q}^{\#}, \boldsymbol{R}^{\#}$ analogously to the two flag case. Let $k_1 < \cdots < k_m$ be the entries of $\boldsymbol{R}^{\#}$ and $c_1, \ldots, c_m$ their cells. The games $\boldsymbol{P}^{\#\#} = \mathrm{ftype}_{\hat{0}}(f^{\#} \wedge A^{\vec{t}}h) = \widetilde{\mathcal{D}}_{c_1} \cdots \widetilde{\mathcal{D}}_{c_m}(\boldsymbol{P})$ and $\boldsymbol{Q}^{\#\#} = \mathrm{ftype}_{\hat{0}}(g^{\#} \wedge A^{\vec{t}}h) = \widetilde{\mathcal{D}}_{c_1} \cdots \widetilde{\mathcal{D}}_{c_m}(\boldsymbol{Q})$ each result in tableaux of the same shape (the inner shape of $\boldsymbol{R}^{\#}$). Perform the skew Robinson-Schensted deletion algorithm on $(\boldsymbol{P}^{\#\#}, \boldsymbol{Q}^{\#\#})$, and the final $P$-tableau obtained is $\mathrm{ftype}_{\hat{0}}(f^{\#} \wedge A^{\vec{s}}g \wedge A^{\vec{t}}h)$, so its inner shape is $\mathrm{type}(A^{\vec{r}}f \wedge A^{\vec{s}}g \wedge A^{\vec{t}}h)$.

For two flags $f$ and $g$, the $A$-statistics simultaneously encode the relative positions of the flags and their interval type tables. In Section 7, we were not able to enumerate the number of pairs of flags in given relative positions by the techniques used in Section 5.5 for interval type tables, because our statistics were insufficient to determine the types of length 2 intervals. Since $A$-statistics resolve that difficulty, they should be explored further to see if the number of pairs of flags realizing particular $A$-statistics is in fact enumerable, exactly or asymptotically in the generic case.

The following is a generalization of Conjecture 8.20 to multiple flags.

**Conjecture 8.24.** *Let $\boldsymbol{P}, \boldsymbol{Q}, \ldots$ be standard tableaux, and $\vec{r}, \vec{s}, \ldots$ be reduced sequences. Compute the generic values $\rho_{\mathrm{gen}}^{\vec{r},\vec{s},\ldots}$ of $A^{\vec{r}}f \wedge A^{\vec{r}}g \wedge \cdots$ by using $\mathcal{L}$ and $\mathcal{D}_c$ in place of $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{D}}_c$ in the procedure described above. Then for all flags $f, g, \ldots,$ we have*

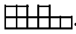
$$\rho_{\mathrm{gen}}^{\vec{r},\vec{s},\ldots} \leq \rho(A^{\vec{r}}f \wedge A^{\vec{s}}g \wedge \cdots).$$

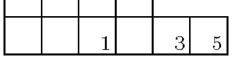
**Corollary 8.25.** *Let $(\boldsymbol{P}, \boldsymbol{Q})$ be a pair of standard tableau of shape $\lambda$. Let $\boldsymbol{P} = \boldsymbol{P}_n, \ldots, \boldsymbol{P}_0 = \emptyset$ be the sequence of tableau obtained in the inverse Robinson-Schensted algorithm. Let $f$ and $g$ be flags with $\mathrm{ftype}_{\hat{0}}\, f = \boldsymbol{P}$ and $\mathrm{ftype}_{\hat{0}}\, g = \boldsymbol{Q}$ in a semi-primary lattice of type $\lambda$. Then $\mathrm{ftype}_{\hat{0}}(f \wedge g_k) \trianglerighteq' \boldsymbol{P}_k$.*

*Conjectural Proof.* We assume the previous conjecture, and assume that compositions of games $\mathcal{D}_c$ give generic values. The $j$th partition in $\mathrm{ftype}(f \wedge g_k)$ is $\mathrm{type}(f_j \wedge g_k)$. The first $i$ columns of it have a total of $\rho(A^i f_0 \wedge f_j \wedge g_k)$ squares. By the conjecture, this is minimized by the generic value, which is the first $i$ columns of the $j$th partition of $\boldsymbol{P}_k$. $\square$

# Part IV. Strongly modular lattices

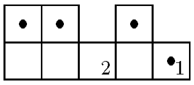
## 9. Strongly modular lattices

**9.1. The Fibonacci lattices $Fib(r)$ and $Z(r)$.** Let $r$ be a positive integer, and $W(r)$ be the set of all finite length words formed from the alphabet $\{1_1, 1_2, \ldots, 1_r, 2\}$, including the empty word $\emptyset$. A digit is "a 1" if it is any $1_i$. When $r = 1$, the alphabet may be abbreviated $\{1, 2\}$. As with partitions, an arbitrary number of 0's may be appended to the end of a word when convenient. Fomin [**5**] introduced a pictorial diagram for words of $W(1)$ that is a rotation of the usual composition diagram; the word 221211 has **snake diagram** . We may expand this to words of $W(r)$ by placing an $i$ in the lower right corner of the cell representing $1_i$, so $221_1 21_3 1_5$ has snake diagram

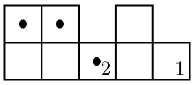



In [**28**], two different partial orders on $W(r)$ are defined by defining the cover relations and then extending to a partial order by transitivity.

In $Fib(r)$, the cover relation $v \gtrdot_F u$ holds when $u$ is obtained from $v$ either by changing a single 2 to a 1 anywhere in $v$, or by deleting the last digit of $v$ if it is a 1. When extended transitively to a full partial order, $v \geq_F u$ if $u$ may be obtained from $v$ by changing some 2's to 1's, and deleting any number of digits from the right end of the word. Thus in $Fib(2)$, the word $221_2 21_1$ covers $1_1 21_2 21_1$, $1_2 21_2 21_1$, $21_1 1_2 21_1$, $21_2 1_2 21_1$, $221_2 1_1 1_1$, $221_2 1_2 1_1$, and $221_2 2$. Pictorially, we may delete any of the marked cells, and if they in the upper row, we may introduce any subscript in the lower cell.




The cover relation $v \gtrdot_Z u$ holds in $Z(r)$ when either of two conditions hold. Either the leftmost 1 of $v$ is deleted to obtain $u$, denoted $v \gtrdot_1 u$, or a 2 in $v$ preceded only by 2's is changed to a 1, denoted $v \gtrdot_2 u$. So in $Z(2)$, we have $221_2 21_1 \gtrdot_1 2221_1$ and $221_2 21_1 \gtrdot_2 1_1 21_2 21_1$, $1_2 21_2 21_1$, $21_1 1_2 21_1$, $21_2 1_2 21_1$. Pictorially, we may delete any marked cell, and when it is a 2, add any subscript to the lower cell.




The total number of upper covers of $u$ is $\kappa^+(u) = (k+1)r$ or $(k+1)r + 1$, where $u = 2^k$ or $2^k 1_i x$, respectively, for arbitrary $k \geq 0$ and $x \in Z(r)$. This is because any of $r$ 1's may be inserted before the first 2 or following any of the first $k$ 2's, or if there is a 1, the first 1 may be changed to a 2. Similarly, the total number of lower covers of $v$ is $\kappa^-(v) = kr$ or $kr + 1$, where $v = 2^k$ or $2^k 1x$, respectively; any one of the first $k$ 2's can replaced by any of $r$ 1's, or the first 1 (if present) can be deleted.

Both $Fib(r)$ and $Z(r)$ are graded lattices with rank function $\rho(a_1 a_2 \cdots a_k) = a_1 + a_2 + \cdots + a_k$, where we add the $a_i$'s as integers by ignoring the subscripts on the 1's. These lattices are called **Fibonacci lattices** because the number of elements of rank $n$ when $r = 1$ is the Fibonacci number $F_{n+1}$ (where $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$).

We will evaluate the operators $A$ and $C$ in $Z(r)$ in Theorems 9.4 and 9.7, and establish in Theorem 9.2 a new criterion for determining whether two elements of $Z(r)$

are comparable under $\geq_Z$ without having to exhibit a chain of elements satisfying the cover relation $\gtrdot_Z$.

**Definition 9.1.** Let $u$ and $v$ be two elements of $Z(r)$ and factor them $u = \mu\tau$ and $v = \nu\tau$, where $\tau$ is as large as possible; that is, one or both of $\mu$ and $\nu$ is empty, or they are both nonempty but end in different digits. Let $\mu_1$, $\mu_2$, and $\mu_{12}$ respectively denote the number of 1's of all kinds in $\mu$; the number of 2's in $\mu$; and the total number of digits in $\mu$, and define similar subscripted quantities for $u, v, \tau, \nu$, and other variables in $Z(r)$ that are introduced later. When dealing with multiple values of $u$ and $v$, all of $\mu, \nu, \tau$ and their subscripted forms may be expressed as functions of $u$ and $v$, so that $u = \mu(u, v)\tau(u, v)$ and $v = \nu(u, v)\tau(u, v)$.

**Theorem 9.2.** *Let $v, u \in Z(r)$. Then $v \geq_Z u$ iff $\nu_2 \geq \mu_{12}$.*

*Proof.* Suppose $\nu_2 \geq \mu_{12}$. First, prepend to $u$ a total of $\nu_2 - \mu_{12}$ 2's by alternately prepending $1_1$ and then changing the $1_1$ to 2. The successive elements obtained increase by alternate cover relations $\gtrdot_1$ and $\gtrdot_2$. So now we have the word $2^{\nu_2-\mu_{12}}\mu\tau$. Next, change the 1's in the $\mu$ part to 2's, sequentially from left to right, to obtain $2^{\nu_2}\tau$; each successive change is a $\gtrdot_2$ relation in $Z(r)$. Finally, by a sequence of $\gtrdot_1$ relations, insert into this $1_i$'s from right to left with the same positions and subscripts as in $\nu$, to obtain $v = \nu\tau$. Thus $v \geq_Z u$.

Conversely, suppose $v \geq_Z u$. Any saturated chain from $u$ to $v$ must pass through an element of the form $2^m\tau$ because the digit preceding $\tau$ is eventually changed, and to change a digit requires all preceding digits to be 2's. We must have $m \geq \mu_{12}$ because the number of digits in a word weakly increases as we go up in $Z(r)$ (so comparing $u$ and $2^m\tau$, this number increases from $\mu_{12} + \tau_{12}$ to $m + \tau_{12}$), and also we must have $\nu_2 \geq m$ because the number of 2's in a word weakly increases as we go up in $Z(r)$ (so comparing $2^m\tau$ and $v$, this increases from $m + \tau_2$ to $\nu_2 + \tau_2$). So $\nu_2 \geq \mu_{12}$. $\square$

**Example 9.3.** Let

$$\begin{aligned} u &= \phantom{2\ 2\ 1_1 2\ 2\ 1_4 2\ 1_1 2\ 2\ 1_2 2\ 2\ 2\ } 2\ 1_2 1_1 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\ v &= 2\ 2\ 1_1 2\ 2\ 1_4 2\ 1_1 2\ 2\ 1_2 2\ 2\ 2\ 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \end{aligned}$$

Factor these as $u = \mu\tau$ and $v = \nu\tau$ with $\tau$ as large as possible.

$$\begin{aligned} \mu &= 2\ 1_2 1_1 2\ 2\ 1_3 1_2 \\ \nu &= 2\ 2\ 1_1 2\ 2\ 1_4 2\ 1_1 2\ 2\ 1_2 2\ 2\ 2 \\ \tau &= 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \end{aligned}$$

Since $\nu_2 = 10 \geq 7 = \mu_{12}$, we should have $v \geq_Z u$. Starting with $u$, first prepend three 2's (since $\nu_2 - \mu_{12} = 3$) by prepending $1_1$ and changing it to 2, yielding alternating

$\gtrdot_1$ and $\gtrdot_2$ cover relations.

$$\begin{array}{r}
2\ 1_2 1_1 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
1_1 2\ 1_2 1_1 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 1_2 1_1 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
1_1 2\ 2\ 1_2 1_1 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 2\ 1_2 1_1 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
1_1 2\ 2\ 2\ 1_2 1_1 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 2\ 2\ 1_2 1_1 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2
\end{array}$$

Then change the $1_i$'s of $\mu$ into 2's, yielding a succession of $\gtrdot_2$ cover relations.

$$\begin{array}{r}
2\ 2\ 2\ 2\ 1_2 1_1 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 2\ 2\ 2\ 1_1 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2
\end{array}$$

Finally, insert the $1_i$'s of $\nu$ into this from right to left, yielding a succession of $\gtrdot_1$ cover relations.

$$\begin{array}{r}
2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 2\ 2\ 2\ 2\ 2\ 1_2 2\ 2\ 2\ 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 2\ 2\ 2\ 1_1 2\ 2\ 1_2 2\ 2\ 2\ 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 2\ 2\ 1_4 2\ 1_1 2\ 2\ 1_2 2\ 2\ 2\ 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 1_1 2\ 2\ 1_4 2\ 1_1 2\ 2\ 1_2 2\ 2\ 2\ 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2
\end{array}$$

We will see below that $\mathrm{Atype}[u,v] = 2^3 1^8$ and $\mathrm{Ctype}[u,v] = 22121^7$.

**Theorem 9.4.** *If $v \geq_Z u$ then $A_v u$ is determined as follows:*

| | |
|---|---|
| $2u$ | *if* $\mu_{12} < \nu_2$*;* |
| *Change the leftmost* 1 *of* $u$ *to* 2 | *if* $\mu_2 < \mu_{12} = \nu_2$*;* |
| $\mu 1_i \tau$ *where* $\nu$ *ends in* $1_i$ | *if* $\mu_2 = \mu_{12} = \nu_2 < \nu_{12}$*;* |
| $u$ | *if* $\mu_2 = \mu_{12} = \nu_2 = \nu_{12} = 0$. |

*The only exception occurs when $r = 1$, $u = \emptyset$, and $\nu_2 > 0$; in this case, $A_v u = 1$.*

*Proof.* If $\mu_{12} < \nu_2$ then $u <_Z 2u \leq_Z v$ because $\nu_2(2u,v) - \mu_{12}(2u,v) < \nu_2(u,v) - \mu_{12}(u,v)$; either $\tau$ and $\nu$ stay the same and $\mu_{12}$ goes up by 1, or $\tau(u,v) = u$, $\tau(2u,v) = 2u$, $\mu(u,v) = \mu(2u,v) = \emptyset$, $\nu(u,v) = 2\nu(2u,v)$ so that $\nu_2$ decreases by 1 and $\mu_{12}$ stays 0. Further, all $\kappa^+(u)$ upper covers of $u$ are lower covers of $2u$, and unless $r = 1$ and $u = \emptyset$, there is more than one upper cover, so their join $Au$ is $2u$. Since $2u \leq v$, also $A_v u = Au \wedge v = Au = 2u$. When $r = 1$ and $u = \emptyset$, there is only one upper cover and $A_v u = Au = 1$.

So assume $\mu_{12} = \nu_2$. Suppose $\mu_2 < \mu_{12}$. What are the atoms $u'$ of $[u,v]$? There is at least one 1 in $\mu$ because $\mu_2 < \mu_{12}$, so any $u' \gtrdot u$ has $\tau(u',v) = \tau(u,v)$, and the altered digit is in the $\mu$ portion. We cannot insert a 1 because then $\mu_{12}(u',v) > \nu_2(u',v)$ so

$u' \not\leq_Z v$. So $u' \gtrdot_2 u$, and the only such $u'$ is obtained by changing the first 1 in $u$ to a 2.

Now suppose $\mu_2 = \mu_{12} = \nu_2$. Then $\mu = 2^{\mu_2}$. If $\nu_2 = \nu_{12}$ then $\mu = \nu$, but since $\mu$ and $\nu$ by definition do not end in the same digit, this requires $\mu = \nu = \emptyset$, so $u = v$ and $A_v u = u$. Only the case $\mu_2 = \mu_{12} = \nu_2 < \nu_{12}$ remains, and $\nu$ must end in some 1; say it ends in $1_i$. Then $v \geq_Z \mu 1_i \tau \gtrdot_Z u$. Are there any other atoms $u'$ of $[u, v]$? Consider another upper cover $u'$ of $u$ in $Z(r)$. What are $\tau(u', v)$, $\mu(u', v)$, and $\nu(u', v)$ as compared with $\tau(u, v)$, $\mu(u, v)$, and $\nu(u, v)$? All digits of $\tau$ left of the altered one in $u \lessdot_Z u'$ are moved to $\mu$ and $\nu$ (if the alteration was in $\tau$), thereby adding at least as many digits to $\mu$ as 2's are added to $\nu$; next, $\mu$ gains a 1 if $u' \gtrdot_1 u$, or $\mu$ gains a 2 and $\nu$ a 1 if $u' \gtrdot_2 u$, thereby adding a digit to $\mu$ but not adding any 2's to $\nu$. So at least one more digit is added to $\mu$ than 2's are added to $\nu$, yielding $\mu_{12}(u', v) \geq \nu_2(u', v) + 1$. Thus, $u' \not\leq_Z v$. So $\mu 1_i \tau$ is the unique atom of $[u, v]$. $\square$

**Corollary 9.5.** $\text{Atype}[u, v] = 2^{\nu_2 - \mu_{12}} 1^{\mu_1 + \nu_1}$, *unless* $r = 1$ *and* $u = \emptyset$ *and* $\nu_2 > 0$, *in which case* $\text{Atype}[u, v] = 12^{\nu_2 - 1} 1^{1 + \nu_1}$ *if* $v$ *ends in* 2, *and* $\text{Atype}[u, v] = 12^{\nu_2} 1^{\nu_1 - 1}$ *if* $v$ *ends in* 1.

*Proof.* Except in the exceptional case, the chain of elements from $u$ to $v$ built in the proof of Theorem 9.2 is precisely $u, A_v u, A_v^2 u, \ldots$ by the preceding theorem; first we prepend $\nu_2 - \mu_{12}$ 2's to $u$, and then change $\mu_1$ 1's to 2's, and then insert $\nu_1$ 1's, and sequentially listing the changes in rank yields $\text{Atype}[u, v] = 2^{\nu_2 - \mu_{12}} 1^{\mu_1 + \nu_1}$. In the exceptional case $r = 1$, $u = \emptyset$, and $\nu_2 > 0$, we have $A_v u = 1$ instead of 2, but the next $\nu_2 - 1$ or $\nu_2$ (depending on whether $v$ ends in 2 or 1) applications of $A_v$ to this do prepend 2's. If $v$ ends in 2, the next application of $A_v$ changes the terminal 1 to a 2. Finally, we insert the remaining 1's into $\nu$. $\square$

**Example 9.6.** In Example 9.3 above, $\text{Atype}[u, v] = 2^3 1^8$. For exceptional cases, $\text{Atype}[\emptyset, 221] = 122$ if $r = 1$ but $\text{Atype}[\emptyset, 221_1] = 221$ if $r > 1$, and $\text{Atype}[\emptyset, 22] = 121$ if $r = 1$ but $\text{Atype}[\emptyset, 22] = 22$ if $r > 1$.

**Theorem 9.7.** *If* $v \geq_Z u$ *then* $C_u v$ *is determined as follows:*

| | |
|---|---|
| *Delete the first digit of* $v$ | *if* $\nu_2 > \mu_{12}$, *or if* $\nu_2 = \mu_{12}$ *and* $v$ *starts with a* 1; |
| *Delete the leftmost* 1 *of* $v$ | *if* $\nu_{12} > \nu_2 = \mu_{12}$; |
| $\nu' 1_i \tau$, *where* $\nu = \nu' 2$ *and* $\mu = \mu' 1_i$ | *if* $\nu_{12} = \nu_2 = \mu_{12} > \mu_2$; |
| $u$ | *if* $\nu_{12} = \nu_2 = \mu_{12} = \mu_2$. |

*The only exception occurs when* $r = 1$; *in this case,* $C_\emptyset 2 = 1$.

*Proof.* If $\nu_2 > \mu_{12}$ and we delete the first digit of $v$ to obtain $v'$, we will decrease $\nu_1$ or $\nu_2$ by 1 but keep $\mu_{12}$ the same, so $v >_Z v' \geq_Z u$. If this first digit is a 1, then $v'$ is the unique lower cover of $v$ in $Z(r)$. If $v = 2v'$, then all $\kappa^-(v)$ lower covers of $v$ are upper covers of $v'$, and there is more than one lower cover unless $r = 1$ and $v = 2$, so the meet of all lower covers of $v$ is $Cv = v' \geq u$. Further, $C_u v = Cv \vee u = v'$. If $v = 2$ and $r = 1$ then $u = \emptyset$ (as $1 = \nu_2 > \mu_{12} \geq 0$) and $C_u v = 1$.

Now suppose $\nu_2 = \mu_{12}$. If $\nu_{12} > \nu_2$ then $\nu$ has 1's in it. Consider lower covers $v'$ of $v$. If $v \gtrdot_2 v'$ then some 2 of $\nu(u, v)$ is changed to a 1 to obtain $\nu(u, v')$, but $\mu(u, v') = \mu(u, v)$, so $\mu_{12}$ stays the same and $\nu_2$ goes down by 1, yielding $\nu_2(u, v') < \mu_{12}(u, v')$, so $u \not\leq_Z v'$. So $v \gtrdot_1 v'$, which uniquely determines $v'$ by deleting the first 1 in $v$. So $C_u v = v'$.

Now suppose $\nu_{12} = \nu_2 = \mu_{12}$. If $\mu_{12} = \mu_2$ then $\mu$ and $\nu$ have the same length and are all 2's, and since they must not end in the same digit, $\mu = \nu = \emptyset$ and $u = v$, so $C_u v = u$. So we are left with the case $\mu_{12} > \mu_2$. All digits of $\nu$ are 2, so $\mu$ has the form $\mu = \mu' 1_i$. Write $\nu = \nu' 2$. Then $\nu' 1_i \tau$ satisfies $u \leq_Z \nu' 1_i \tau \lessdot_2 v$. Do any other lower covers $v'$ of $v$ satisfy $v' \geq_Z u$? To compute $\mu(u, v')$ and $\nu(u, v')$ from $\mu$ and $\nu$, we add to both $\mu$ and $\nu$ all digits of $\tau$ left of the one that changes in $v \gtrdot_Z v'$ (if the change is in $\tau$), and then we add a 1 to $\mu$ if $v \gtrdot_1 v'$ or a 2 to $\mu$ and 1 to $\nu$ if $v \gtrdot_2 v'$. Either way, the number of digits added to $\mu$ goes up by at least one more than the number of 2's added to $\nu$, so $\mu_{12}(u, v') > \nu_2(u, v')$, so $v' \not\geq_Z u$. So the only lower cover of $v$ that exceeds $u$ is $\nu' 1_i \tau$, whence $C_u v = \nu' 1_i \tau$. $\square$

**Corollary 9.8.** Ctype$[u, v]$ *is found as follows. Factor* $\nu = st$ *where* $t_2 = \mu_{12}$*; the factorization isn't unique, as there may be some 1's that can be placed either at the right of* $s$ *or the left of* $t$*. Form* $s'$ *from* $s$ *by dropping the subscripts on all the 1's. Then* Ctype$[u, v] = s' 1^{t_1 + \mu_1}$*, except when* $r = 1$*,* $u = \emptyset$*, and* $\nu_2 > 0$*. When* $r = 1$*,* Ctype$[\emptyset, v] = v$ *unless it ends in* 2*, in which case the final* 2 *is changed to* 11.

*Proof.* We will consider $\mu(u, v')$ and $\nu(u, v')$ as $v'$ takes on successive values $v$, $C_u v$, $C_u^2 v, \ldots$. Initially set $v'$ to $v$. The first $s_{12}$ applications of $C_u$ simply remove initial digits $s$ of $v$; remove initial digits one by one as long as doing so maintains $\nu_2(u, v') \geq \mu_{12}(u, v')$. Then remove 1's from left to right as long as $\nu_{12} > \mu_{12}$ (a total of $t_1$ 1's). Now $u$ and $v'$ have the same number of digits, and where they disagree, $u$ has some $1_i$ and $v'$ has a 2, so from right to left, lower the 2's of $v'$ to the $1_i$'s found in $u$ (a total of $\mu_1$ times). Finally we obtain $u$.

Each digit removed or lowered is precisely an application of $C_u$ by the preceding theorem, except for the exceptional case. List the successive drops in rank to obtain Ctype$[u, v]$ as stated above: removing a 1 or 2 decreases the rank by 1 or 2, respectively, and lowering a 2 to a 1 decreases the rank by 1.

The only exceptional case is when $r = 1$, $u = \emptyset$, and $v$ ends in a 2. When $r = 1$ and $u = \emptyset$, successive applications of $C_u = C$ remove leading digits of $v$ one at a time, until all that is left is the terminal digit, and then $C2 = 1$ and $C1 = \emptyset$, instead of $C2 = \emptyset$ as happens with $r > 1$. $\square$

**Theorem 9.9.** *Let* $u, v \in Z(r)$*. Without loss of generality,* $\mu_2 \leq \nu_2$*. Let* $m = \max\{0, \mu_{12} - \nu_2\}$*. Then* $u \wedge v$ *is obtained from* $u$ *by deleting the first* $m$ *1's.*

*Without loss of generality,* $\mu_{12} \leq \nu_{12}$*. Let* $m = \max\{0, \mu_{12} - \nu_2\}$*. Then* $u \vee v$ *is obtained from* $v$ *by changing the first* $m$ *1's to 2's.*

*Finally,*

$$\begin{aligned} \rho(u \wedge v) &= \min\{\rho(u), \rho(v), \mu_2 + \nu_2 + \rho(\tau)\}, \\ \rho(u \vee v) &= \max\{\rho(v), \rho(v), \mu_{12} + \nu_{12} + \rho(\tau)\}. \end{aligned}$$

*Proof.* We compute the meet; the join is computed similarly. Let $\alpha$ be the result of deleting the first $m$ 1's from $u$. Then $\alpha \leq_Z u$. Either $m = 0$, or $m = \mu_{12} - \nu_2 \leq \mu_{12} - \mu_2 = \mu_1$, so these 1's are all contained in $\mu$. Let $\beta$ be the result of deleting the first $m$ 1's from $\mu$, so $\alpha = \beta\tau$. To prove $\alpha \leq_Z v$, we show $\beta \leq_Z \nu$, for if this holds, append $\tau$ to both sides to obtain $\alpha \leq_Z v$. If $\beta$ and $\nu$ do not end in the same digit then $\beta_{12} = \mu_{12} - m = \min\{\mu_{12}, \nu_2\} \le \nu_2$, so $\beta \le \nu$. If they do end in the same digit then it is a 2 and we deleted all the 1's from $\mu$, because $\mu$ and $\nu$ do not end in the same digit. So $\beta = 2^{\mu_2}$, and $\nu$ has at least $\mu_2$ 2's (since $\mu_2 \le \nu_2$), so $\beta \le_Z \mu$. Thus, $\alpha \le_Z u \wedge v$.

If $\alpha = u$ or $v$ then $\alpha$ is certainly the meet, so assume $\alpha <_Z u, v$. Since $\alpha$ is obtained from $u$ by deleting 1's, the only upper cover $\alpha^+$ of $\alpha$ with $\alpha^+ \le_Z u$ is $A_u\alpha$, obtained by reinserting the rightmost deleted 1 in the $\beta$ portion. Then $\nu(\alpha^+, v) = \nu$, $\tau(\alpha^+, v) = \tau$, and $\mu(\alpha^+, v) = \beta^+$ is $\beta$ with a 1 suitably inserted. We have $\beta_{12} = \min\{\mu_{12}, \nu_2\} < \mu_{12}$ because if $\beta_{12} = \mu_{12}$, inserting a 1 would yield more digits than $u$ has. Thus, $\beta_{12} = \nu_2$ and $\beta^+_{12} = \nu_2 + 1 \not\le \nu_2$, so $\alpha^+ \not\le_Z v$, and we conclude $\alpha = u \wedge v$.

The rank of the meet is

$$\begin{aligned}
\rho(\alpha) &= \rho(u) - m \\
&= \min\{\rho(u), \rho(u) - \mu_{12} + \nu_2\} \\
&= \min\{\rho(u), \mu_2 + \mu_{12} - \mu_{12} + \nu_2\} \\
&= \min\{\rho(u), \mu_2 + \nu_2\}.
\end{aligned}$$

Combining this with the like expression for the case $\mu_2 \ge \nu_2$ gives the value stated in the theorem. □

**9.2. Strongly modular lattices.** We now introduce a new class of lattices.

**Theorem 9.10.** *The following conditions on a lattice are equivalent.*

(1) *The lattice is modular, and all complemented intervals have length at most 2.*
(2) *For any set $S$ of at least two elements of the lattice, the meet of all the elements of $S$ is covered by each element in $S$ iff each element in $S$ is covered by the join of all elements of $S$.*

*Proof.*

**(1) ⇒ (2):** The function $Au$ is defined for all $u$ even when the lattice has infinite length: if there are an infinite number of elements covering some $u$ and their join is unbounded, a finite subcollection of them yields joins of any finite rank, and the interval from $u$ to these joins is atomic and hence complemented. Thus, complemented intervals of all finite ranks exist, violating (1).

Let $S$ be a set of at least two elements of the lattice such that the meet $u$ of all elements of $S$ is covered by each element of $S$. Then $Au$ is the join of all the elements of $S$ and possibly other elements of the lattice. Since there are at least two elements in the join, $Au$ is strictly larger than either, and $[u, Au]$ is an atomic (hence complemented) interval of length at least 2, so by (1), its length is exactly 2. Since $Au$ is larger than all elements of $S$ and has rank 1 larger, it covers them all.

The converse statement in (2) is proven dually, using $C$ instead of $A$.

**(2) ⇒ (1):** If a lattice satisfies (2), it is modular, for given any two elements, they cover their meet iff they are covered by their join. Further, suppose $[x, y]$ is a complemented interval of length greater than 1. There must be at least two atoms, for if there is only one atom, $v$, it has no complement: the only element disjoint with it is $x$, but $x \vee v = v < \hat{1}$. The meet of all the atoms of $[x, y]$ is $x$, which is a lower cover of every atom; thus, by (2), the join of all atoms must cover each atom, so $[x, A_y x]$ has length 2. If $A_y x < y$ then $A_y x$ has no complement, because the only element disjoint from it is $x$, and $x \vee A_y x = A_y x < \hat{1}$. So $A_y x = y$, and $[x, y]$ has length 2. $\square$

**Definition 9.11.** A **strongly modular lattice** is a lattice satisfying either of the above equivalent conditions.

A semi-primary lattice of type $\lambda$ with $\lambda'_1 \leq 2$ is a strongly modular lattice.

**Theorem 9.12 (see [28,** Prop. 5.4]**).** *$Z(r)$ is a strongly modular lattice.*

*Proof.* If $S$ is a set of at least two elements whose meet $x$ is a lower cover of each element of $S$, then $Ax = 2x$ is an upper cover of each element of $S$. If $S$ is a set of at least two elements whose join $y$ is an upper cover of each element of $S$, then $y$ has the form $2x$ (since words not beginning with 2 have at most one lower cover), and $x$ is a lower cover of each element of $S$. $\square$

**Theorem 9.13.** *Let $\mathcal{L}$ be a strongly modular lattice. For any closed interval $[x, y]$, both* $\mathrm{Atype}[x, y]$ *and* $\mathrm{Ctype}[x, y]$ *are in $W(1)$.*

*Proof.* The parts of $\mathrm{Atype}[x, y]$ are the successive lengths of the intervals $[x, A_y x]$, $[A_y x, A_y^2 x]$, $[A_y^2 x, A_y^3 x], \ldots$. Each such interval is atomic and hence complemented, so for $i$ with $A_y^i x < y$, the interval $[A_y^i x, A_y^{i+1} x]$ has length 1 or 2, while all further intervals have length 0. Dualize this argument for Ctype. $\square$

**Theorem 9.14.** *If $x \lessdot v \leq y$ in $\mathcal{L}$ then* $\mathrm{Atype}[v, y] \lessdot_Z \mathrm{Atype}[x, y]$ *in $Z(1)$. Dually, if $x \leq v \lessdot y$, then* $\mathrm{Ctype}[x, v] \lessdot_Z \mathrm{Ctype}[x, y]$ *in $Z(1)$.*

*Proof.* We show the first cover relation by inducting on the length of the interval $[x, y]$.

Suppose $\mathrm{Atype}[x, y]$ begins with a 1. Then $A_y x$ is the unique atom of $[x, y]$, which must be $v$, so $\mathrm{Atype}[x, y]$ is $\mathrm{Atype}[v, y]$ with a 1 prepended.

Suppose $\mathrm{Atype}[x, y]$ and $\mathrm{Atype}[v, y]$ both begin with a 2. Since $v \geq x \Rightarrow A_y v \geq A_y x$ and $v$ has rank one greater than $x$ and $A_y$ increases both ranks by 2, we must have $A_y v \gtrdot A_y x$. By induction, $\mathrm{Atype}[A_y v, y] \lessdot_Z \mathrm{Atype}[A_y x, y]$, and we prepend 2 to both sides to obtain $\mathrm{Atype}[v, y] \lessdot_Z \mathrm{Atype}[x, y]$.

Finally, if $\mathrm{Atype}[x, y]$ begins with 2 and $\mathrm{Atype}[v, y]$ begins with 1, then $A_y v \geq A_y x$, and both sides have the same rank, so they are equal. So all other entries in the types are the same. $\square$

**Theorem 9.15.** *Let $x \lessdot v \leq y$. If* $\mathrm{Atype}[v, y] \lessdot_1 \mathrm{Atype}[x, y]$ *with a 1 inserted after the first $k$ 2's, then $v = C^k A_y^{k+1} x$. If* $\mathrm{Atype}[v, y] \lessdot_2 \mathrm{Atype}[x, y]$ *with the 1 changed to a 2 in the $k$th position, then $x = C^k A_y^k v$.*

*Dually, let $x \leq v \lessdot y$. If* $\mathrm{Ctype}[x, v] \lessdot_1 \mathrm{Ctype}[x, y]$ *with a 1 inserted after the first $k$ 2's, then $v = A^k C_x^{k+1} y$. If* $\mathrm{Ctype}[x, v] \lessdot_2 \mathrm{Ctype}[x, y]$ *with the 1 changed to a 2 in the $k$th position, then $y = A^k C_x^k v$.*

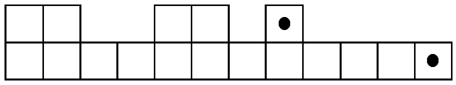
*Only uniqueness of $v$, $x$, or $y$ in each case is asserted, not existence.*

*Proof.* Let $x \lessdot v \leq y$. Suppose $\mathrm{Atype}[v,y] \lessdot_1 \mathrm{Atype}[x,y]$ with a 1 inserted after the first $k$ 2's. Then $A_y^k x \lessdot A_y^k v$ since each side has rank $2k$ more than in $x \lessdot v$. Now $A_y^k v \leq A_y(A_y^k x) = A_y^{k+1} x$ because the middle term is a join of elements which includes the left term. The left and right terms both have the same rank by examining $\mathrm{Atype}[v,y]$ and $\mathrm{Atype}[x,y]$, so in fact, $A_y^k v = A_y^{k+1} x$. Next, $C_v^k A_y^k v \leq v$ by Theorem 3.3(2), and also the left side is less than $v$ because $C_v$ is bounded below by $v$. So $v = C_v^k A_y^k v = C_v^k A_y^{k+1} x$. This is bounded below by $C^k A_y^{k+1} x$. Each of the $k$ applications of $C_v$ must have decreased the rank by 2, and so the applications of $C$ can decrease the rank no further. Thus, $v = C_x^k A_y^{k+1} x$.

Now suppose $\mathrm{Atype}[v,y] \lessdot_2 \mathrm{Atype}[x,y]$ with the 1 changed to a 2 in the $k$th position. Then $A_y^k x = A_y^k v$ since the right side is a weak upper bound for the left, and both have the same rank (by adding the first $k$ digits of the types to the ranks of $x \lessdot v$). Then $x = C_x^k A_y^k x = C_x^k A_y^k v$, and since each application of $C_x$ reduces the rank by 2, it could be replaced by $C$, to obtain $x = C^k A_y^k v$.

The proofs for Ctype are dual. $\square$

To examine the relation of $\mathrm{Ctype}[x,y]$ and $\mathrm{Ctype}[v,y]$ when $v$ is an atom of $[x,y]$ requires a new graded poset $Z_*$. Its elements are those of $W(1)$, and the cover relations are $u \lessdot_{1_*} u1$ and $u11^m \lessdot_{2_*} u21^m$, where $u \in W(1)$ and $m \geq 0$ are arbitrary. In other words, $u \lessdot_{Z_*} v$ when either a 1 is inserted in $u$ in a position followed only by 1's (equivalently, when a 1 is appended to $u$), or when a 1 followed only by 1's is changed to a 2. In terms of snake diagrams, this may be visualized by the following example, in which $v$ is shown, and either marked cell may be removed to obtain some $u \lessdot_{Z_*} v$.



Although we will use $Z_*$ in comparing the types as stated above, we will not have any other use for $Z_*$. Note that $Z_*$ is not a lattice, since upper bounds may not exist; for example, 12 and 2 have no upper bound, because in $Z_*$, all words larger than 12 start with 12 and all words larger than 2 start with 2. It is a meet semilattice: given two words $u = xz$ and $v = xy$ where $z$ or $y$ is empty, or they are both nonempty and start with different digits, then $u \wedge v = x1^m$ where $m$ is the smaller of the number of digits of $y$ or $z$.

**Theorem 9.16.** *If $x \lessdot v \leq y$ in $\mathcal{L}$, then $\mathrm{Ctype}[v,y] \lessdot_{Z_*} \mathrm{Ctype}[x,y]$; further, if $\mathrm{Ctype}[v,y] \lessdot_{1_*} \mathrm{Ctype}[x,y]$, then $v = C_x^{k-1} y$, where $k = \mathrm{arank}[x,y]$. Dually, if $x \leq v \lessdot y$, then $\mathrm{Atype}[x,v] \lessdot_{Z_*} \mathrm{Atype}[x,y]$, and if $\mathrm{Atype}[x,v] \lessdot_{1_*} \mathrm{Atype}[x,y]$, then $v = A_y^{k-1} x$, where $k = \mathrm{arank}[x,y]$.*

*Proof.* We induct on the length of the interval $[x,y]$, and set $\alpha = \mathrm{Ctype}[x,y]$ and $\beta = \mathrm{Ctype}[v,y]$.

If $\alpha$ and $\beta$ begin with the same digit, then $C_x y = C_v y$, and if we replace $y$ by $C_x y$, we get a smaller interval with the first digit of $\alpha$ and $\beta$ deleted; the theorem holds in the smaller interval, and by prepending the common deleted digit, it holds in $[x,y]$.

So assume $\alpha$ and $\beta$ do not start with the same digit. Then $C_x y \neq C_v y = v \vee C_x y$, so $v \not\leq C_x y$. Join $v$ elementwise with the chain $C_x y \geq C_x^2 y \geq \cdots$ to obtain $C_v y \geq C_v^2 y \geq \cdots$; the ranks of the elements in the second chain are one higher than in the first, because the chains are in $[x,y]$ and $v$ is an atom of $[x,y]$, with $v$ not smaller than the top element of the chain. Thus, successive differences in ranks in the two chains are the same, so only the first digit of $\alpha$ and $\beta$ differ. If $\alpha = 1$ and $\beta = \emptyset$ we are done. If $\alpha = 1\gamma$ and $\beta = 2\gamma$ for some word $\gamma$, then $\rho(\alpha) \lessdot \rho(\beta)$; however, $[x, y]$ is an interval of length one higher than $[v, y]$, so this cover relation should be reversed. Thus, this situation doesn't occur. We are left with $\alpha = 2\gamma$ and $\beta = 1\gamma$. Can any digit of $\beta$ be 2? If the $j$th digit of $\beta$ is 2, then $C_v^j y = C^j y$ has rank 2 lower than $C_v^{j-1} y$, and $v \le C^j y \le C_x^j y$, violating the fact that $v \not\le C_x^j y$. So all digits of $\beta$ are 1, and all digits of $\alpha$ but the first are 1.

Note that if $\alpha = \beta 1$ then $v$ is uniquely determined by Theorem 3.8. $\square$

**9.3. Interval type tables in strongly modular lattices.** Let $f = (f_0 \lessdot f_1 \lessdot \cdots \lessdot f_n)$ be a saturated flag in a strongly modular lattice.

The interval Ctype table of a saturated flag $f$ in a strongly modular lattice is the triangular array $(\mathrm{Ctype}[f_i, f_j])_{0 \le i \le j \le n}$. The interval Atype table is defined dually. We will be interested in determining the possible interval type tables of a flag given just some of the entries of the table, such as the first row, $(\mathrm{Ctype}[f_0, f_j])_{0 \le j \le n}$, and more generally, any sequence $\mathrm{Ctype}[f_{i_0}, f_{j_0}], \ldots, \mathrm{Ctype}[f_{i_n}, f_{j_n}]$ where $i_0 = j_0$, $i_n = 0$, $j_n = n$, and for each $k > 0$, either $i_k = i_{k-1} - 1$ and $j_k = j_{k-1}$, or $i_k = i_{k-1}$ and $j_k = j_{k-1} + 1$.

We will develop necessary (but not sufficient) conditions for a triangular array $\Lambda = (\lambda^{(ij)})_{0 \le i \le j \le n}$ of words of $W(1)$ to be realizable as the interval type table of some flag in an arbitrary strongly modular lattice, and then will focus on $Z(r)$.

By Theorem 9.14, $\lambda^{(ij)}$ increases in $Z(1)$ for fixed $i$ and increasing $j$. By Theorem 9.16, it increases in $Z_*$ for fixed $j$ and decreasing $i$.

Let $i < j$ and consider the types of the intervals determined by the four elements $f_i \lessdot f_{i+1} \le f_{j-1} \lessdot f_j$. The types are denoted as shown in this table, and have the cover relations shown by Theorems 9.14 and 9.16.

| $\mathrm{Ctype}[f_a, f_b]$ | $b = j-1$ | | $b = j$ |
|---|---|---|---|
| $a = i$ | $\gamma$ | $\lessdot_Z$ | $\delta$ |
| | $\rotatebox[origin=c]{90}{$\lessdot$}_{Z_*}$ | | $\rotatebox[origin=c]{90}{$\lessdot$}_{Z_*}$ |
| $a = i+1$ | $\alpha$ | $\lessdot_Z$ | $\beta$ |

**Proposition 9.17.** *The local configurations permitted in an interval Ctype table are as follows, where $x \in Z(1)$ and $k, m \ge 0$ are arbitrary. The label at the center of each is for later reference.*

$$\begin{array}{ccc} \gamma & \lessdot_Z & \delta \\ \rotatebox[origin=c]{90}{$\lessdot$}_{Z_*} & & \rotatebox[origin=c]{90}{$\lessdot$}_{Z_*} \\ \alpha & \lessdot_Z & \beta \end{array} \qquad \begin{array}{ccc} 2^k x 1 & \lessdot_1 & 2^k 1 x 1 \\ \rotatebox[origin=c]{90}{$\lessdot$}_{1_*} & (1) & \rotatebox[origin=c]{90}{$\lessdot$}_{1_*} \\ 2^k x & \lessdot_1 & 2^k 1 x \end{array} \qquad \begin{array}{ccc} 2^k 2 & \lessdot_1 & 2^k 21 \\ \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} & (2) & \rotatebox[origin=c]{90}{$\lessdot$}_{1_*} \\ 2^k 1 & \lessdot_2 & 2^k 2 \end{array}$$

$$\begin{array}{ccc} 2^k x 21^m & \lessdot_1 & 2^k 1 x 21^m \\ \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} & (3) & \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} \\ 2^k x 11^m & \lessdot_1 & 2^k 1 x 11^m \end{array} \qquad \begin{array}{ccc} 2^k 21^m & \lessdot_1 & 2^k 211^m \\ \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} & (4) & \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} \\ 2^k 11^m & \lessdot_1 & 2^k 111^m \end{array} \qquad \begin{array}{ccc} 2^k 1 x 21^m & \lessdot_2 & 2^k 2 x 21^m \\ \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} & (5) & \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} \\ 2^k 1 x 11^m & \lessdot_2 & 2^k 2 x 11^m \end{array}$$

$$\begin{array}{ccc} 2^k 211^m & \lessdot_2 & 2^k 221^m \\ \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} & (6) & \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} \\ 2^k 111^m & \lessdot_2 & 2^k 211^m \end{array} \qquad \begin{array}{ccc} 2^k 1 x 1 & \lessdot_2 & 2^k 2 x 1 \\ \rotatebox[origin=c]{90}{$\lessdot$}_{1_*} & (7) & \rotatebox[origin=c]{90}{$\lessdot$}_{1_*} \\ 2^k 1 x & \lessdot_2 & 2^k 2 x \end{array} \qquad \begin{array}{ccc} 2^k 1 & \lessdot_2 & 2^k 2 \\ \rotatebox[origin=c]{90}{$\lessdot$}_{1_*} & (8) & \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} \\ 2^k & \lessdot_1 & 2^k 1 \end{array}$$

*Proof.* In the left column of a configuration, given $\gamma$, there is at most one $\alpha$ with $\alpha \lessdot_{1_*} \gamma$ (namely, if $\gamma$ ends in 1 then remove it), and at most one $\alpha$ with $\alpha \lessdot_{2_*} \gamma$ (if $\gamma$ has a 2, change the last 2 to 1). In the right column, there are similar possibilities for obtaining $\beta$ from $\delta$. In the top row, if $\gamma \lessdot_1 \delta$ then $\gamma$ is uniquely determined from $\delta$ by removing the first 1, while if $\gamma \lessdot_2 \delta$ then $\delta$ is uniquely determined from $\gamma$ by changing the first 1 to 2. A similar relationship holds in the bottom row. This allows us to explicitly list all forms this table may take on by considering the kind of cover relation we have in each of the four positions. Given these four cover relations, if the top row has $\lessdot_1$ then $\delta$ determines all three of the other entries of the table, while if it has $\lessdot_2$ then $\gamma$ determines the other three. Exhaustively listing all possibilities gives the configurations in the statement of the theorem, except for (2) and (8).

In place of (2), we actually obtain

$$\boxed{\begin{array}{ccc} 2^k21^m & \lessdot_1 & 2^k211^m \\ \vee_{2_*} & (2') & \vee_{1_*} \\ 2^k11^m & \lessdot_2 & 2^k21^m \end{array}}$$

Assume we have elements in configuration $(2')$. We show $m = 0$, which is configuration (2). Let $z = C_{f_i}^{k+1} f_j$. Since $\beta$ and $\delta$ both start with $2^{k+1}$, the first $k+1$ applications of $C_{f_i}$ and $C_{f_{i+1}}$ to $f_j$ agree, and $z = C_{f_{i+1}}^{k+1} f_j \geq f_{i+1}$. Since the remaining digits of $\delta$ are 1's, $\mathrm{Ctype}[f_i, z] = 1^{m+1}$, so $[f_i, z]$ is a chain and $f_{i+1}$ is its atom. Next, since $\delta = \gamma 1$ and $\gamma$ has $k+m+1$ digits, the first $k+m+1$ applications of $C_{f_i}$ to $f_{j-1} \lessdot f_j$ maintain a cover relation: $C_{f_i}^t f_{j-1} \lessdot C_{f_i}^t f_j$ for $0 \leq t \leq k+m+1$. When $t \leq k+m$, both sides are strictly larger than $f_i$, hence weakly larger than the unique cover $f_{i+1}$ of $f_i$, so we can replace $C_{f_i}$ with $C_{f_{i+1}}$. Thus, the first $k+m$ digits of $\alpha$, $\beta$, $\gamma$, $\delta$ should all agree. But when $m > 0$, the $(k+1)$th digit is 1 in $\alpha$ and 2 in the others, a contradiction. So only $m = 0$ is possible.

In place of (8), we actually obtain

$$\boxed{\begin{array}{ccc} 2^k11^m & \lessdot_2 & 2^k21^m \\ \vee_{1_*} & (8') & \vee_{2_*} \\ 2^k1^m & \lessdot_1 & 2^k11^m \end{array}}$$

Assume we have elements in this configuration and that $m > 0$. There are at least $k+1$ digits in each of $\alpha, \beta, \gamma, \delta$. We have $C_{f_i}^{k+1} f_{j-1} \leq C_{f_i}^{k+1} f_j$, and the ranks of both sides are the same since the first $k+1$ digits of $\gamma$ and $\delta$ differ in sum by 1, as do the ranks of $f_{j-1}$ and $f_j$. So $C_{f_i}^{k+1} f_{j-1} = C_{f_i}^{k+1} f_j$. Join both sides with $f_{i+1}$ to obtain $C_{f_{i+1}}^{k+1} f_{j-1} = C_{f_{i+1}}^{k+1} f_j$. However, the first $k+1$ digits of $\alpha$ and $\beta$ agree, so this should be $\lessdot$, a contradiction. Thus, we cannot have $m > 0$, so only $m = 0$ is conceivable. $\square$

Whether these configurations of types are realized in a particular strongly modular lattice depends on the lattice.

Next, we examine determination of $\beta$ or $\gamma$ from the other three types.

Given $\alpha \lessdot_Z \gamma \lessdot_{Z_*} \delta$, there is always some $\beta$ consistent with them (if configuration (2) is removed, the rest yield unique solutions covering all cases), and it is usually unique. Similarly, given $\alpha \lessdot_{Z_*} \beta \lessdot_Z \delta$, there is always some $\gamma$ consistent with them (omit configurations (2) and (6) to get unique solutions covering all cases), usually unique. Usually (odd numbered configurations), both vertical covers $\alpha \lessdot_{Z_*} \gamma$ and $\beta \lessdot_{Z_*} \delta$ entail changing a digit in the same position from the right end of the word, and both horizontal covers $\alpha \lessdot_Z \beta$ and $\gamma \lessdot_Z \delta$ entail changing the same position from the left end of the word. When $\alpha$ has the form $2\cdots 21\cdots 1$ (even numbered configurations), however, the positions near the central 21 are accessible for alterations by both $\lessdot_Z$ and $\lessdot_{Z_*}$.

Examination of configurations (1)–(8) yields the following as the only possibilities for $\beta$ or $\gamma$ not being uniquely determined by the other three types.

$$\boxed{\begin{array}{ccc} 2^k2 & \lessdot_1 & 2^k21 \\ \vee_{2_*} & (9) & \vee_{Z_*} \\ 2^k1 & \lessdot_Z & \beta \end{array}} \qquad \begin{array}{l} (4)\ \beta = 2^k11;\ \text{generic} \\ (2)\ \beta = 2^k2;\ \text{degenerate: } f_{i+1} = C^{k+1}f_j \end{array}$$

$$\boxed{\begin{array}{ccc} \gamma & \lessdot_Z & 2^k21 \\ \vee_{Z_*} & (10) & \vee_{1_*} \\ 2^k1 & \lessdot_2 & 2^k2 \end{array}} \qquad \begin{array}{l} (7)\ \gamma = 2^k11;\ \text{generic} \\ (2)\ \gamma = 2^k2;\ \text{degenerate: } f_{j-1} = A^{k+1}f_i \end{array}$$

$$\boxed{\begin{array}{ccc} \gamma & \lessdot_Z & 2^k221^m \\ \vee_{Z_*} & (11) & \vee_{2_*} \\ 2^k111^m & \lessdot_2 & 2^k211^m \end{array}} \qquad \begin{array}{l} (6)\ \gamma = 2^k211^m;\ \text{more likely in } Z(r) \\ (5)\ \gamma = 2^k121^m;\ \text{less likely in } Z(r) \end{array}$$

Given $f_i, f_j$ with $\mathrm{Ctype}[f_i, f_j] = 2^k21$, the only $f_{i+1}$ with $\mathrm{Ctype}[f_{i+1}, f_j] = 2^k2$ is $f_{i+1} = C^{k+1}f_j$, and the only $f_{j-1}$ with $\mathrm{Ctype}[f_i, f_{j-1}] = 2^k2$ is $f_{j-1} = A^{k+1}f_i$. Thus, in (9) and (10), both these conditions holding simultaneously is "degenerate," while one holding and the other element being chosen freely from among potentially many possibilities is "generic." Of course, the number of "generic" possibilities depends on the lattice. In $Z(r)$, the generic possibilities truly do occur with higher frequency than the degenerate ones. Choose any $f_i, f_j$ with $\mathrm{Ctype}[f_i, f_j] = 2^k21$. Then $f_j = 2^k2x$ for some $x \in Z(r)$. To obtain configuration (9), let $f_{j-1} = 2^k2f_i$, and $f_{i+1}$ be any upper cover of $f_i$ except for $x$. There are $\kappa^+(f_i) - 1$ choices, and $\kappa^+(f_i) - 1 \geq r - 1$. To obtain configuration (10), let $f_{i+1} = x$ and $f_{j-1}$ be any lower cover of $f_j$ except for $2^k2f_i$. There are $\kappa^-(f_j) - 1 \geq (k+1)r - 1$ choices.

Last, we consider configuration (11). Choose any $f_i, f_j$ with $\mathrm{Ctype}[f_i, f_j] = 2^k221^m$. Since $\delta$ begins with $2^k22$, we have $f_j = 2^k22x$ with $x = C^{k+2}f_j \geq f_i$. Choose any $f_i < f_{i+1} \leq f_j$ with $\mathrm{Ctype}[f_i, f_j] = \delta$ and $\mathrm{Ctype}[f_{i+1}, f_j] = \beta$. Since $\delta$ begins with $2^k22$, we can write $f_j = 2^k22x$, with $x \geq f_i$ and $\mathrm{Ctype}[f_i, x] = 1^m$. Since $\beta$ begins $2^k21$, we have $2x \geq f_{i+1}$ but $x \not\geq f_{i+1}$, and $\mathrm{Ctype}[f_{i+1}, 2x] = 11^m$. All lower covers of $f_{j-1}$ exceed $f_{i+1}$ because they all exceed $2^k2x$. Since $\alpha = 2^k111^m$, the form of $f_{j-1}$ is $2^kx'$ where $\mathrm{Ctype}[f_{i+1}, x'] = 111^m$. Also, since $C_{f_i}^{k+2}f_j = C_{f_i}^{k+2}f_{j-1}$ by rank considerations, $C^2x' = x$. Either $\mathrm{Ctype}[x, x'] = 12$ so that $x' = 1_s2x$ in one of $r$ ways, or $\mathrm{Ctype}[x, x'] = 21$, so that $x'$ is obtained from $x$ by prepending 2 to any of the $\kappa^+(x)$ upper covers of $x$. In the first case, $\gamma = 2^k121^m$, and in the second, $\gamma = 2^k211^m$ unless $x' = 22C_{f_i}x$, when $\gamma = 2^k221^{m-1}$. In the second case, $\kappa^+(x) > r$ unless $x = \emptyset$, so that $f_i = \emptyset$, $m = 0$, and $\kappa^+(x) = r$; since $m = 0$, the exception $\gamma = 2^k221^{m-1}$ cannot occur. Thus for all $f_i, f_{i+1}, f_j$, there are at least as many choices of $f_{j-1}$ with $\gamma = 2^k211^m$ as there are with $\gamma = 2^k121^m$. Note that the ratio $\kappa^+(x)/r$ is asymptotically an integer as $r \to \infty$, rather than a power of $r$, so this more frequent choice is not generic in the sense that term has been used elsewhere.

**Theorem 9.18.** *Let $\Lambda = (\lambda^{(ij)})_{0\le i\le j\le n}$ be a triangular array whose local configurations satisfy Proposition* 9.17*. There is a polynomial $p_\Lambda(r)$ such that the number of flags $\hat{0} = f_0 \lessdot \cdots \lessdot f_n$ in $Z(r)$ with* $\mathrm{Ctype}[f_i, f_j] = \lambda^{(ij)}$ *for all $i, j$ is $p_\Lambda(r)$ when $r > 1$. When configuration (2) is not present, this polynomial is monic, with degree equal to the number of digits of $\lambda^{(0n)}$. When configuration (2) is present, the degree is lower.*

*Proof.* A necessary condition for $\Lambda$ to be the interval Ctype table of a flag is that Proposition 9.17 applies to every local portion of $\Lambda$.

Let $\boldsymbol{P} = (\emptyset = \lambda^{(0)} \lessdot \cdots \lessdot \lambda^{(n)})$ be a chain in $Z(1)$. Let $J = \left\{ j : \lambda^{(j-1)} \lessdot_1 \lambda^{(j)} \right\}$. Let $\pi = \{B_1, \ldots, B_k\}$ be a set partition of $J$, and $\vec{c} = (c_1, \ldots, c_k)$ be distinct numbers in $\{1, \ldots, r\}$. Given $\pi$, there are $r(r-1)\cdots(r-k+1)$ choices of $\vec{c}$. Let $f(\boldsymbol{P}, \pi, \vec{c})$ be the flag $\emptyset = f_0 \lessdot \cdots \lessdot f_n$ in which $\mathrm{Ctype}[\emptyset, f_j] = \lambda^{(j)}$ and when $j \in B_m$, a $1_{c_m}$ is inserted in $f_{j-1}$ to obtain $f_j$. The specification of $\boldsymbol{P}$, $\pi$, and $\vec{c}$ uniquely determines $f$. Conversely, all flags are expressible in this form.

The computation of $\mathrm{Ctype}[f_i, f_j]$ in Corollary 9.8 only depends on knowing $\lambda^{(i)}$, $\lambda^{(j)}$, and whether subscripts on particular 1's agree or disagree, which is all encoded in $\boldsymbol{P}$ and $\pi$. Thus, $\Lambda(f(\boldsymbol{P}, \pi, \vec{c}))$ is independent of $\vec{c}$, so we can call it $\Lambda(\boldsymbol{P}, \pi)$.

Now take any $\Lambda$ locally satisfying Proposition 9.17, and let $\boldsymbol{P} = (\lambda^{(00)} \lessdot \cdots \lessdot \lambda^{(0n)})$. Then

$$p_\Lambda(r) = \sum_{\{\,\pi\,:\,\Lambda(\boldsymbol{P},\pi) = \Lambda\,\}} r(r-1)\cdots(r-k+1)$$

where $\pi$ is a partition of $J$ and $k$ is the number of blocks of $\pi$.

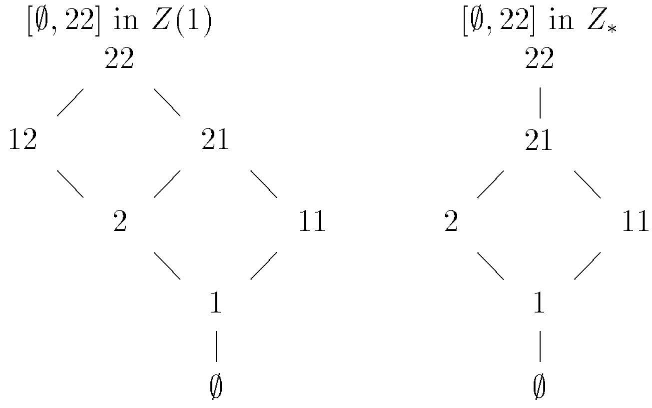
The total number of flags with $\mathrm{Ctype}[\emptyset, f_j] = \lambda^{(j)}$ for all $j$ is $r^{\#J}$, obtained by making independent choices of which 1 to insert when $f_{j-1} \lessdot_1 f_j$. The discussion before this theorem showed that for all but $O(r^{-1})$ flags, configuration (2) does not arise in the interval Ctype table, and that without configuration (2), all other entries are uniquely determined by specifying the top row $\boldsymbol{P}$. Thus, for each $\boldsymbol{P}$, there is some $\Lambda$ with $\deg(p_\Lambda) = \#J$; this can only be $\Lambda = \Lambda(\boldsymbol{P}, \pi)$, where $\pi$ is the partition of $J$ into singleton blocks, and $p_\Lambda$ is monic because all other $\pi$ with $\Lambda = \Lambda(\boldsymbol{P}, \pi)$ have fewer blocks and hence contribute only smaller degree terms. $\square$

*Note.*

(1) The restriction $r > 1$ is necessary for the trivial reason that $C2 = 1$ in $Z(1)$ and $C2 = \emptyset$ for $r > 1$.

(2) Recall that ev is an involution on chains in Young's lattice, and generically gives the cotype of a flag of given type. The preceding theorem similarly gives the generic value of $(\mathrm{Ctype}[f_i, f_n])_{0\le i\le n}$ when $(\mathrm{Ctype}[f_0, f_i])_{0\le i\le n}$ is specified, for $\emptyset = f_0 \lessdot \cdots \lessdot f_n$ in $Z(r)$. However, it does not give a bijection between saturated chains in $Z(1)$ and in $Z_*$ from $\emptyset$ to the same maximum element: configuration (11) gives $\alpha, \delta$ with two different $\gamma$ necessarily mapping to the same $\beta$, so we cannot specify a chain in $Z_*$ and apply certain of configurations (1)–(8) to recover a unique chain in $Z(1)$. In fact, there usually aren't the same number of saturated chains in $Z(1)$ and $Z_*$ from $\emptyset$ to $x \in W(1)$. For

example, if $x = 22$, there are 3 chains in the first and 2 in the second.




(3) Stanley [**28**, Prop. 5.4] showed that the number of saturated chains from  to $x$ in $Z(r)$ is the same as in $Fib(r)$. In the same spirit as ev and the preceding theorem, it is tempting to try to form a triangular array $\Lambda = (\lambda^{(ij)})_{0 \le i \le j \le n}$ of words of $W(r)$ of rank $j - i$, increasing in $Z(r)$ as $j$ increases and increasing in $Fib(r)$ as $i$ decreases. We initialize $\lambda^{(ii)} = \emptyset$, and then $\lambda^{(0j)}$'s to a chain in $Z(r)$, or $\lambda^{(in)}$'s to a chain in $Fib(r)$, and have a propagation rule on local $2 \times 2$ configurations to determine the remaining entries. Unfortunately, this scheme does not work. One problematic configuration is

$$\boxed{\begin{array}{ccc} \gamma & \lessdot_Z & \delta \\ \rotatebox{90}{$\lessdot$}_F & & \rotatebox{90}{$\lessdot$}_F \\ \alpha & \lessdot_Z & \beta \end{array}} = \boxed{\begin{array}{ccc} 2^k 2 2^l x & \lessdot_Z & 2^k 2 2^l 1_1 x \\ \rotatebox{90}{$\lessdot$}_F & & \rotatebox{90}{$\lessdot$}_F \\ 2^k 1_2 2^l x & \lessdot_Z & ? \end{array}}$$

When $x = (1_1)^m$, the only possible solution is $\beta = 2^k 2 2^l x$. When $x$ is not of this form, there are no solutions.

**9.4. Fomin's Robinson-Schensted algorithm for** $Z(1)$**.** Fomin [**4**] developed an analogue of the Robinson-Schensted correspondence for $Z(1)$: a bijection between pairs of saturated chains from $\emptyset$ to the same endpoint, and permutations in $S_n$. See Fomin [**6**] and Roby [**21**] for a unified treatment of these and similar correspondences. The bijection is as follows.

Form an array $\Lambda = (\lambda^{(ij)})_{0 \le i,j \le n}$. A local configuration in it is

$$\Lambda^{(ij)} = \boxed{\begin{array}{ccc} \lambda^{(i,j-1)} & \leq_Z & \lambda^{(ij)} \\ \rotatebox{90}{$\leq$}_Z & & \rotatebox{90}{$\leq$}_Z \\ \lambda^{(i-1,j-1)} & \leq_Z & \lambda^{(i-1,j)} \end{array}} = \boxed{\begin{array}{ccc} \gamma & \leq_Z & \delta \\ \rotatebox{90}{$\leq$}_Z & & \rotatebox{90}{$\leq$}_Z \\ \alpha & \leq_Z & \beta \end{array}}$$

**Fomin's Robinson-Schensted correspondence in $Z(1)$.** Let $\sigma \in S_n$. Form the array $\Lambda$ as follows. Initialize the left and bottom edges of $\Lambda$ to $\emptyset$: $\lambda^{(i0)} = \lambda^{(0i)} = \emptyset$

for $i = 0, \ldots, n$. Working in some order of increasing $i$ and $j$, set

$$\delta = \begin{cases} 1\alpha & \text{if } \alpha = \beta = \gamma \text{ and } \sigma(j) = i \\ \alpha & \text{if } \alpha = \beta = \gamma \text{ and } \sigma(j) \neq i \\ 2\alpha & \text{if } \alpha \lessdot_Z \beta, \gamma \\ \beta & \text{if } \beta \gtrdot_Z \gamma = \alpha \\ \gamma & \text{if } \gamma \gtrdot_Z \beta = \alpha \end{cases}$$

As we construct $\lambda^{(ij)}$, it is a weakly increasing function of $i$ and $j$, and $\rho(\lambda^{(ij)}) = \#\{\, 0 < j' \le j : \sigma(j') \le i \,\}$. Finally, the chains on the right and top edges of $\Lambda$ are $\boldsymbol{P} = (\lambda^{(0n)}, \ldots, \lambda^{(nn)})$ and $\boldsymbol{Q} = (\lambda^{(n0)}, \ldots, \lambda^{(nn)})$.

**The inverse correspondence.** Given saturated chains $\boldsymbol{P}$ and $\boldsymbol{Q}$ in $Z(1)$ from $\emptyset$ to the same top element, form the array $\Lambda_{\mathrm{FRS}}(\boldsymbol{P}, \boldsymbol{Q})$ as follows. Initialize the chains on the right and top edges to $(\lambda^{(0n)}, \ldots, \lambda^{(nn)}) = \boldsymbol{P}$ and $(\lambda^{(n0)}, \ldots, \lambda^{(nn)}) = \boldsymbol{Q}$. Working in some order of decreasing $i$ and $j$, set

$$\alpha = \begin{cases} \beta & \text{if } \beta \lessdot_Z \gamma = \delta \\ \gamma & \text{if } \gamma \lessdot_Z \beta = \delta \\ \text{delete first digit of } \delta & \text{otherwise} \end{cases}$$

We have $\lambda^{(n,j-1)} \lessdot_Z \lambda^{(nj)}$ and $\lambda^{(0,j-1)} = \lambda^{(0j)}$. If $\lambda^{(i,j-1)} = \lambda^{(ij)}$, the same is true upon replacing $i$ with $i-1$, so for each $j$ there is a unique $i$ with $\alpha = \beta = \gamma \lessdot \delta$, and we set $\sigma(j) = i$.

**9.5. Relative positions of flags in strongly modular lattices.** Let $f = (f_0 \lessdot \cdots \lessdot f_n)$ and $f' = (f'_0 \lessdot \cdots \lessdot f'_n)$ be two saturated flags in a strongly modular lattice with the same endpoints, $f_0 = f'_0$, $f_n = f'_n$, where $f_0$ has finite rank. Let $x_{ij} = f_i \wedge f'_j$. We consider the meet type table $\Lambda(f, f') = (\lambda^{(ij)})$, where $\lambda^{(ij)} = \mathrm{Ctype}[\hat{0}, x_{ij}]$ for $0 \le i, j \le n$. Since $x_{i-1,j-1} = x_{i-1,j} \wedge x_{i,j-1}$, we may examine a local portion of $\Lambda$:

$$\Lambda^{(ij)} = \begin{array}{l|ccc} \mathrm{Ctype}[\emptyset, f_a \wedge f'_b] & b = j-1 & & b = j \\ \hline a = i & \gamma & \lessdot_Z & \delta \\ & \lessdot_Z\!\!\!\!\!\!\!\!\vee & & \vee\!\!\!\!\lessdot_Z \\ a = i-1 & \alpha & \lessdot_Z & \beta \end{array}$$

We seek possible values of $\alpha$ given the other three.

If $x_{i,j-1}$ or $x_{i-1,j}$ equals $x_{ij}$, we obtain one of the following configurations.

$$(1)\ \boxed{\begin{array}{ccc} \gamma & \lessdot & \delta \\ \shortparallel & & \shortparallel \\ (\gamma) & \lessdot & \delta \end{array}} \qquad (2)\ \boxed{\begin{array}{ccc} \delta & = & \delta \\ \gtrdot & & \gtrdot \\ (\beta) & = & \beta \end{array}} \qquad (3)\ \boxed{\begin{array}{ccc} \gamma & = & \gamma \\ \shortparallel & & \shortparallel \\ (\gamma) & = & \gamma \end{array}}$$

These all occur in $\Lambda_{\mathrm{FRS}}$ too.

We consider configurations with $\beta, \gamma \lessdot_Z \delta$. If $x_{i-1,j}$ and $x_{i,j-1}$ are distinct lower covers of $x_{ij}$, the interval $[x_{i-1,j} \wedge x_{i,j-1}, x_{ij}]$ is complemented of length 2, so $Cx_{ij} =$

$x_{i-1,j-1}$, and $\delta = 2\alpha$.

$$
(4)\ \boxed{\begin{array}{ccc}
\gamma & \lessdot_Z & 2\alpha \\
& \diagdown & \\
\rotatebox{90}{$\lessdot$}_Z & \neq & \rotatebox{90}{$\lessdot$}_Z \\
& \diagdown & \\
(\alpha) & \lessdot_Z & \beta
\end{array}}
\qquad
(5)\ \boxed{\begin{array}{ccc}
\gamma & \lessdot_2 & 2\alpha \\
& \diagdown & \\
\rotatebox{90}{$\lessdot$}_1 & = & \rotatebox{90}{$\lessdot$}_2 \\
& \diagdown & \\
(\alpha) & \lessdot_1 & \gamma
\end{array}}
$$

Either $\beta \neq \gamma$, or we have two distinct lower covers of equal type. By Theorem 9.15, when $\gamma \lessdot_1 \delta$, there is at most one lower cover of $x_{ij}$ of type $\gamma$, so to have two distinct lower covers requires $\gamma \lessdot_2 \delta$. This restriction on a local configuration does not exist in $\Lambda_{\mathrm{FRS}}$.

Finally, if $x_{ij} \gtrdot x_{i-1,j} = x_{i,j-1}$, we have

$$
(6)\ \boxed{\begin{array}{ccc}
\gamma & \lessdot_1 & \delta \\
\shortparallel & & \rotatebox{90}{$\lessdot$}_1 \\
(\gamma) & = & \gamma
\end{array}}
\qquad
(7)\ \boxed{\begin{array}{ccc}
\gamma & \lessdot_2 & \delta \\
\shortparallel & & \rotatebox{90}{$\lessdot$}_2 \\
(\gamma) & = & \gamma
\end{array}}
$$

Unlike $\Lambda_{\mathrm{FRS}}$, this does not require $\delta = 1\gamma$.

In all cases, if $\alpha = \beta \wedge \gamma$ then any solutions of $\alpha \leq \beta \leq \delta$ and $\alpha \leq \gamma \leq \delta$ fall into one of these configurations. The only other case is $\alpha \lessdot \beta = \gamma \lessdot \delta$, when we are restricted to (5) and not $\alpha \lessdot_2 \beta = \gamma \lessdot_1 \delta$.

Given the chains along the right and top boundaries of $\Lambda$, if we propagate in some order of decreasing $i$ and $j$ to fill $\Lambda$ using rules (1)–(7), the only ambiguity arises when $\gamma = \beta \lessdot_2 \delta$, since either (5) or (7) can be used; all other arrangements of $\beta, \gamma, \delta$ admit exactly one choice of $\alpha$, namely $\alpha = \beta \wedge \gamma$. Let $\Lambda_{\mathrm{ZRS}}(\boldsymbol{P}, \boldsymbol{Q})$ be the array obtained by using (5) when the ambiguity arises, and $\sigma_{\mathrm{ZRS}}(\boldsymbol{P}, \boldsymbol{Q})$ be the associated permutation. This array is not the same as $\Lambda_{\mathrm{FRS}}(\boldsymbol{P}, \boldsymbol{Q})$ because their local rules differ. Using rule (7) instead of rule (5) would give $\lambda^{(ij)} = \lambda^{(nj)} \wedge \lambda^{(in)}$ with associated permutation $\sigma(\boldsymbol{P}, \boldsymbol{Q})$, and provides a lower bound of the relative position in the Bruhat order by Theorem 7.7.

*Note.* When $\boldsymbol{P}$ and $\boldsymbol{Q}$ consist only of words of the form $2^k 1^m$, so do all entries of $\Lambda$, no matter whether (5) or (7) is used. In fact, this is exactly the same as the local rule version of the ordinary Robinson-Schensted algorithm in Young's lattice when we interpret $2^k 1^m \in Z(1)$ as the two row partition $(k+m, m)$ in Young's lattice, for each word in $\boldsymbol{P}$, $\boldsymbol{Q}$, and $\Lambda$. Rule (5) gives the generic choice and rule (7) the degenerate choice for Young's lattice as well.

**Theorem 9.19.** *Let $f, f'$ be saturated chains with the same endpoints in a strongly modular lattice, and $\boldsymbol{P} = (\mathrm{Ctype}[f_0, f_i])_{0 \leq i \leq n}$, $\boldsymbol{Q} = (\mathrm{Ctype}[f'_0, f'_j])_{0 \leq j \leq n}$. Then $\sigma(\boldsymbol{P}, \boldsymbol{Q}) \leq \sigma(f, f') \leq \sigma_{\mathrm{ZRS}}(\boldsymbol{P}, \boldsymbol{Q})$ in the Bruhat order.*

*Proof.* The lower bound holds by Theorem 7.7.

Let $\Lambda$ be a meet type table. We show how to alter certain $\lambda^{(ij)}$ with $0 \leq i, j < n$ so that these $\lambda^{(ij)}$ decrease in $Z(1)$, and all other $\lambda^{(ij)}$ stay intact. Such alterations can be carried out iff configuration (7) appears somewhere in $\Lambda$, so the stable table obtained at the end is $\Lambda_{\mathrm{ZRS}}(\boldsymbol{P}, \boldsymbol{Q})$, and is componentwise less than $\Lambda$ in $Z(1)$; thus, the permutation $\Lambda$ defines is smaller than the one $\Lambda_{\mathrm{ZRS}}(\boldsymbol{P}, \boldsymbol{Q})$ defines in the Bruhat order.

So suppose $\Lambda^{(i',j')}$ is in configuration (7). Then we can write $\delta = 2x$. Define a new meet type table $M = (\mu^{(ij)})_{0 \leq i,j \leq n}$ by

$$\mu^{(ij)} = \begin{cases} \lambda^{(ij)} & \text{if } i \geq i' \text{ or } j \geq j' \\ \lambda^{(ij)} \wedge x & \text{if } i < i' \text{ and } j < j' \end{cases}$$

We show that all local configurations of $M$ are of the forms (1)–(7), and that $\mu^{(ij)} = \lambda^{(ij)}$ or $\mu^{(ij)} \lessdot_1 \lambda^{(ij)}$ (denoted $\mu^{(ij)} \leq_1 \lambda^{(ij)}$) for all $i, j$. The local configurations are denoted as follows.

$$M^{(ij)} = \boxed{\begin{array}{ccc} \mu^{(i,j-1)} & \leq_Z & \mu^{(ij)} \\ \forall\!\!\mid_Z & & \forall\!\!\mid_Z \\ \mu^{(i-1,j-1)} & \leq_Z & \mu^{(i-1,j)} \end{array}} = \boxed{\begin{array}{ccc} \tilde{\gamma} & \leq_Z & \tilde{\delta} \\ \forall\!\!\mid_Z & & \forall\!\!\mid_Z \\ \tilde{\alpha} & \leq_Z & \tilde{\beta} \end{array}}$$

First, since $x \lessdot_1 \lambda^{(i'-1,j'-1)} \lessdot_2 \lambda^{(i',j')}$, we have $\mu^{(i'-1,j'-1)} = x \lessdot_1 \lambda^{(i'-1,j'-1)}$.

For some $i < i'$ and $j < j'$, let $\beta = \lambda^{(i-1,j)}$, $\delta = \lambda^{(ij)}$, $\tilde{\beta} = \mu^{(i-1,j)}$, $\tilde{\delta} = \mu^{(ij)}$. Suppose $\tilde{\delta} \leq_1 \delta$. We show $\mu^{(i-1,j)} \leq_1 \lambda^{(i-1,j)}$, that is, $\tilde{\beta} \leq_1 \beta$. We have

$$\tilde{\beta} = x \wedge \beta = x \wedge (\delta \wedge \beta) = (x \wedge \delta) \wedge \beta = \tilde{\delta} \wedge \beta.$$

- $\boldsymbol{\tilde{\delta} = \delta}$**:** Then $\tilde{\beta} = \delta \wedge \beta = \beta$.
- $\boldsymbol{\beta = \tilde{\delta} \lessdot_1 \delta}$**:** Then $\tilde{\beta} = \tilde{\delta} \wedge \beta = \beta$.
- $\boldsymbol{\tilde{\delta} \lessdot_1 \delta = \beta}$**:** Then $\tilde{\beta} = \tilde{\delta} \lessdot_1 \delta = \beta$.
- $\boldsymbol{\tilde{\delta} \lessdot_1 \delta}$ **and** $\boldsymbol{\tilde{\delta} \neq \beta \lessdot \delta}$**:** We have $\delta = 2y \gtrdot_1 \tilde{\delta} \gtrdot_2 y$ and $\delta \gtrdot_2 \beta \gtrdot_1 y$ (there's only one $\gtrdot_1$ lower cover, and $\tilde{\delta} \neq \beta$), so $\tilde{\beta} = y \lessdot_1 \beta$.

Similarly, $\mu^{(i,j-1)} \leq_1 \lambda^{(i,j-1)}$, so we can propagate to obtain $\mu^{(ij)} \leq_1 \lambda^{(ij)}$ for all $i$ and $j$.

Next, we verify that all $M^{(ij)}$ are valid configurations.

When $i > i'$ or $j > j'$ we have $M^{(ij)} = \Lambda^{(ij)}$, so the configuration is valid.

Let $i \leq i'$ and $j = j'$, so $\alpha = \beta = \tilde{\beta}$ and $\gamma = \delta = \tilde{\delta}$. Then

$$\tilde{\alpha} = \alpha \wedge x = (\gamma \wedge \alpha) \wedge x = (\alpha \wedge x) \wedge \gamma = (\beta \wedge x) \wedge \gamma = \beta \wedge \tilde{\gamma} = \tilde{\beta} \wedge \tilde{\gamma},$$

so $M^{(ij)}$ is in a valid configuration.

Similarly, $i = i'$ and $j \leq j'$ yields $M^{(ij)}$ in a valid configuration.

Finally, consider $i < i'$ and $j < j'$. If $\Lambda^{(ij)}$ is not in configuration (5) then $\alpha = \beta \wedge \gamma$ and so $\tilde{\alpha} = \tilde{\beta} \wedge \tilde{\gamma}$ so $M^{(ij)}$ is valid. If $\Lambda^{(ij)}$ is in configuration (5), either $\tilde{\delta} = \delta$ so $M^{(ij)} = \Lambda^{(ij)}$ is valid, or else $\tilde{\delta} \lessdot_1 \delta$. Then $\tilde{\delta} \neq \gamma$ since $\gamma \lessdot_2 \delta$, so $\tilde{\alpha} = \tilde{\beta} = \tilde{\gamma} = \alpha$ and $M^{(ij)}$ is in configuration (7) (since $\alpha \lessdot_2 \tilde{\delta} \lessdot_1 \delta = 2\alpha$). □

**Corollary 9.20.** *Conjecture* 7.6 *holds in semi-primary lattices whose type has two rows.*

*Proof.* That conjecture for semi-primary lattices is identical to the preceding theorem for strongly modular lattices; they coincide for semi-primary lattices whose type has two rows. □

**Theorem 9.21.** *Let $\Lambda = (\lambda^{(ij)})_{0 \le i,j \le n}$ be a meet type table. There is a polynomial $p_\Lambda(r)$ such that the number of pairs of saturated chains $f, f'$ in $Z(r)$ with $f_0 = f'_0 = \emptyset$ and $f_n = f'_n$ and* $\mathrm{Ctype}[\emptyset, f_i \wedge f'_j] = \lambda^{(ij)}$ *for all $0 \le i, j \le n$, is $p_\Lambda(r)$ when $r > 1$. The degree is the number of configuration (6)'s appearing in $\Lambda$, and the maximum degree $n$ is obtained precisely when $\Lambda$ has the form $\Lambda_{\mathrm{ZRS}}(\boldsymbol{P}, \boldsymbol{Q})$ for saturated chains $\boldsymbol{P}$, $\boldsymbol{Q}$ in $Z(1)$ of length $n$ from $\emptyset$ to the same endpoint.*

*Proof.* We form a matrix $X = (x_{ij})_{0 \le i,j \le n}$ of entries in $Z(r)$ that increase in $Z(r)$ as $i$ and $j$ do; have $\mathrm{Ctype}[\emptyset, x_{ij}] = x_{ij}$; and have $x_{i-1,j} \wedge x_{i,j-1} = x_{i-1,j-1}$ for $0 < i, j \le n$. Let $x_{i0} = x_{0i} = \emptyset$ for $0 \le i \le n$. Propagate from the perimeter for increasing $i$ and $j$ to compute the remaining entries of $X$. When $\Lambda^{(ij)}$ is in local configuration $(m)$, propagate according to rule $(m')$ below.

$$(1')\ \boxed{\begin{array}{ccc} x & \lessdot & (y) \\ \| & & \| \\ x & \lessdot & y \end{array}} \qquad (2')\ \boxed{\begin{array}{ccc} y & = & (y) \\ \dot\vee & & \dot\vee \\ x & = & x \end{array}} \qquad (3')\ \boxed{\begin{array}{ccc} x & = & (x) \\ \| & & \| \\ x & = & x \end{array}}$$

$$(4',5')\ \boxed{\begin{array}{ccc} z & \lessdot & (2x) \\ & \diagdown & \\ \dot\vee & \ne & \dot\vee \\ & \diagdown & \\ x & \lessdot & y \end{array}} \qquad (6')\ \boxed{\begin{array}{ccc} 2^k x & \lessdot & 2^k(1_{c_j} x) \\ \| & & \dot\vee \\ 2^k x & = & 2^k x \end{array}} \qquad (7)\ \boxed{\begin{array}{ccc} 2^k 1_m x & \lessdot_2 & (2^k 2x) \\ \| & & \dot\vee_2 \\ 2^k 1_m x & = & 2^k 1_m x \end{array}}$$

In $(1')$, $(2')$, $(3')$, $(4')$, and $(7')$, the value given in the upper right corner is clearly the unique solution of $x_{ij}$ given the other three values and $\mathrm{Ctype}[\emptyset, x_{ij}] = \delta$. In $(6')$, the subscript $c_j$ may be chosen arbitrarily from $\{1, \ldots, r\}$. In $(4')$, since $\beta \ne \gamma$, we already have $x_{i-1,j} \ne x_{i,j-1}$. In $(5')$, we require $x_{i-1,j} \ne x_{i,j-1}$, meaning that the subscripts on their first 1's are different; these words were given to us from earlier propogation steps, so it's a bit late to impose this restriction. Proceed with the construction by ignoring this restriction, and we'll fix it up later. All conditions we require on the entries of $X$ are met except that $x_{i,j-1} \wedge x_{i-1,j} = x_{i-1,j-1}$ may fail when $\Lambda^{(ij)}$ is in configuration (5).

Let $J = \left\{ j : \Lambda^{(ij)} \text{ is in configuration (6) for some } i \right\}$. There is a subscript $c_j$ for each $j \in J$. The restrictions from configuration (5) say that certain $c_j$'s should be unequal; let $R = \{ \{j_1, j_2\} : (5) \Rightarrow c_{j_1} \ne c_{j_2} \}$. Then

$$p_\Lambda(r) = \sum_\pi r(r-1)\cdots(r-k+1)$$

where $\pi$ runs over all set partitions of $J$ in which no block contains both elements of a restricted pair in $R$, and $k$ is the number of blocks of $\pi$. Assign values $1, \ldots, r$ to the $c_j$'s so that $c_{j_1} = c_{j_2}$ iff $j_1$ and $j_2$ are in the same block of $\pi$; the number of ways to do this assignment for a given $\pi$ is $r(r-1)\cdots(r-k+1)$.

The partition into singleton blocks yields the term of maximum degree in the sum, so $\deg(p_\Lambda) = \#J$.

The maximum degree $p_\Lambda$ can have is $n$, when $J = \{1, \ldots, n\}$. The total number of configuration (6) and (7)'s in $\Lambda$ is $n$. For any $\boldsymbol{P}$ and $\boldsymbol{Q}$, the unique $\Lambda$ with right and top perimeter chains $\boldsymbol{P}$ and $\boldsymbol{Q}$ and $n$ configuration (6)'s (hence no (7)'s) is $\Lambda_{\mathrm{ZRS}}(\boldsymbol{P},\boldsymbol{Q})$. $\square$

## Appendix A. Statistical summary of interval type tables computed with Mathematica®

The **weight** $f_\Lambda(q)$ of an interval type table $\Lambda$ is the number of flags realizing it in a $q$-regular semi-primary lattice; it is the polynomial of Conjecture 5.25. Empirically, it usually has a factorization involving powers of $q$ and the following functions.

$$\begin{aligned}
[n] &= (q-1)(q-2)\cdots(q-n)\\
\langle n\rangle &= 1+q+\cdots+q^{n-1} = \frac{q^n-1}{q-1}\\
\langle n\rangle! &= \langle 1\rangle\langle 2\rangle\cdots\langle n\rangle\\
\langle 0\rangle! &= 1\\
M_\lambda(q) &= \prod_j \langle m_j(\lambda)\rangle!
\end{aligned}$$

For $q$-regular semi-primary lattices of various types, the weights of all interval type tables were computed by the method developed in Section 5.5 with Mathematica® (a symbolic mathematics program, ©1993 Wolfram Research, Inc.). We present a brief summary and a longer summary of the results. In all cases, the parameter $q_0$ associated with these polynomials has value 1, so the polynomials enumerate the interval type tables in all $q$-regular semi-primary lattices. The Mathematica computations also show that for the lattice types listed, if $f_\Lambda(q)$ is identically 0 as a polynomial, then $\Lambda$ is not realized even in irregular semi-primary lattices.

### A.1. Brief summary of results.

Key:

- $\lambda$ Type of lattice. Let $n = |\lambda|$.
- $N_r$ Number of interval type tables realized over all lattices of type $\lambda$.
- $N_s$ Number of realized interval type tables that are symmetric.
- $N_h$ Number of realized interval type tables that are hereditary, equal to the number of unordered set partitions of $\{1,\ldots,n\}$ of type $\lambda$. The order total is the $n$th Bell number.
- $N_{sh}$ Number of realized interval type tables that are symmetric and hereditary, equal to the number of unordered set partitions of $\{1,\ldots,n\}$ of type $\lambda$ invariant under the permutation $i \mapsto n+1-i$.
- min Minimum degree of weights of interval type tables, equal to
$$n(\lambda) - n_0(\lambda) = \sum_i (i-1)\lambda_i - \sum_j \binom{m_j(\lambda)}{2}.$$
- max Maximum degree of weights of interval type tables, equal to
$$n(\lambda) = \sum_i (i-1)\lambda_i.$$
- $f^\lambda$ Number of standard Young tableaux of shape $\lambda$.

| $\lambda$ | $N_r$ | $N_s$ | $N_h$ | $N_{sh}$ | min | max | $M_\lambda(q)$ | $f^\lambda$ |
|---|---|---|---|---|---|---|---|---|
| $\emptyset$ | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| **Order 0** | **1** | **1** | **1** | **1** | | | | |
| $(1)$ | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| **Order 1** | **1** | **1** | **1** | **1** | | | | |
| $(2)$ | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| $(1,1)$ | 1 | 1 | 1 | 1 | 1 | 1 | $\langle 2\rangle$ | 1 |
| **Order 2** | **2** | **2** | **2** | **2** | | | | |
| $(3)$ | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| $(2,1)$ | 3 | 1 | 3 | 1 | 0 | 1 | 1 | 2 |
| $(1^3)$ | 1 | 1 | 1 | 1 | 3 | 3 | $\langle 3\rangle!$ | 1 |
| **Order 3** | **5** | **3** | **5** | **3** | | | | |
| $(4)$ | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| $(3,1)$ | 5 | 1 | 4 | 0 | 0 | 1 | 1 | 3 |
| $(2,2)$ | 3 | 3 | 3 | 3 | 1 | 2 | $\langle 2\rangle$ | 2 |
| $(2,1,1)$ | 6 | 2 | 6 | 2 | 1 | 3 | $\langle 2\rangle$ | 3 |
| $(1^4)$ | 1 | 1 | 1 | 1 | 6 | 6 | $\langle 4\rangle!$ | 1 |
| **Order 4** | **16** | **8** | **15** | **7** | | | | |
| $(5)$ | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| $(4,1)$ | 7 | 1 | 5 | 1 | 0 | 1 | 1 | 4 |
| $(3,2)$ | 12 | 2 | 10 | 2 | 0 | 2 | 1 | 5 |
| $(3,1,1)$ | 15 | 3 | 10 | 2 | 1 | 3 | $\langle 2\rangle$ | 6 |
| $(2,2,1)$ | 15 | 3 | 15 | 3 | 1 | 4 | $\langle 2\rangle$ | 5 |
| $(2,1^3)$ | 10 | 2 | 10 | 2 | 3 | 6 | $\langle 3\rangle!$ | 4 |
| $(1^5)$ | 1 | 1 | 1 | 1 | 10 | 10 | $\langle 5\rangle!$ | 1 |
| **Order 5** | **61** | **13** | **52** | **12** | | | | |
| $(6)$ | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| $(5,1)$ | 9 | 1 | 6 | 0 | 0 | 1 | 1 | 5 |
| $(4,2)$ | 26 | 6 | 15 | 3 | 0 | 2 | 1 | 9 |
| $(4,1,1)$ | 28 | 4 | 15 | 3 | 1 | 3 | $\langle 2\rangle$ | 10 |
| $(3,3)$ | 12 | 6 | 10 | 4 | 1 | 3 | $\langle 2\rangle$ | 5 |
| $(3,2,1)$ | 87 | 3 | 60 | 0 | 0 | 4 | 1 | 16 |
| $(3,1^3)$ | 35 | 3 | 20 | 0 | 3 | 6 | $\langle 3\rangle!$ | 10 |
| $(2^3)$ | 15 | 7 | 15 | 7 | 3 | 6 | $\langle 3\rangle!$ | 5 |
| $(2,2,1,1)$ | 45 | 9 | 45 | 9 | 2 | 7 | $\langle 2\rangle^2$ | 9 |
| $(2,1^4)$ | 15 | 3 | 15 | 3 | 6 | 10 | $\langle 4\rangle!$ | 5 |
| $(1^6)$ | 1 | 1 | 1 | 1 | 15 | 15 | $\langle 6\rangle!$ | 1 |
| **Order 6** | **274** | **44** | **203** | **31** | | | | |

| $\lambda$ | $N_r$ | $N_s$ | $N_h$ | $N_{sh}$ | min | max | $M_\lambda(q)$ | $f^\lambda$ |
|---|---|---|---|---|---|---|---|---|
| $(7)$ | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| $(6,1)$ | 11 | 1 | 7 | 1 | 0 | 1 | 1 | 6 |
| $(5,2)$ | 44 | 4 | 21 | 3 | 0 | 2 | 1 | 14 |
| $(5,1,1)$ | 45 | 5 | 21 | 3 | 1 | 3 | $\langle 2\rangle$ | 15 |
| $(4,3)$ | 57 | 3 | 35 | 3 | 0 | 3 | 1 | 14 |
| $(4,2,1)$ | 240 | 6 | 105 | 3 | 0 | 4 | 1 | 35 |
| $(4,1^3)$ | 84 | 4 | 35 | 3 | 3 | 6 | $\langle 3\rangle!$ | 20 |
| $(3,3,1)$ | 128 | 10 | 70 | 4 | 1 | 5 | $\langle 2\rangle$ | 21 |
| $(3,2,2)$ | 148 | 10 | 105 | 9 | 1 | 6 | $\langle 2\rangle$ | 21 |
| $(3,2,1,1)$ | 357 | 9 | 210 | 6 | 1 | 7 | $\langle 2\rangle$ | 35 |
| $(3,1^4)$ | 70 | 6 | 35 | 3 | 6 | 10 | $\langle 4\rangle!$ | 15 |
| $(2^3,1)$ | 105 | 7 | 105 | 7 | 3 | 9 | $\langle 3\rangle!$ | 14 |
| $(2,2,1^3)$ | 105 | 9 | 105 | 9 | 4 | 11 | $\langle 3\rangle!\,\langle 2\rangle$ | 14 |
| $(2,1^5)$ | 21 | 3 | 21 | 3 | 10 | 15 | $\langle 5\rangle!$ | 6 |
| $(1^7)$ | 1 | 1 | 1 | 1 | 21 | 21 | $\langle 7\rangle!$ | 1 |
| **Order 7** | **1417** | **79** | **877** | **59** | | | | |
| $(8)$ | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| $(7,1)$ | 13 | 1 | 8 | 0 | 0 | 1 | 1 | 7 |
| $(6,2)$ | 66 | 8 | 28 | 4 | 0 | 2 | 1 | 20 |
| $(6,1,1)$ | 66 | 6 | 28 | 4 | 1 | 3 | $\langle 2\rangle$ | 21 |
| $(5,3)$ | 145 | 9 | 56 | 0 | 0 | 3 | 1 | 28 |
| $(5,2,1)$ | 493 | 5 | 168 | 0 | 0 | 4 | 1 | 64 |
| $(5,1^3)$ | 165 | 5 | 56 | 0 | 3 | 6 | $\langle 3\rangle!$ | 35 |
| $(4,4)$ | 57 | 19 | 35 | 11 | 1 | 4 | $\langle 2\rangle$ | 14 |
| $(4,3,1)$ | 769 | 7 | 280 | 0 | 0 | 5 | 1 | 70 |
| $(4,2,2)$ | 580 | 38 | 210 | 18 | 1 | 6 | $\langle 2\rangle$ | 56 |
| $(4,2,1,1)$ | 1224 | 30 | 420 | 12 | 1 | 7 | $\langle 2\rangle$ | 90 |
| $(4,1^4)$ | 210 | 10 | 70 | 6 | 6 | 10 | $\langle 4\rangle!$ | 35 |
| $(3,3,2)$ | 475 | 27 | 280 | 16 | 1 | 7 | $\langle 2\rangle$ | 42 |
| $(3,3,1,1)$ | 725 | 37 | 280 | 16 | 2 | 8 | $\langle 2\rangle^2$ | 56 |
| $(3,2,2,1)$ | 1394 | 18 | 840 | 0 | 1 | 9 | $\langle 2\rangle$ | 70 |
| $(3,2,1^3)$ | 1092 | 12 | 560 | 0 | 3 | 11 | $\langle 3\rangle!$ | 64 |
| $(3,1^5)$ | 126 | 6 | 56 | 0 | 10 | 15 | $\langle 5\rangle!$ | 21 |
| $(2^4)$ | 105 | 25 | 105 | 25 | 6 | 12 | $\langle 4\rangle!$ | 14 |
| $(2^3,1,1)$ | 420 | 28 | 420 | 28 | 4 | 13 | $\langle 3\rangle!\,\langle 2\rangle$ | 28 |
| $(2,2,1^4)$ | 210 | 18 | 210 | 18 | 7 | 16 | $\langle 4\rangle!\,\langle 2\rangle$ | 20 |
| $(2,1^6)$ | 28 | 4 | 28 | 4 | 15 | 21 | $\langle 6\rangle!$ | 7 |
| $(1^8)$ | 1 | 1 | 1 | 1 | 28 | 28 | $\langle 8\rangle!$ | 1 |
| **Order 8** | **8365** | **315** | **4140** | **164** | | | | |

| $\lambda$ | $N_r$ | $N_s$ | $N_h$ | $N_{sh}$ | min | max | $M_\lambda(q)$ | $f^\lambda$ |
|---|---|---|---|---|---|---|---|---|
| $(9)$ | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| $(6,3)$ | 278 | 6 | 84 | 4 | 0 | 3 | 1 | 48 |
| $(5,4)$ | 303 | 7 | 126 | 6 | 0 | 4 | 1 | 42 |
| $(5,1^4)$ | 495 | 15 | 126 | 6 | 6 | 10 | $\langle 4\rangle!$ | 70 |
| $(4,1^5)$ | 462 | 10 | 126 | 6 | 10 | 15 | $\langle 5\rangle!$ | 56 |
| $(3^3)$ | 475 | 27 | 280 | 16 | 3 | 9 | $\langle 3\rangle!$ | 42 |
| $(2^4,1)$ | 945 | 25 | 945 | 25 | 6 | 16 | $\langle 4\rangle!$ | 42 |
| $(1^9)$ | 1 | 1 | 1 | 1 | 36 | 36 | $\langle 9\rangle!$ | 1 |
| **Order 9** | **incomplete** | | | | | | | |
| $(10)$ | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| $(5,5)$ | 303 | 45 | 126 | 16 | 1 | 5 | $\langle 2\rangle$ | 42 |
| $(5,1^5)$ | 1287 | 15 | 252 | 0 | 10 | 15 | $\langle 5\rangle!$ | 126 |
| $(1^{10})$ | 1 | 1 | 1 | 1 | 45 | 45 | $\langle 10\rangle!$ | 1 |
| **Order 10** | **incomplete** | | | | | | | |

**A.2. Special classes of types.** For hooks $\lambda = (i, 1^j)$ (with $i > 0$ and $j \geq 0$), the formula $N_r = \binom{2i+j-2}{j}$ empirically holds when $1 \leq i \leq 5$ and $0 \leq j \leq 5$.

For two column partitions $\lambda = (i, j)'$ (with $i \geq j \geq 0$), all interval type tables are hereditary by Theorem 5.53, so $N_r = N_h = n!/(j!(i-j)!2^j)$.

For two row partitions $(i, j)$ (with $i \geq j \geq 0$), the following values of $N_r$ were computed.

| $j \backslash i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | | 1 | 3 | 5 | 7 | 9 | 11 |
| 2 | | | 3 | 12 | 26 | 44 | 66 |
| 3 | | | | 12 | 57 | 145 | 278 |
| 4 | | | | | 57 | 303 | n/a |
| 5 | | | | | | 303 | n/a |

**A.3. Extended summary of results.** We now list all the weights occurring for each type. The individual interval type tables $\Lambda$ are not listed, but counts of how many $\Lambda$ share common characteristics are given.

- $f_\Lambda(q)$ Number of flags realizing interval type table $\Lambda$ in a $q$-regular semi-primary lattice.
- $d$ Degree of $f_\Lambda(q)$.
- $N_r$ Number of realized interval type tables with this weight.
- $N_s$ Number of realized interval type tables with this weight that are symmetric.

The tables below have the following components.

| $M_\lambda(q)$ | | | $\lambda$ |
|---|---|---|---|
| degree of $f_\Lambda(q)$ | $f_\Lambda(q)/M_\lambda(q)$ | # of $\Lambda$ of this form | # of symmetric $\Lambda$ |
| degree of $f_\Lambda(q)$ | $f_\Lambda(q)/M_\lambda(q)$ | # of $\Lambda$ of this form | # of symmetric $\Lambda$ |
| ... | ... | ... | ... |

When $M_\lambda(q) = 1$, it is omitted.

| $M_\lambda$ | | | $\lambda$ |
|---|---|---|---|
| $d$ | $f_\Lambda/M_\lambda$ | $N_r$ | $N_s$ |
| | | | $\emptyset$ |
| 0 | 1 | 1 | 1 |
| | | | $(1)$ |
| 0 | 1 | 1 | 1 |
| | | | $(2)$ |
| 0 | 1 | 1 | 1 |
| | $\langle 2\rangle$ | | $(1,1)$ |
| 1 | 1 | 1 | 1 |
| | | | $(3)$ |
| 0 | 1 | 1 | 1 |
| | | | $(2,1)$ |
| 1 | $q$ | 2 | 0 |
| 0 | 1 | 1 | 1 |
| | $\langle 3\rangle!$ | | $(1^3)$ |
| 3 | 1 | 1 | 1 |
| | | | $(4)$ |
| 0 | 1 | 1 | 1 |
| | | | $(3,1)$ |
| 1 | $q$ | 2 | 0 |
| 1 | $[1]$ | 1 | 1 |
| 0 | 1 | 2 | 0 |
| | $\langle 2\rangle$ | | $(2,2)$ |
| 2 | $q$ | 2 | 2 |
| 1 | 1 | 1 | 1 |
| | $\langle 2\rangle$ | | $(2,1,1)$ |
| 3 | $q^2$ | 3 | 1 |
| 2 | $q$ | 2 | 0 |
| 1 | 1 | 1 | 1 |
| | $\langle 4\rangle!$ | | $(1^4)$ |
| 6 | 1 | 1 | 1 |
| | | | $(5)$ |
| 0 | 1 | 1 | 1 |

| $M_\lambda$ | | | $\lambda$ |
|---|---|---|---|
| $d$ | $f_\Lambda/M_\lambda$ | $N_r$ | $N_s$ |
| | | | $(4,1)$ |
| 1 | $q$ | 2 | 0 |
| 1 | $[1]$ | 2 | 0 |
| 0 | 1 | 3 | 1 |
| | | | $(3,2)$ |
| 2 | $q^2$ | 3 | 1 |
| 2 | $q[1]$ | 2 | 0 |
| 1 | $q$ | 6 | 0 |
| 0 | 1 | 1 | 1 |
| | $\langle 2\rangle$ | | $(3,1,1)$ |
| 3 | $q^2$ | 3 | 1 |
| 3 | $q[1]$ | 3 | 1 |
| 2 | $q$ | 4 | 0 |
| 2 | $[1]$ | 2 | 0 |
| 1 | 1 | 3 | 1 |
| | $\langle 2\rangle$ | | $(2,2,1)$ |
| 4 | $q^3$ | 5 | 1 |
| 3 | $q^2$ | 6 | 0 |
| 2 | $q$ | 3 | 1 |
| 1 | 1 | 1 | 1 |
| | $\langle 3\rangle!$ | | $(2,1^3)$ |
| 6 | $q^3$ | 4 | 0 |
| 5 | $q^2$ | 3 | 1 |
| 4 | $q$ | 2 | 0 |
| 3 | 1 | 1 | 1 |
| | $\langle 5\rangle!$ | | $(1^5)$ |
| 10 | 1 | 1 | 1 |
| | | | $(6)$ |
| 0 | 1 | 1 | 1 |
| | | | $(5,1)$ |
| 1 | $q$ | 2 | 0 |
| 1 | $[1]$ | 3 | 1 |
| 0 | 1 | 4 | 0 |

| $M_\lambda$ | | | $\lambda$ |
|---|---|---|---|
| $d$ | $f_\Lambda/M_\lambda$ | $N_r$ | $N_s$ |
| | | | $(4,2)$ |
| 2 | $q^2$ | 3 | 1 |
| 2 | $q[1]$ | 5 | 1 |
| 2 | $[2]$ | 1 | 1 |
| 1 | $q$ | 8 | 0 |
| 1 | $[1]$ | 5 | 1 |
| 0 | 1 | 4 | 2 |
| | $\langle 2\rangle$ | | $(4,1,1)$ |
| 3 | $q^2$ | 3 | 1 |
| 3 | $q[1]$ | 6 | 0 |
| 3 | $[1]^2$ | 1 | 1 |
| 2 | $q$ | 6 | 0 |
| 2 | $[1]$ | 6 | 0 |
| 1 | 1 | 6 | 2 |
| | $\langle 2\rangle$ | | $(3,3)$ |
| 3 | $q^2$ | 3 | 1 |
| 3 | $q[1]$ | 2 | 2 |
| 2 | $q$ | 6 | 2 |
| 1 | 1 | 1 | 1 |
| | | | $(3,2,1)$ |
| 4 | $q^4$ | 8 | 0 |
| 4 | $q^3[1]$ | 8 | 0 |
| 3 | $q^3$ | 20 | 0 |
| 3 | $q^2[1]$ | 12 | 2 |
| 2 | $q^2$ | 20 | 0 |
| 2 | $q[1]$ | 6 | 0 |
| 1 | $q$ | 10 | 0 |
| 1 | $[1]$ | 1 | 1 |
| 0 | 1 | 2 | 0 |
| | $\langle 3\rangle!$ | | $(3,1^3)$ |
| 6 | $q^3$ | 4 | 0 |
| 6 | $q^2[1]$ | 6 | 2 |
| 5 | $q^2$ | 6 | 0 |
| 5 | $q[1]$ | 6 | 0 |
| 4 | $q$ | 6 | 0 |
| 4 | $[1]$ | 3 | 1 |
| 3 | 1 | 4 | 0 |

| $M_\lambda$ | | | $\lambda$ |
|---|---|---|---|
| $d$ | $f_\Lambda/M_\lambda$ | $N_r$ | $N_s$ |
| $\langle 3\rangle!$ | | | $(2^3)$ |
| 6 | $q^3$ | 5 | 3 |
| 5 | $q^2$ | 6 | 2 |
| 4 | $q$ | 3 | 1 |
| 3 | 1 | 1 | 1 |
| $\langle 2\rangle^2$ | | | $(2,2,1,1)$ |
| 7 | $q^5$ | 9 | 3 |
| 6 | $q^4$ | 13 | 1 |
| 5 | $q^3$ | 12 | 2 |
| 4 | $q^2$ | 7 | 1 |
| 3 | $q$ | 3 | 1 |
| 2 | 1 | 1 | 1 |
| $\langle 4\rangle!$ | | | $(2,1^4)$ |
| 10 | $q^4$ | 5 | 1 |
| 9 | $q^3$ | 4 | 0 |
| 8 | $q^2$ | 3 | 1 |
| 7 | $q$ | 2 | 0 |
| 6 | 1 | 1 | 1 |
| $\langle 6\rangle!$ | | | $(1^6)$ |
| 15 | 1 | 1 | 1 |
| | | | $(7)$ |
| 0 | 1 | 1 | 1 |
| | | | $(6,1)$ |
| 1 | $q$ | 2 | 0 |
| 1 | $[1]$ | 4 | 0 |
| 0 | 1 | 5 | 1 |
| | | | $(5,2)$ |
| 2 | $q^2$ | 3 | 1 |
| 2 | $q[1]$ | 8 | 0 |
| 2 | $[1]^2$ | 1 | 1 |
| 2 | $[2]$ | 2 | 0 |
| 1 | $q$ | 10 | 0 |
| 1 | $[1]$ | 12 | 0 |
| 0 | 1 | 8 | 2 |

| $M_\lambda$ | | | $\lambda$ |
|---|---|---|---|
| $d$ | $f_\Lambda/M_\lambda$ | $N_r$ | $N_s$ |
| $\langle 2\rangle$ | | | $(5,1,1)$ |
| 3 | $q^2$ | 3 | 1 |
| 3 | $q[1]$ | 9 | 1 |
| 3 | $[1]^2$ | 3 | 1 |
| 2 | $q$ | 8 | 0 |
| 2 | $[1]$ | 12 | 0 |
| 1 | 1 | 10 | 2 |
| | | | $(4,3)$ |
| 3 | $q^3$ | 4 | 0 |
| 3 | $q^2[1]$ | 8 | 0 |
| 3 | $q[2]$ | 2 | 0 |
| 2 | $q^2$ | 16 | 2 |
| 2 | $q[1]$ | 12 | 0 |
| 1 | $q$ | 14 | 0 |
| 0 | 1 | 1 | 1 |
| | | | $(4,2,1)$ |
| 4 | $q^4$ | 8 | 0 |
| 4 | $q^3[1]$ | 19 | 1 |
| 4 | $q^2[1]^2$ | 4 | 0 |
| 4 | $q[1]^3$ | 2 | 0 |
| 4 | $q^2[2]$ | 2 | 0 |
| 3 | $q^3$ | 26 | 0 |
| 3 | $q^2[1]$ | 44 | 0 |
| 3 | $q[1]^2$ | 8 | 0 |
| 3 | $q[2]$ | 4 | 0 |
| 2 | $q^2$ | 37 | 1 |
| 2 | $q[1]$ | 38 | 0 |
| 2 | $[1]^2$ | 2 | 0 |
| 2 | $[2]$ | 1 | 1 |
| 1 | $q$ | 26 | 0 |
| 1 | $[1]$ | 11 | 1 |
| 0 | 1 | 8 | 2 |

| $M_\lambda$ | | | $\lambda$ |
|---|---|---|---|
| $d$ | $f_\Lambda/M_\lambda$ | $N_r$ | $N_s$ |
| $\langle 3\rangle!$ | | | $(4,1^3)$ |
| 6 | $q^3$ | 4 | 0 |
| 6 | $q^2[1]$ | 12 | 0 |
| 6 | $q[1]^2$ | 4 | 0 |
| 5 | $q^2$ | 9 | 1 |
| 5 | $q[1]$ | 18 | 0 |
| 5 | $[1]^2$ | 3 | 1 |
| 4 | $q$ | 12 | 0 |
| 4 | $[1]$ | 12 | 0 |
| 3 | 1 | 10 | 2 |
| $\langle 2\rangle$ | | | $(3,3,1)$ |
| 5 | $q^4$ | 7 | 1 |
| 5 | $q^3[1]$ | 9 | 1 |
| 5 | $q\,(q^2-q+1)$ $\times[1]$ | 1 | 1 |
| 5 | $q^2[1]^2$ | 4 | 0 |
| 4 | $q^3$ | 22 | 0 |
| 4 | $q^2[1]$ | 20 | 0 |
| 4 | $q[1]^2$ | 5 | 1 |
| 3 | $q^2$ | 22 | 0 |
| 3 | $q[1]$ | 17 | 3 |
| 2 | $q$ | 16 | 2 |
| 2 | $[1]$ | 2 | 0 |
| 1 | 1 | 3 | 1 |
| $\langle 2\rangle$ | | | $(3,2,2)$ |
| 6 | $q^5$ | 10 | 2 |
| 6 | $q^4[1]$ | 10 | 0 |
| 6 | $q^3[1]^2$ | 1 | 1 |
| 5 | $q^4$ | 27 | 1 |
| 5 | $q^3[1]$ | 20 | 0 |
| 4 | $q^3$ | 37 | 3 |
| 4 | $q^2[1]$ | 10 | 0 |
| 3 | $q^2$ | 23 | 1 |
| 3 | $q[1]$ | 2 | 0 |
| 2 | $q$ | 7 | 1 |
| 1 | 1 | 1 | 1 |

| $M_\lambda$ | | | $\lambda$ |
|---|---|---|---|
| $d$ | $f_\Lambda/M_\lambda$ | $N_r$ | $N_s$ |
| | $\langle 2\rangle$ | | $(3,2,1,1)$ |
| 7 | $q^6$ | 15 | 1 |
| 7 | $q^5[1]$ | 20 | 0 |
| 6 | $q^5$ | 40 | 0 |
| 6 | $q^4[1]$ | 42 | 0 |
| 5 | $q^4$ | 54 | 2 |
| 5 | $q^3[1]$ | 46 | 2 |
| 4 | $q^3$ | 54 | 0 |
| 4 | $q^2[1]$ | 26 | 0 |
| 3 | $q^2$ | 30 | 2 |
| 3 | $q[1]$ | 11 | 1 |
| 2 | $q$ | 14 | 0 |
| 2 | $[1]$ | 2 | 0 |
| 1 | 1 | 3 | 1 |
| | $\langle 4\rangle!$ | | $(3,1^4)$ |
| 10 | $q^4$ | 5 | 1 |
| 10 | $q^3[1]$ | 10 | 2 |
| 9 | $q^3$ | 8 | 0 |
| 9 | $q^2[1]$ | 12 | 0 |
| 8 | $q^2$ | 9 | 1 |
| 8 | $q[1]$ | 9 | 1 |
| 7 | $q$ | 8 | 0 |
| 7 | $[1]$ | 4 | 0 |
| 6 | 1 | 5 | 1 |
| | $\langle 3\rangle!$ | | $(2^3,1)$ |
| 9 | $q^6$ | 14 | 0 |
| 8 | $q^5$ | 28 | 2 |
| 7 | $q^4$ | 28 | 0 |
| 6 | $q^3$ | 20 | 2 |
| 5 | $q^2$ | 10 | 2 |
| 4 | $q$ | 4 | 0 |
| 3 | 1 | 1 | 1 |
| | $\langle 3\rangle!\langle 2\rangle$ | | $(2,2,1^3)$ |
| 11 | $q^7$ | 14 | 2 |
| 10 | $q^6$ | 22 | 0 |
| 9 | $q^5$ | 24 | 2 |
| 8 | $q^4$ | 21 | 1 |
| 7 | $q^3$ | 13 | 1 |
| 6 | $q^2$ | 7 | 1 |
| 5 | $q$ | 3 | 1 |
| 4 | 1 | 1 | 1 |

| $M_\lambda$ | | | $\lambda$ |
|---|---|---|---|
| $d$ | $f_\Lambda/M_\lambda$ | $N_r$ | $N_s$ |
| | $\langle 5\rangle!$ | | $(2,1^5)$ |
| 15 | $q^5$ | 6 | 0 |
| 14 | $q^4$ | 5 | 1 |
| 13 | $q^3$ | 4 | 0 |
| 12 | $q^2$ | 3 | 1 |
| 11 | $q$ | 2 | 0 |
| 10 | 1 | 1 | 1 |
| | $\langle 7\rangle!$ | | $(1^7)$ |
| 21 | 1 | 1 | 1 |
| | | | $(8)$ |
| 0 | 1 | 1 | 1 |
| | | | $(7,1)$ |
| 1 | $q$ | 2 | 0 |
| 1 | $[1]$ | 5 | 1 |
| 0 | 1 | 6 | 0 |
| | | | $(6,2)$ |
| 2 | $q^2$ | 3 | 1 |
| 2 | $q[1]$ | 11 | 1 |
| 2 | $[1]^2$ | 3 | 1 |
| 2 | $[2]$ | 3 | 1 |
| 1 | $q$ | 12 | 0 |
| 1 | $[1]$ | 21 | 1 |
| 0 | 1 | 13 | 3 |
| | $\langle 2\rangle$ | | $(6,1,1)$ |
| 3 | $q^2$ | 3 | 1 |
| 3 | $q[1]$ | 12 | 0 |
| 3 | $[1]^2$ | 6 | 2 |
| 2 | $q$ | 10 | 0 |
| 2 | $[1]$ | 20 | 0 |
| 1 | 1 | 15 | 3 |

| $M_\lambda$ | | | $\lambda$ |
|---|---|---|---|
| $d$ | $f_\Lambda/M_\lambda$ | $N_r$ | $N_s$ |
| | | | $(5,3)$ |
| 3 | $q^3$ | 4 | 0 |
| 3 | $q^2[1]$ | 14 | 2 |
| 3 | $q[1]^2$ | 3 | 1 |
| 3 | $q[2]$ | 6 | 0 |
| 3 | $[3]$ | 1 | 1 |
| 2 | $q^2$ | 18 | 0 |
| 2 | $q[1]$ | 37 | 1 |
| 2 | $[2]$ | 9 | 1 |
| 1 | $q$ | 26 | 0 |
| 1 | $[1]$ | 19 | 3 |
| 0 | 1 | 8 | 0 |
| | | | $(5,2,1)$ |
| 4 | $q^4$ | 8 | 0 |
| 4 | $q^3[1]$ | 30 | 0 |
| 4 | $q^2[1]^2$ | 16 | 0 |
| 4 | $q[1]^3$ | 4 | 0 |
| 4 | $q^2[2]$ | 4 | 0 |
| 4 | $q[1][2]$ | 2 | 0 |
| 3 | $q^3$ | 32 | 0 |
| 3 | $q^2[1]$ | 88 | 2 |
| 3 | $q[1]^2$ | 32 | 0 |
| 3 | $[1]^3$ | 1 | 1 |
| 3 | $q[2]$ | 10 | 0 |
| 2 | $q^2$ | 58 | 0 |
| 2 | $q[1]$ | 96 | 0 |
| 2 | $[1]^2$ | 10 | 0 |
| 2 | $[2]$ | 2 | 0 |
| 1 | $q$ | 52 | 0 |
| 1 | $[1]$ | 30 | 2 |
| 0 | 1 | 18 | 0 |
| | $\langle 3\rangle!$ | | $(5,1^3)$ |
| 6 | $q^3$ | 4 | 0 |
| 6 | $q^2[1]$ | 18 | 2 |
| 6 | $q[1]^2$ | 12 | 0 |
| 6 | $[1]^3$ | 1 | 1 |
| 5 | $q^2$ | 12 | 0 |
| 5 | $q[1]$ | 36 | 0 |
| 5 | $[1]^2$ | 12 | 0 |
| 4 | $q$ | 20 | 0 |
| 4 | $[1]$ | 30 | 2 |
| 3 | 1 | 20 | 0 |

| $M_\lambda$ | | | $\lambda$ |
|---|---|---|---|
| $d$ | $f_\Lambda/M_\lambda$ | $N_r$ | $N_s$ |
| | $\langle 2\rangle$ | | $(4,4)$ |
| 4 | $q^3$ | 4 | 2 |
| 4 | $q^2[1]$ | 8 | 2 |
| 4 | $q[2]$ | 2 | 2 |
| 3 | $q^2$ | 16 | 2 |
| 3 | $q[1]$ | 12 | 4 |
| 2 | $q$ | 14 | 6 |
| 1 | $1$ | 1 | 1 |
| | | | $(4,3,1)$ |
| 5 | $q^5$ | 10 | 0 |
| 5 | $q^4[1]$ | 27 | 1 |
| 5 | $q^2\,(q^2-q+1)$ | | |
| | $\times[1]$ | 2 | 0 |
| 5 | $q^3[1]^2$ | 18 | 0 |
| 5 | $q^2[1]^3$ | 5 | 1 |
| 5 | $q^3[2]$ | 4 | 0 |
| 5 | $q\,(q^2-2q+2)$ | | |
| | $\times[1]^2$ | 2 | 0 |
| 5 | $q^2[1][2]$ | 2 | 0 |
| 4 | $q^4$ | 48 | 0 |
| 4 | $q^3[1]$ | 100 | 0 |
| 4 | $q\,(q^2-q+1)$ | | |
| | $\times[1]$ | 2 | 0 |
| 4 | $q^2[1]^2$ | 46 | 0 |
| 4 | $q[1]^3$ | 8 | 0 |
| 4 | $q^2[2]$ | 6 | 0 |
| 4 | $q[1][2]$ | 6 | 0 |
| 3 | $q^3$ | 88 | 0 |
| 3 | $q^2[1]$ | 124 | 4 |
| 3 | $q[1]^2$ | 44 | 0 |
| 3 | $q[2]$ | 6 | 0 |
| 2 | $q^2$ | 80 | 0 |
| 2 | $q[1]$ | 84 | 0 |
| 1 | $q$ | 50 | 0 |
| 1 | $[1]$ | 3 | 1 |
| 0 | $1$ | 4 | 0 |

| $M_\lambda$ | | | $\lambda$ |
|---|---|---|---|
| $d$ | $f_\Lambda/M_\lambda$ | $N_r$ | $N_s$ |
| | $\langle 2\rangle$ | | $(4,2,2)$ |
| 6 | $q^5$ | 10 | 2 |
| 6 | $q^4[1]$ | 26 | 2 |
| 6 | $q^3[1]^2$ | 11 | 1 |
| 6 | $q^2[1]^3$ | 3 | 1 |
| 6 | $(q^3-2q^2+q-1)$ | | |
| | $\times q[1]$ | 1 | 1 |
| 6 | $q^3[2]$ | 3 | 1 |
| 6 | $q^2[1][2]$ | 2 | 0 |
| 5 | $q^4$ | 35 | 1 |
| 5 | $q^3[1]$ | 79 | 1 |
| 5 | $q^2[1]^2$ | 23 | 1 |
| 5 | $q[1]^3$ | 6 | 2 |
| 5 | $q^2[2]$ | 12 | 0 |
| 4 | $q^3$ | 63 | 5 |
| 4 | $q^2[1]$ | 100 | 2 |
| 4 | $q[1]^2$ | 15 | 1 |
| 4 | $q[2]$ | 10 | 2 |
| 3 | $q^2$ | 60 | 2 |
| 3 | $q[1]$ | 61 | 3 |
| 3 | $[1]^2$ | 2 | 2 |
| 3 | $[2]$ | 2 | 0 |
| 2 | $q$ | 33 | 5 |
| 2 | $[1]$ | 14 | 0 |
| 1 | $1$ | 9 | 3 |

| $M_\lambda$ | | | $\lambda$ |
|---|---|---|---|
| $d$ | $f_\Lambda/M_\lambda$ | $N_r$ | $N_s$ |
| | $\langle 2\rangle$ | | $(4,2,1,1)$ |
| 7 | $q^6$ | 15 | 1 |
| 7 | $q^5[1]$ | 46 | 2 |
| 7 | $q^4[1]^2$ | 18 | 0 |
| 7 | $q^2\,(q^2-q+1)$ | | |
| | $\times[1]^2$ | 1 | 1 |
| 7 | $q^3[1]^3$ | 7 | 1 |
| 7 | $q^4[2]$ | 3 | 1 |
| 6 | $q^5$ | 52 | 0 |
| 6 | $q^4[1]$ | 133 | 1 |
| 6 | $q^3[1]^2$ | 46 | 0 |
| 6 | $q^2[1]^3$ | 11 | 1 |
| 6 | $q^3[2]$ | 8 | 0 |
| 5 | $q^4$ | 93 | 3 |
| 5 | $q^3[1]$ | 190 | 2 |
| 5 | $q^2[1]^2$ | 48 | 2 |
| 5 | $q[1]^3$ | 2 | 0 |
| 5 | $q^2[2]$ | 12 | 2 |
| 4 | $q^3$ | 116 | 0 |
| 4 | $q^2[1]$ | 156 | 2 |
| 4 | $q[1]^2$ | 22 | 0 |
| 4 | $q[2]$ | 4 | 0 |
| 3 | $q^2$ | 87 | 5 |
| 3 | $q[1]$ | 72 | 0 |
| 3 | $[1]^2$ | 5 | 1 |
| 3 | $[2]$ | 1 | 1 |
| 2 | $q$ | 44 | 0 |
| 2 | $[1]$ | 19 | 1 |
| 1 | $1$ | 13 | 3 |
| | $\langle 4\rangle\,!$ | | $(4,1^4)$ |
| 10 | $q^4$ | 5 | 1 |
| 10 | $q^3[1]$ | 20 | 0 |
| 10 | $q^2[1]^2$ | 10 | 2 |
| 9 | $q^3$ | 12 | 0 |
| 9 | $q^2[1]$ | 36 | 0 |
| 9 | $q[1]^2$ | 12 | 0 |
| 8 | $q^2$ | 18 | 2 |
| 8 | $q[1]$ | 36 | 0 |
| 8 | $[1]^2$ | 6 | 2 |
| 7 | $q$ | 20 | 0 |
| 7 | $[1]$ | 20 | 0 |
| 6 | $1$ | 15 | 3 |

| $M_\lambda$ | | | $\lambda$ |
|---|---|---|---|
| $d$ | $f_\Lambda/M_\lambda$ | $N_r$ | $N_s$ |
| | $\langle 2\rangle$ | | $(3,3,2)$ |
| 7 | $q^6$ | 12 | 2 |
| 7 | $q^5[1]$ | 18 | 2 |
| 7 | $q^3(q^2-q+1)$ $\times[1]$ | 3 | 1 |
| 7 | $q^4[1]^2$ | 9 | 1 |
| 6 | $q^5$ | 54 | 2 |
| 6 | $q^4[1]$ | 54 | 0 |
| 6 | $q^3[1]^2$ | 18 | 2 |
| 5 | $q^4$ | 81 | 3 |
| 5 | $q^3[1]$ | 72 | 4 |
| 4 | $q^3$ | 87 | 5 |
| 4 | $q^2[1]$ | 18 | 0 |
| 3 | $q^2$ | 36 | 2 |
| 3 | $q[1]$ | 3 | 1 |
| 2 | $q$ | 9 | 1 |
| 1 | 1 | 1 | 1 |

| $M_\lambda$ | | | $\lambda$ |
|---|---|---|---|
| $d$ | $f_\Lambda/M_\lambda$ | $N_r$ | $N_s$ |
| | $\langle 2\rangle^2$ | | $(3,3,1,1)$ |
| 8 | $q^6$ | 12 | 2 |
| 8 | $q^5[1]$ | 24 | 2 |
| 8 | $q^3(q^2-q+1)$ $\times[1]$ | 2 | 0 |
| 8 | $q^4[1]^2$ | 17 | 3 |
| 8 | $q^3[1]^3$ | 1 | 1 |
| 7 | $q^5$ | 42 | 2 |
| 7 | $q^4[1]$ | 70 | 0 |
| 7 | $q^2(q^2-q+1)$ $\times[1]$ | 4 | 0 |
| 7 | $q^3[1]^2$ | 39 | 3 |
| 6 | $q^4$ | 65 | 3 |
| 6 | $q^3[1]$ | 105 | 3 |
| 6 | $q(q^2-q+1)$ $\times[1]$ | 2 | 0 |
| 6 | $q^2[1]^2$ | 32 | 2 |
| 6 | $q[1]^3$ | 1 | 1 |
| 5 | $q^3$ | 75 | 3 |
| 5 | $q^2[1]$ | 78 | 0 |
| 5 | $q[1]^2$ | 18 | 2 |
| 4 | $q^2$ | 49 | 1 |
| 4 | $q[1]$ | 45 | 3 |
| 4 | $[1]^2$ | 1 | 1 |
| 3 | $q$ | 31 | 3 |
| 3 | $[1]$ | 6 | 0 |
| 2 | 1 | 6 | 2 |

| $M_\lambda$ | | | $\lambda$ |
|---|---|---|---|
| $d$ | $f_\Lambda/M_\lambda$ | $N_r$ | $N_s$ |
| | $\langle 2\rangle$ | | $(3,2,2,1)$ |
| 9 | $q^8$ | 30 | 0 |
| 9 | $q^7[1]$ | 38 | 2 |
| 9 | $q^6[1]^2$ | 2 | 0 |
| 8 | $q^7$ | 102 | 0 |
| 8 | $q^6[1]$ | 110 | 0 |
| 8 | $q^5[1]^2$ | 4 | 0 |
| 7 | $q^4(2q-1)$ $\times[1]$ | 1 | 1 |
| 7 | $q^6$ | 176 | 0 |
| 7 | $q^5[1]$ | 154 | 6 |
| 7 | $q^4[1]^2$ | 2 | 0 |
| 6 | $q^5$ | 208 | 0 |
| 6 | $q^4[1]$ | 132 | 2 |
| 5 | $q^4$ | 168 | 0 |
| 5 | $q^3[1]$ | 73 | 3 |
| 4 | $q^3$ | 98 | 0 |
| 4 | $q^2[1]$ | 30 | 2 |
| 3 | $q^2$ | 44 | 0 |
| 3 | $q[1]$ | 7 | 1 |
| 2 | $q$ | 12 | 0 |
| 2 | $[1]$ | 1 | 1 |
| 1 | 1 | 2 | 0 |
| | $\langle 3\rangle!$ | | $(3,2,1^3)$ |
| 11 | $q^8$ | 24 | 0 |
| 11 | $q^7[1]$ | 40 | 0 |
| 10 | $q^7$ | 66 | 0 |
| 10 | $q^6[1]$ | 97 | 3 |
| 9 | $q^6$ | 100 | 0 |
| 9 | $q^5[1]$ | 132 | 0 |
| 8 | $q^5$ | 122 | 0 |
| 8 | $q^4[1]$ | 123 | 5 |
| 7 | $q^4$ | 112 | 0 |
| 7 | $q^3[1]$ | 80 | 0 |
| 6 | $q^3$ | 74 | 0 |
| 6 | $q^2[1]$ | 41 | 3 |
| 5 | $q^2$ | 40 | 0 |
| 5 | $q[1]$ | 16 | 0 |
| 4 | $q$ | 18 | 0 |
| 4 | $[1]$ | 3 | 1 |
| 3 | 1 | 4 | 0 |

| $M_\lambda$ | | | $\lambda$ |
|---|---|---|---|
| $d$ | $f_\Lambda/M_\lambda$ | $N_r$ | $N_s$ |
| | $\langle 5\rangle!$ | | $(3,1^5)$ |
| 15 | $q^5$ | 6 | 0 |
| 15 | $q^4[1]$ | 15 | 3 |
| 14 | $q^4$ | 10 | 0 |
| 14 | $q^3[1]$ | 20 | 0 |
| 13 | $q^3$ | 12 | 0 |
| 13 | $q^2[1]$ | 18 | 2 |
| 12 | $q^2$ | 12 | 0 |
| 12 | $q[1]$ | 12 | 0 |
| 11 | $q$ | 10 | 0 |
| 11 | $[1]$ | 5 | 1 |
| 10 | 1 | 6 | 0 |
| | $\langle 4\rangle!$ | | $(2^4)$ |
| 12 | $q^6$ | 14 | 6 |
| 11 | $q^5$ | 28 | 6 |
| 10 | $q^4$ | 28 | 4 |
| 9 | $q^3$ | 20 | 4 |
| 8 | $q^2$ | 10 | 2 |
| 7 | $q$ | 4 | 2 |
| 6 | 1 | 1 | 1 |
| | $\langle 3\rangle!\langle 2\rangle$ | | $(2^3,1,1)$ |
| 13 | $q^9$ | 28 | 4 |
| 12 | $q^8$ | 64 | 2 |
| 11 | $q^7$ | 88 | 6 |
| 10 | $q^6$ | 87 | 3 |
| 9 | $q^5$ | 68 | 4 |
| 8 | $q^4$ | 45 | 3 |
| 7 | $q^3$ | 24 | 2 |
| 6 | $q^2$ | 11 | 3 |
| 5 | $q$ | 4 | 0 |
| 4 | 1 | 1 | 1 |

| $M_\lambda$ | | | $\lambda$ |
|---|---|---|---|
| $d$ | $f_\Lambda/M_\lambda$ | $N_r$ | $N_s$ |
| | $\langle 4\rangle!\langle 2\rangle$ | | $(2,2,1^4)$ |
| 16 | $q^9$ | 20 | 4 |
| 15 | $q^8$ | 33 | 1 |
| 14 | $q^7$ | 39 | 3 |
| 13 | $q^6$ | 39 | 1 |
| 12 | $q^5$ | 33 | 3 |
| 11 | $q^4$ | 22 | 2 |
| 10 | $q^3$ | 13 | 1 |
| 9 | $q^2$ | 7 | 1 |
| 8 | $q$ | 3 | 1 |
| 7 | 1 | 1 | 1 |
| | $\langle 6\rangle!$ | | $(2,1^6)$ |
| 21 | $q^6$ | 7 | 1 |
| 20 | $q^5$ | 6 | 0 |
| 19 | $q^4$ | 5 | 1 |
| 18 | $q^3$ | 4 | 0 |
| 17 | $q^2$ | 3 | 1 |
| 16 | $q$ | 2 | 0 |
| 15 | 1 | 1 | 1 |
| | $\langle 8\rangle!$ | | $(1^8)$ |
| 28 | 1 | 1 | 1 |
| | | | $(9)$ |
| 0 | 1 | 1 | 1 |
| | | | $(6,3)$ |
| 3 | $q^3$ | 4 | 0 |
| 3 | $q^2[1]$ | 20 | 0 |
| 3 | $q[1]^2$ | 10 | 0 |
| 3 | $q[2]$ | 10 | 0 |
| 3 | $[1][2]$ | 2 | 0 |
| 3 | $[3]$ | 2 | 0 |
| 2 | $q^2$ | 21 | 1 |
| 2 | $q[1]$ | 66 | 0 |
| 2 | $[1]^2$ | 10 | 2 |
| 2 | $[2]$ | 20 | 0 |
| 1 | $q$ | 38 | 0 |
| 1 | $[1]$ | 54 | 0 |
| 0 | 1 | 21 | 3 |

| $M_\lambda$ | | | $\lambda$ |
|---|---|---|---|
| $d$ | $f_\Lambda/M_\lambda$ | $N_r$ | $N_s$ |
| | | | $(5,4)$ |
| 4 | $q^4$ | 5 | 1 |
| 4 | $q^3[1]$ | 20 | 0 |
| 4 | $q^2[1]^2$ | 5 | 1 |
| 4 | $q^2[2]$ | 10 | 0 |
| 4 | $q[3]$ | 2 | 0 |
| 3 | $q^3$ | 30 | 0 |
| 3 | $q^2[1]$ | 70 | 0 |
| 3 | $q[2]$ | 20 | 0 |
| 2 | $q^2$ | 60 | 4 |
| 2 | $q[1]$ | 50 | 0 |
| 1 | $q$ | 30 | 0 |
| 0 | 1 | 1 | 1 |
| | $\langle 4\rangle!$ | | $(5,1^4)$ |
| 10 | $q^4$ | 5 | 1 |
| 10 | $q^3[1]$ | 30 | 2 |
| 10 | $q^2[1]^2$ | 30 | 2 |
| 10 | $q[1]^3$ | 5 | 1 |
| 9 | $q^3$ | 16 | 0 |
| 9 | $q^2[1]$ | 72 | 0 |
| 9 | $q[1]^2$ | 48 | 0 |
| 9 | $[1]^3$ | 4 | 0 |
| 8 | $q^2$ | 30 | 2 |
| 8 | $q[1]$ | 90 | 2 |
| 8 | $[1]^2$ | 30 | 2 |
| 7 | $q$ | 40 | 0 |
| 7 | $[1]$ | 60 | 0 |
| 6 | 1 | 35 | 3 |

| $M_\lambda$ | | | $\lambda$ |
| --- | --- | --- | --- |
| $d$ | $f_\Lambda/M_\lambda$ | $N_r$ | $N_s$ |
| | $\langle 5\rangle!$ | | $(4,1^5)$ |
| 15 | $q^5$ | 6 | 0 |
| 15 | $q^4[1]$ | 30 | 0 |
| 15 | $q^3[1]^2$ | 20 | 0 |
| 14 | $q^4$ | 15 | 1 |
| 14 | $q^3[1]$ | 60 | 0 |
| 14 | $q^2[1]^2$ | 30 | 2 |
| 13 | $q^3$ | 24 | 0 |
| 13 | $q^2[1]$ | 72 | 0 |
| 13 | $q[1]^2$ | 24 | 0 |
| 12 | $q^2$ | 30 | 2 |
| 12 | $q[1]$ | 60 | 0 |
| 12 | $[1]^2$ | 10 | 2 |
| 11 | $q$ | 30 | 0 |
| 11 | $[1]$ | 30 | 0 |
| 10 | 1 | 21 | 3 |
| | $\langle 3\rangle!$ | | $(3^3)$ |
| 9 | $q^6$ | 12 | 2 |
| 9 | $q^5[1]$ | 18 | 2 |
| 9 | $q^3(q^2-q+1)$ $\times[1]$ | 3 | 1 |
| 9 | $q^4[1]^2$ | 9 | 1 |
| 8 | $q^5$ | 54 | 2 |
| 8 | $q^4[1]$ | 54 | 0 |
| 8 | $q^3[1]^2$ | 18 | 2 |
| 7 | $q^4$ | 81 | 3 |
| 7 | $q^3[1]$ | 72 | 4 |
| 6 | $q^3$ | 87 | 5 |
| 6 | $q^2[1]$ | 18 | 0 |
| 5 | $q^2$ | 36 | 2 |
| 5 | $q[1]$ | 3 | 1 |
| 4 | $q$ | 9 | 1 |
| 3 | 1 | 1 | 1 |

| $M_\lambda$ | | | $\lambda$ |
| --- | --- | --- | --- |
| $d$ | $f_\Lambda/M_\lambda$ | $N_r$ | $N_s$ |
| | $\langle 4\rangle!$ | | $(2^4,1)$ |
| 16 | $q^{10}$ | 42 | 2 |
| 15 | $q^9$ | 120 | 0 |
| 14 | $q^8$ | 180 | 4 |
| 13 | $q^7$ | 195 | 3 |
| 12 | $q^6$ | 165 | 3 |
| 11 | $q^5$ | 117 | 5 |
| 10 | $q^4$ | 70 | 2 |
| 9 | $q^3$ | 35 | 3 |
| 8 | $q^2$ | 15 | 1 |
| 7 | $q$ | 5 | 1 |
| 6 | 1 | 1 | 1 |
| | $\langle 9\rangle!$ | | $(1^9)$ |
| 36 | 1 | 1 | 1 |
| | | | $(10)$ |
| 0 | 1 | 1 | 1 |
| | $\langle 2\rangle$ | | $(5,5)$ |
| 5 | $q^4$ | 5 | 1 |
| 5 | $q^3[1]$ | 20 | 4 |
| 5 | $q^2[1]^2$ | 5 | 1 |
| 5 | $q^2[2]$ | 10 | 2 |
| 5 | $q[3]$ | 2 | 2 |
| 4 | $q^3$ | 30 | 2 |
| 4 | $q^2[1]$ | 70 | 6 |
| 4 | $q[2]$ | 20 | 4 |
| 3 | $q^2$ | 60 | 6 |
| 3 | $q[1]$ | 50 | 10 |
| 2 | $q$ | 30 | 6 |
| 1 | 1 | 1 | 1 |

| $M_\lambda$ | | | $\lambda$ |
| --- | --- | --- | --- |
| $d$ | $f_\Lambda/M_\lambda$ | $N_r$ | $N_s$ |
| | $\langle 5\rangle!$ | | $(5,1^5)$ |
| 15 | $q^5$ | 6 | 0 |
| 15 | $q^4[1]$ | 45 | 3 |
| 15 | $q^3[1]^2$ | 60 | 0 |
| 15 | $q^2[1]^3$ | 15 | 3 |
| 14 | $q^4$ | 20 | 0 |
| 14 | $q^3[1]$ | 120 | 0 |
| 14 | $q^2[1]^2$ | 120 | 0 |
| 14 | $q[1]^3$ | 20 | 0 |
| 13 | $q^3$ | 40 | 0 |
| 13 | $q^2[1]$ | 180 | 4 |
| 13 | $q[1]^2$ | 120 | 0 |
| 13 | $[1]^3$ | 10 | 2 |
| 12 | $q^2$ | 60 | 0 |
| 12 | $q[1]$ | 180 | 0 |
| 12 | $[1]^2$ | 60 | 0 |
| 11 | $q$ | 70 | 0 |
| 11 | $[1]$ | 105 | 3 |
| 10 | 1 | 56 | 0 |
| | $\langle 10\rangle!$ | | $(1^{10})$ |
| 45 | 1 | 1 | 1 |

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