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Isotypic Decompositions of Lattice Determinants

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The q,t-Macdonald polynomials are conjectured by Garsia and Haiman to have a representation theoretic interpretation in terms of the S_n-module $\mathbf{M}_\mu$ spanned by the derivatives of a certain polynomial $\Delta_\mu(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n)$. The diagonal action of a permutation $\sigma \in S_n$ on a polynomial $P = P(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n)$ is defined by setting $\sigma P = P(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}; y_{\sigma_1}, y_{\sigma_2}, \ldots, y_{\sigma_n})$. Since the polynomial Δ_μ alternates under the diagonal action, $\mathbf{M}_\mu$ is S_n-invariant. We analyze here the diagonal action of S_n on $\mathbf{M}_\mu$ and show that its decomposition into irreducibles is equivalent to a certain isotypic expansion for the translate $\Delta_\mu(x_1+x'_1, x_2+x'_2, \ldots, x_n+x'_n; y_1+y'_1, y_2+y'_2, \ldots, y_n+y'_n)$ of the polynomial Δ_μ.

1. Introduction

A *lattice diagram* is a finite set $L = \{(h_1, k_1), \ldots, (h_n, k_n)\}$ of cells in the positive plane quadrant (or a positive orthant in higher dimensions). Of particular interest is the *Ferrer's Diagram* of a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ of an integer n:

$$L_\mu \;=\; \{\,(i,j) : 0 \le i \le k-1, \quad 0 \le j \le \mu_{i+1} - 1\,\} \;.$$

The *lattice determinant* of L is

$$\Delta_L(x_1, \ldots, x_n; y_1, \ldots, y_n) \;=\; \Delta_L(\mathbf{x}; \mathbf{y}) \;=\; \det[{x_i}^{h_j}\, {y_i}^{k_j}]_{i,j=1}^n \;. \tag{1}$$

This is well-defined aside from an overall sign determined by the ordering of the cells. Note that we abbreviate the parameter lists $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$. This polynomial is homogeneous under the *bidegree grading* on polynomials

$$\operatorname{bideg} P(\mathbf{x};\mathbf{y}) \;=\; (\sum_{i=1}^n \deg_{x_i} P \;,\; \sum_{i=1}^n \deg_{y_i} P) \;.$$

In particular, the degree of (1) is $\vec{\mathbf{n}}(L)$ where

$$\vec{\mathbf{n}}(L) \;=\; (h_1 + \cdots + h_n, k_1 + \cdots + k_n) \;,$$

and for partitions, $\vec{\mathbf{n}}(\mu) = (n(\mu), n(\mu'))$ where $n(\mu) = \sum_{i>1} (i-1)\mu_i$.

We adopt the French notation for partitions and a reverse-Cartesian notation for the coordinates of the cells to be consistent with the literature on Δ_μ by Garsia; the cells of partition $(3,2)$ and the lattice determinant $\Delta_{(3,2)}$ are as follows.

$$\begin{array}{|c|c|c}
\hline
(1,0) & (1,1) & \multicolumn{1}{c}{} \\
\hline
(0,0) & (0,1) & \multicolumn{1}{c|}{(0,2)} \\
\hline
\end{array}
\qquad
\det \begin{bmatrix}
{x_1}^0 {y_1}^0 & {x_1}^0 {y_1}^1 & {x_1}^0 {y_1}^2 & {x_1}^1 {y_1}^0 & {x_1}^1 {y_1}^1 \\
\vdots & & & & \vdots \\
{x_5}^0 {y_5}^0 & {x_5}^0 {y_5}^1 & {x_5}^0 {y_5}^2 & {x_5}^1 {y_5}^0 & {x_5}^1 {y_5}^1
\end{bmatrix}$$

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Let $\mathbf{K}$ be a field of characteristic 0. We consider two vector spaces associated with $\Delta_L(\mathbf{x};\mathbf{y})$; the space $\mathbf{M}_L$ spanned by all partial derivatives of Δ_L, and the space $\mathbf{M}_L^+$ spanned by all *translates* of Δ_L. For partitions we write $\mathbf{M}_\mu$ in place of $\mathbf{M}_{L_\mu}$.

$$\mathbf{M}_L \;=\; \operatorname{span}\left\{\, \partial_{x_1}^{a_1}\cdots\partial_{x_n}^{a_n}\partial_{y_1}^{b_1}\cdots\partial_{y_n}^{b_n}\,\Delta_L(\mathbf{x};\mathbf{y}) \,:\, a_1,\ldots,b_n\geq 0 \,\right\} \tag{2}$$

$$\mathbf{M}_L^+ \;=\; \operatorname{span}\left\{\, \Delta_L(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}') \,:\, \mathbf{x}',\mathbf{y}'\in\mathbf{K}^n \,\right\} \tag{3}$$

It can be seen by Taylor's theorem (Section 2) that these spaces are equal. There is an action of the symmetric group S_n on these two spaces called the *diagonal action* (Section 3), defined by setting, for $\sigma\in S_n$,

$$\sigma f(x_1,\ldots,x_n;y_1,\ldots,y_n) \;=\; f(x_{\sigma(1)},\ldots,x_{\sigma(n)};y_{\sigma(1)},\ldots,y_{\sigma(n)})\,.$$

The spaces can be decomposed (Section 4) into their isotypic components under the diagonal action, and further be decomposed into components that are homogeneous under the bidegree grading.

Our main result is a more precise relationship between the two spaces; specifically, in Section 5, we expand the translates in terms of the derivatives:

$$\Delta_L(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}') \;=\; \sum_{r,s\geq 0}\sum_{\lambda\vdash n}\sum_{T\in SYT(\lambda)}\sum_{m=1}^{N_L(r,s;\lambda)} \phi_m^{r,s;T}(\mathbf{x};\mathbf{y})\cdot\psi_m^{r,s;T}(\mathbf{x}';\mathbf{y}')\,, \tag{4}$$

where for integers $r,s\geq 0$ and a partition λ of n,

$$\left\{\, \phi_m^{r,s;T} \;:\; T\in SYT(\lambda)\,,\; 1\leq m\leq N_L(r,s;\lambda)\,\right\}$$

is a basis of the homogeneous (degree (r,s)) isotypic component (type λ) of $\mathbf{M}_L$, and

$$\left\{\, \psi_m^{r,s;T} \;:\; T\in SYT(\lambda)\,,\; 1\leq m\leq N_L(r,s;\lambda)\,\right\}$$

is a basis of the complementary degree and conjugate character component. Note that $SYT(\lambda)$ is the set of Standard Young Tableaux of shape λ.

It develops that in the case of partitions μ, the "$n!$-conjecture" [3] identifies the dimensions $N_\mu(r,s;\lambda)$ of these components as the coefficients that arise in Macdonald's "2-parameter polynomials," and in particular gives a combinatorial explanation of Macdonald's conjecture that these coefficients are positive ([4, p. 355]). If the $n!$-conjecture is true, the Macdonald Polynomials are a generating function for the dimensions of these homogeneous isotypic components of $\mathbf{M}_\mu$, while (4) may be viewed as a generating function of bases of these components of $\mathbf{M}_\mu$.

In Section 6, we generalize expansion (4) to multidimensional lattice diagrams; permanents; multiple summands; and "delivation," an analogue of differentiation. In Section 7 we describe a method of computing these decompositions. And finally, in Section 8, we work out the decomposition (4) for a family of multidimensional partitions.

Example 1. Let $\mu=(2,1)$. Then $\Delta_{(2,1)}(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}')$ is a sum of six terms; the term in (4) indexed by r,s,λ,T is shown as $t^r\,q^s\,T$.

$t^0\,q^0$	$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$	$(1)\,(x_2'\,y_3' - x_3'\,y_3' - x_3'\,y_3' + x_1'\,y_2' + y_1'\,x_3' - y_1'\,x_2')$
$t^0\,q^1$	$\begin{bmatrix} 3 & \\ 1 & 2 \end{bmatrix}$	$(y_3 - y_1)\,(x_2' - x_1')$
$t^0\,q^1$	$\begin{bmatrix} 2 & \\ 1 & 3 \end{bmatrix}$	$(y_1 - y_2)\,(x_3' - x_1')$
$t^1\,q^0$	$\begin{bmatrix} 3 & \\ 1 & 2 \end{bmatrix}$	$(x_1 - x_3)\,(y_2' - y_1')$
$t^1\,q^0$	$\begin{bmatrix} 2 & \\ 1 & 3 \end{bmatrix}$	$(x_2 - x_1)\,(y_3' - y_1')$
$q^1\,t^1$	$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$	$(x_2\,y_3 - x_3\,y_2 - x_1\,y_3 + x_1\,y_2 + y_1\,x_3 - y_1\,x_2)\,(1)$

Note that there is a degree-complementing, sign-twisting automorphism of $\mathbf{M}_\mu$ defined by

$$\mathbf{flip}_\mu(f) \;=\; f(\partial_{x_1},\ldots,\partial_{x_n};\partial_{y_1},\ldots,\partial_{y_n})\Delta_\mu .$$

Even though ϕ and ψ in each term have complementary degrees and conjugate characters, they are not related by flip:

$$(\partial_{y_3} - \partial_{y_1})\Delta_\mu(\mathbf{x};\mathbf{y}) \;=\; 2x_2 - x_1 - x_3\ .$$

Further, if we replace differentiation by delivation (Section 6), this same example will show that neither ϕ nor ψ is the flip of the other.

2. Taylor expansion of translates

We shall work in a field $\mathbf{K}$ of characteristic 0.

Let $x_1,\ldots,x_n$ be indeterminates.

Let $f(x_1,\ldots,x_n) \in \mathbf{K}[x_1,\ldots,x_n]$ be a polynomial.

Let $\mathbf{M}_f = \operatorname{span}\left\{\, \partial_{x_1}^{r_1}\cdots\partial_{x_n}^{r_n} f \;:\; r_1,\ldots,r_n \geq 0 \,\right\}$.

Theorem 1. *Let $\{\phi_1,\ldots,\phi_N\}$ be a basis of $\mathbf{M}_f$. There exist unique polynomials $\psi_1,\ldots,\psi_N \in \mathbf{K}[x_1,\ldots,x_n]$ such that*

$$f(x_1+x_1',\ldots,x_n+x_n') = \sum_{m=1}^{N} \phi_m(x_1,\ldots,x_n)\psi_m(x_1',\ldots,x_n').$$

Further, $\{\psi_1,\ldots,\psi_N\}$ is also a basis of $\mathbf{M}_f$.

Proof: By Taylor's theorem we may write

$$f(x_1+x_1',\ldots,x_n+x_n') \;=\; \sum_{r_1,\ldots,r_n\geq 0} {x_1'}^{r_1}\cdots{x_n'}^{r_n} \left[\frac{\partial_{x_1}}{r_1!}\cdots\frac{\partial_{x_n}}{r_n!} f(x_1,\ldots,x_n)\right]\ . \tag{5}$$

The bracketed expression is in $\mathbf{M}_f$ and hence may be expressed in terms of the ϕ_m's;

$$\frac{\partial_{x_1}}{r_1!}\cdots\frac{\partial_{x_n}}{r_n!} f(x_1,\ldots,x_n) \;=\; \sum_{m=1}^{N} d_m(r_1,\ldots,r_n)\phi_m(x_1,\ldots,x_n)\ , \tag{6}$$

where $d_m(r_1, \ldots, r_n) \in \mathbf{K}$ are unique because the ϕ_m's form a basis. Substituting (6) into (5), we obtain

$$\begin{aligned} f(x_1 + x_1', \ldots, x_n + x_n') &= \sum_{r_1, \ldots, r_n \geq 0} {x_1'}^{r_1} \cdots {x_n'}^{r_n} \sum_{m=1}^{N} d_m(r_1, \ldots, r_n) \phi_m(x_1, \ldots, x_n) \\ &= \sum_{m=1}^{N} \phi_m(x_1, \ldots, x_n) \sum_{r_1, \ldots, r_n \geq 0} d_m(r_1, \ldots, r_n) {x_1'}^{r_1} \cdots {x_n'}^{r_n} \\ &= \sum_{m=1}^{N} \phi_m(x_1, \ldots, x_n) \psi_m(x_1', \ldots, x_n') , \end{aligned} \tag{7}$$

where we have defined

$$\psi_m(x_1', \ldots, x_n') \; = \sum_{r_1, \ldots, r_n \geq 0} d_m(r_1, \ldots, r_n) {x_1'}^{r_1} \cdots {x_n'}^{r_n} .$$

We must show that each $\psi_m(x_1, \ldots, x_n) \in \mathbf{M}_f$. To do this, we expand ϕ_m,

$$\phi_m(x_1, \ldots, x_n) \; = \sum_{r_1, \ldots, r_n \geq 0} d_m'(r_1, \ldots, r_n) {x_1}^{r_1} \cdots {x_n}^{r_n} \tag{8}$$

and plug this into (7).

$$\begin{aligned} f(x_1 + x_1', \ldots, x_n + x_n') &= \sum_{m=1}^{N} \psi_m(x_1', \ldots, x_n') \sum_{r_1, \ldots, r_n \geq 0} d_m'(r_1, \ldots, r_n) {x_1}^{r_1} \cdots {x_n}^{r_n} \\ &= \sum_{r_1, \ldots, r_n \geq 0} {x_1}^{r_1} \cdots {x_n}^{r_n} \sum_{m=1}^{N} d_m'(r_1, \ldots, r_n) \, \psi_m(x_1', \ldots, x_n') \end{aligned}$$

Compare this with the other Taylor expansion of f:

$$f(x_1 + x_1', \ldots, x_n + x_n') \; = \sum_{r_1, \ldots, r_n} {x_1}^{r_1} \cdots {x_n}^{r_n} \left[\frac{\partial_{x_1'}}{r_1!} \cdots \frac{\partial_{x_n'}}{r_n!} f(x_1', \ldots, x_n') \right] . \tag{9}$$

Equating the coefficients of like powers of $x_1, \ldots, x_n$, we have

$$\frac{\partial_{x_1'}}{r_1!} \cdots \frac{\partial_{x_n'}}{r_n!} f(x_1', \ldots, x_n') \; = \; \sum_{m=1}^{N} d_m'(r_1, \ldots, r_n) \, \psi_m(x_1', \ldots, x_n') ,$$

so $\mathbf{M}_f \subset \operatorname{span}\{\psi_1, \ldots, \psi_N\}$. But $\{\psi_1, \ldots, \psi_N\}$ spans a space of dimension at most N, while $\mathbf{M}_f$ has dimension N, so in fact, $\{\psi_1, \ldots, \psi_N\}$ is a basis of $\mathbf{M}_f$.

We have actually proved an additional result that merits its own statement.

Proposition 1. *If there is an expansion*

$$f(x_1 + x_1', \ldots, x_n + x_n') \; = \; \sum_{m=1}^{N} \phi_m(x_1, \ldots, x_m) \, \psi_m(x_1', \ldots, x_m')$$

with ϕ_m *and* ψ_m *polynomials over* $\mathbf{K}$, *then* $\operatorname{span}\{\phi_1, \ldots, \phi_N\}$ *and* $\operatorname{span}\{\psi_1, \ldots, \psi_N\}$ *both contain* $\mathbf{M}_f$, *and* $N \geq \dim \mathbf{M}_f$.

Remark. Let $\left\{\widehat{\phi}_1, \ldots, \widehat{\phi}_N\right\}$ be a second basis of $\mathbf{M}_f$ related to the ϕ's by a $\mathbf{K}$-linear transformation U where $\widehat{\phi}_m = U\phi_m$. On setting $\widehat{\psi}_m = U^{-1}\psi_m$, we obtain

$$f(x_1 + x'_1, \ldots, x_n + x'_n) \;=\; \sum_{m=1}^{N} \widehat{\phi}_m(x_1, \ldots, x_n)\, \widehat{\psi}_m(x'_1, \ldots, x'_n)\;.$$

Remark. Theorem 1 shows that $\mathbf{M}_f^+ \subset \mathbf{M}_f$. Let $E_x^{(x')}$ be the operator $E_x^{(x')} f(x) = f(x + x')$. In operator notation, Taylor's theorem may be expressed

$$E_x{}^{(x')} f(x) \;=\; f(x + x') \;=\; e^{x'\,\partial_x} f(x)\;.$$

On formally inverting this, we obtain for any $x' \neq 0$,

$$\partial_x \;=\; \frac{1}{x'} \ln(E_x^{(x')}) \;=\; \frac{1}{x'} \ln(1 - (1 - E_x^{(x')})) \;=\; -\frac{1}{x'} \sum_{k>0} (1 - E_x^{(x')})^k / k\;.$$

This expands derivatives in terms of translates, so that $\mathbf{M}_f \subset \mathbf{M}_f^+$. Note that the sum may be terminated at any k larger than the degree of the polynomial to which it is applied, because the operator $(1 - E_x^{(x')})$ lowers the x-degree of a polynomial by 1.

3. Actions of S_n on polynomials

For a permutation $\sigma \in S_n$, we define several actions on polynomials

$$f \;=\; f(\mathbf{x}; \mathbf{y}; \mathbf{x}'; \mathbf{y}') \;=\; f(x_1, \ldots, x_n; y_1, \ldots, y_n; x'_1, \ldots, x'_n; y'_1, \ldots, y'_n)\;.$$

They are

$$\sigma f \;=\; f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}; y_{\sigma(1)}, \ldots, y_{\sigma(n)}; x'_1, \ldots, x'_n; y'_1, \ldots, y'_n) \tag{10}$$

$$\sigma' f \;=\; f(x_1, \ldots, x_n; y_1, \ldots, y_n; x'_{\sigma(1)}, \ldots, x'_{\sigma(n)}; y'_{\sigma(1)}, \ldots, y'_{\sigma(n)}) \tag{11}$$

$$\sigma^{(4)} f \;=\; f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}; y_{\sigma(1)}, \ldots, y_{\sigma(n)}; x'_{\sigma(1)}, \ldots, x'_{\sigma(n)}; y'_{\sigma(1)}, \ldots, y'_{\sigma(n)}) \tag{12}$$

The first of these is called the *diagonal action.* These notations extend to elements

$$\theta \;=\; \sum_{\sigma \in S_n} \theta_\sigma \sigma \tag{13}$$

of the group algebra $\mathbf{K}\langle S_n\rangle$ (where $\theta_\sigma \in \mathbf{K}$) via $\theta f = \sum_\sigma \theta_\sigma\, \sigma f$, and so forth.

Alain Lascoux (private communication) has found a critical relationship between diagonal actions on $(\mathbf{x}; \mathbf{y})$ and on $(\mathbf{x}'; \mathbf{y}')$ in the context of alternating polynomials. For any θ as in (13), define

$$\bar{\theta} \;=\; \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \theta_\sigma \sigma^{-1}\;. \tag{14}$$

Note that for products in the group algebra,

$$\overline{\theta\tau} = \bar{\tau}\,\bar{\theta} \tag{15}$$

because taking inverse permutations reverses the order of multiplication.

Proposition 2 (Lascoux). *Let $f(\mathbf{x}; \mathbf{y})$ be alternating under the diagonal action of S_n. For any $\theta \in \mathbf{K}\langle S_n\rangle$, we have*

$$\theta f(\mathbf{x} + \mathbf{x}'; \mathbf{y} + \mathbf{y}') \;=\; \bar{\theta}' f(\mathbf{x} + \mathbf{x}'; \mathbf{y} + \mathbf{y}')\;. \tag{16}$$

Proof: By linearity in the group algebra, it suffices to prove this for $\theta = \sigma$, a single permutation. Since $f(\mathbf{x};\mathbf{y})$ is alternating under the diagonal action (10) of S_n, we have under the action (12) that

$$\sigma^{(4)} f(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}') \;=\; \mathrm{sign}(\sigma)\, f(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}')\;.$$

However, $\sigma^{(4)} = \sigma\,\sigma' = \sigma'\,\sigma$, so we have

$$\sigma'\,\sigma f(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}') \;=\; \mathrm{sign}(\sigma)\, f(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}')\;,$$

whence

$$\sigma f(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}') \;=\; \mathrm{sign}(\sigma)\,(\sigma^{-1})'\, f(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}') \;=\; \bar{\sigma}'\, f(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}')\;.$$

4. Young's natural representation of S_n

We review Young's *natural representation* of S_n so that we may apply it to decompose $\Delta_\mu(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}')$. See [8, pp. 260–266], [5, p. 133], [6, p. 362], [7, p. 16], and [9, pp. 256–258], all of which are in [10]. This presentation of it is in Garsia [2].

Let f_λ be the number of standard tableaux of shape λ. Let S_i^λ $(i = 1, \ldots, f_\lambda)$ be the standard tableaux of shape λ in Young's First Letter Order: $T_1 < T_2$ when the smallest number that is in a different cell of the two tableaux is in a higher numbered row of T_1 than in T_2. So S_1^λ is the column superstandard tableau and $S_{f_\lambda}^\lambda$ is the row superstandard one. Let σ_{ij}^λ be the permutation such that $S_i^\lambda = \sigma_{ij}^\lambda S_j^\lambda$, where the action σT denotes replacing k in T by $\sigma(k)$.

For any tableau T, let $N(T) = \sum_\sigma \mathrm{sign}(\sigma)\sigma$ where σ runs over all permutations that leave the entries in the same column, and $P(T) = \sum_\sigma \sigma$ where σ runs over all permutations that leave the entries in the same row. Let h_T be the product of the hook lengths of the shape λ of T. Let $\gamma_T = N(T)P(T)/h_T$ and $\gamma_i^\lambda = \gamma_{S_i^\lambda}$. Let

$$e_{ij}^\lambda \;=\; \sigma_{ij}^\lambda \gamma_j^\lambda (1-\gamma_{j+1}^\lambda)\cdots(1-\gamma_{f_\lambda}^\lambda)\;. \tag{17}$$

Theorem 2 (Young).

1. *For two standard young tableaux on $1, \ldots, n$, $\gamma_{T_1}\gamma_{T_2} = 0$ when the tableaux have different shapes, or when they have the same shape and there is a row of T_1 and a column of T_2 that share two or more entries in common. As a special case, when $T_1 > T_2$ in Young's First Letter Order, this product is 0.*

   Otherwise, $\gamma_{T_1}\gamma_{T_2} = \pm\sigma_{T_1,T_2}\gamma_{T_2}$, and in particular, $\gamma_T\gamma_T = \gamma_T$.
2. $\left\{ e_{ij}^\lambda \,:\, \lambda \vdash n\,,\ 1 \le i,j \le f_\lambda \right\}$ *is a basis of $\mathbf{K}\langle S_n\rangle$.*
3. $e_{ij}^\lambda\, e_{rs}^\mu \;=\; \begin{cases} e_{is}^\lambda & \textit{if } \lambda=\mu \textit{ and } j=r; \\ 0 & \textit{otherwise.} \end{cases}$
4. *The operator that projects into the isotypic component of type λ is $\rho_\lambda = \sum_{i=1}^{f_\lambda} e_{ii}^\lambda$.*
5. *The identity permutation is $1 = \sum_{\lambda\vdash n}\sum_{i=1}^{f_\lambda} e_{ii}^\lambda$.*

When we apply (14) to $N(T)$, $P(T)$, and γ_T, we obtain the following simple forms.

Proposition 3. *For any injective tableau T,*

$$\overline{P(T)} = N(T^t) \qquad , \qquad \overline{N(T)} = P(T^t) \qquad , \qquad \overline{\gamma_T} = \gamma_{T^t}\;,$$

where T^t is the transpose of T.

Proof: The permutations that occur in the summation for $P(T)$ and $\overline{P(T)}$ do not change, because when a permutation fixes the rows of T, so does its inverse. However, a sign is placed onto each term, resulting in $\overline{P(T)} = N(T^t)$. The second statement is proved similarly.

For the third, we have

$$\overline{\gamma_T} \;=\; h_T{}^{-1}\overline{\Big(N(T)P(T)\Big)} \;=\; h_T{}^{-1}\Big(\overline{P(T)}\Big)\Big(\overline{N(T)}\Big) \;=\; h_{T^t}{}^{-1}N(T^t)P(T^t) \;=\; \gamma_{T^t}\;.$$

5. Decomposing the translates by S_n

Theorem 3. *Consider $\mathcal{H}_{r,s}(e^{\lambda}_{ii}\mathbf{M}_L)$, the component of $\mathbf{M}_L$ that is homogeneous of bidegree (r,s) and invariant under the idempotent e^{λ}_{ii}. Let $N_L(r,s;S^{\lambda}_i)$ denote its dimension, and $\phi^{r,s;S^{\lambda}_i}_m(\mathbf{x};\mathbf{y})$ $(m=1,\ldots,N_L(r,s;S^{\lambda}_i))$ be a basis of this component. Then there exist unique polynomials $\psi^{r,s;S^{\lambda}_i}_m$ such that*

$$\Delta_L(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}') \;=\; \sum_{r,s}\sum_{\lambda\vdash n}\sum_{i=1}^{f_\lambda}\sum_{m=1}^{N_L(r,s;S^{\lambda}_i)} \phi^{r,s;S^{\lambda}_i}_m(\mathbf{x};\mathbf{y})\psi^{r,s;S^{\lambda}_i}_m(\mathbf{x}';\mathbf{y}')\,. \tag{18}$$

Further, $\psi^{r,s;S^{\lambda}_i}_m(\mathbf{x}';\mathbf{y}')$ has bidegree $\vec{\mathbf{n}}(L)-(r,s)$ complementary to (r,s), and is in the isotypic component of character λ'. In particular,

$$(\overline{e^{\lambda}_{ii}})'\psi^{r,s;S^{\lambda}_i}_m(\mathbf{x}';\mathbf{y}') \;=\; \psi^{r,s;S^{\lambda}_i}_m(\mathbf{x}';\mathbf{y}')\,.$$

Proof: By Theorem 1, there is a decomposition of the form (18), and the polynomials ψ are unique. We must establish that they have the stated properties. The left side of the equation is bihomogeneous in total $\mathbf{x}+\mathbf{x}'$ degree and total $\mathbf{y}+\mathbf{y}'$ degree; therefore, restricting every term on the right side to its component of these degrees would maintain the equality, but since the ψ's are unique, the restriction doesn't actually change any term. So the ψ's have complementary bidegree to the ϕ's.

Next, we apply the idempotent e^{λ}_{ii} to equation (18), and restrict to the component of $(\mathbf{x};\mathbf{y})$ bidegree (r,s). We define

$$a^{r,s;S^{\lambda}_i}(\mathbf{x};\mathbf{y};\mathbf{x}';\mathbf{y}') \;=\; e^{\lambda}_{ii}\Delta_L(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}')\Big|_{(\mathbf{x};\,\mathbf{y})\text{-bidegree }(r,s)}\,. \tag{19}$$

Each $\phi^{r,s;S^{\nu}_j}_m$ satisfies $\phi^{r,s;S^{\nu}_j}_m \;=\; e^{\nu}_{jj}\phi^{r,s;S^{\nu}_j}_m$, so that

$$e^{\lambda}_{ii}\phi^{r,s;S^{\nu}_j}_m \;=\; e^{\lambda}_{ii}e^{\nu}_{jj}\phi^{r,s;S^{\nu}_j}_m \;=\; \begin{cases}\phi^{r,s;S^{\nu}_j}_m & \text{if } \lambda=\nu \text{ and } i=j;\\ 0 & \text{otherwise,}\end{cases}$$

because the idempotents e^{λ}_{ii} are orthogonal. On substituting (18) into (19), we obtain

$$a^{r,s;S^{\lambda}_i}(\mathbf{x};\mathbf{y};\mathbf{x}';\mathbf{y}') \;=\; \sum_{m=1}^{N_L(r,s;S^{\lambda}_i)} \phi^{r,s;S^{\lambda}_i}_m(\mathbf{x};\mathbf{y})\psi^{r,s;S^{\lambda}_i}_m(\mathbf{x}';\mathbf{y}') \tag{20}$$

as the innermost summation. However,

$$\begin{aligned}a^{r,s;S^{\lambda}_i} &= e^{\lambda}_{ii}e^{\lambda}_{ii}\Delta_L(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}')\big|_{(r,s)} \;=\; e^{\lambda}_{ii}(\overline{e^{\lambda}_{ii}})'\Delta_L(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}')\big|_{(r,s)}\\ &= (\overline{e^{\lambda}_{ii}})'e^{\lambda}_{ii}\Delta_L(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}')\big|_{(r,s)}\\ &= (\overline{e^{\lambda}_{ii}})'a^{r,s;S^{\lambda}_i}\,,\end{aligned}$$

so each ψ in (20) must be invariant under $(\overline{e^{\lambda}_{ii}})'$.

Now, it turns out that the quantities $N_L(r,s;S^{\lambda}_i)$ depend on λ but not on S^{λ}_i. Let $N_L(r,s;\lambda)=N_L(r,s;S^{\lambda}_1)$.

Proposition 4. *Fix r,s,λ,i, and any decomposition of the form (20). Pick any $1\le j\le f_\lambda$. Then*

$$a^{r,s;S^{\lambda}_j}(\mathbf{x};\mathbf{y};\mathbf{x}';\mathbf{y}') \;=\; \sum_{m=1}^{N_L(r,s;\lambda)} \left(e^{\lambda}_{ji}\phi^{r,s;S^{\lambda}_i}_m(\mathbf{x};\mathbf{y})\right)\left((\overline{e^{\lambda}_{ij}})'\psi^{r,s;S^{\lambda}_i}_m(\mathbf{x}';\mathbf{y}')\right)\,. \tag{21}$$

Proof: We have

$$\begin{aligned}
a^{r,s;S_j^\lambda} &= e_{jj}^\lambda\, \Delta_L(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}')\big|_{(r,s)} \\
&= e_{ji}^\lambda\, e_{ii}^\lambda\, e_{ij}^\lambda\, \Delta_L(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}')\big|_{(r,s)} \\
&= e_{ji}^\lambda\, e_{ii}^\lambda\, (\overline{e_{ij}^\lambda})'\, \Delta_L(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}')\big|_{(r,s)} \\
&= (\overline{e_{ij}^\lambda})'\, e_{ji}^\lambda\, e_{ii}^\lambda\, \Delta_L(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}')\big|_{(r,s)} \\
&= (\overline{e_{ij}^\lambda})'\, e_{ji}^\lambda a^{r,s;S_i^\lambda} \\
&= \sum_{m=1}^{N_L(r,s;S_i^\lambda)} \left(e_{ji}^\lambda \phi_m^{r,s;S_i^\lambda}(\mathbf{x};\mathbf{y})\right) \left((\overline{e_{ij}^\lambda})' \psi_m^{r,s;S_i^\lambda}(\mathbf{x}';\mathbf{y}')\right) .
\end{aligned}$$

In view of Proposition 1 we then have $N_L(r,s;S_i^\lambda) \geq N_L(r,s;S_j^\lambda)$ for all i,j, and hence these numbers are equal.

Let $\mathcal{H}_{r,s}(\mathbf{M})$ be the component of the graded module $\mathbf{M}$ that is homogeneous of total degree r in $\mathbf{x}$ and total degree s in $\mathbf{y}$. The t,q-graded *Frobenius characteristic* of an S_n module $\mathbf{M}$ is the generating function of the dimensions of $\mathbf{M}$ split into these components and further split into isotypic components:

$$\mathrm{F\,ch}\,\mathbf{M} \;=\; \sum_{r,s}\sum_{\lambda\vdash n} \dim \mathcal{H}_{r,s}(\gamma_{f^\lambda}^\lambda \mathbf{M}) t^r q^s s_\lambda \;=\; \sum_{r,s}\sum_{\lambda\vdash n} \frac{\dim \mathcal{H}_{r,s}(\rho_\lambda \mathbf{M}_L)}{f_\lambda} t^r q^s s_\lambda \;,$$

where s_λ are Schur functions.

Corollary 1. *The quantities $N_L(r,s;\lambda)$ are given by*

$$\mathrm{F\,ch}\,\mathbf{M}_L \;=\; \sum_{r,s}\sum_{\lambda\vdash n} N_L(r,s;\lambda) t^r q^s\, s_\lambda \;.$$

This leads us to conjectured values of $N_L(r,s;\lambda)$ for partitions and partitions with one cell removed. The "n!-conjecture" [3] is that for all partitions μ of n, $\dim \mathbf{M}_\mu = n!$, and further,

$$\mathrm{F\,ch}\,\mathbf{M}_\mu \;=\; \tilde{H}_\mu(x;q,t) \;=\; \sum_{|\lambda|=|\mu|} \tilde{K}_{\lambda,\mu}(q,t) s_\lambda(x) \;,$$

where $\tilde{H}_\mu(x;q,t)$ and $\tilde{K}_{\lambda,\mu}(q,t)$ are variants of Macdonald's "2-parameter polynomials" and the "q,t-Kostka coefficients" that arise in their expansion. These are related to Macdonald's q,t-Kostka coefficients $K_{\lambda,\mu}(q,t)$ via

$$\tilde{K}_{\lambda,\mu}(q,t) = t^{n(\mu)} K_{\lambda,\mu}(q,t^{-1}) \;.$$

The following is therefore equivalent to the $n!$-conjecture.

Conjecture 1. *The variant Macdonald Polynomials and q,t-Kostka coefficients are given by*

$$\tilde{H}_\mu \;=\; \sum_{r,s,\lambda} N_\mu(r,s;\lambda) t^r q^s\, s_\lambda \qquad \text{and} \qquad \tilde{K}_{\lambda,\mu}(q,t) \;=\; \sum_{r,s} N_\mu(r,s;\lambda)\, t^r q^s \;. \tag{22}$$

An extension of the $n!$-conjecture has been developed for *punctured diagrams μ/ij* formed by removing one cell (i,j) from a two-dimensional partition μ; see [1].

Conjecture 2. *Let μ/ij be a punctured diagram. Then $N_{\mu/ij}(r,s;\lambda)$ is given by*

$$C_{\mu/ij} \;=\; \sum_{r,s,\lambda} N_{\mu/ij}(r,s;\lambda) t^r q^s\, s_\lambda \;,$$

where conjectural formulas for $C_{\mu/ij}$ are given in [1].

6. Generalizations

Multidimensional lattice diagrams. The results of the preceding section extend to multidimensional lattice diagrams, but we do not have conjectured values of N_L's in this case. Let L be a subset of $\mathbf{N}^d$ with n cells, and $\vec{\mathbf{n}}(L)$ be the component-wise sum of all the coordinates:

$$\begin{aligned} L &= \{(h_1,k_1,\ldots,m_1),\ldots,(h_n,k_n,\ldots,m_n)\} \\ \vec{\mathbf{n}}(L) &= (h_1+h_2+\cdots+h_n,\ldots,m_1+m_2+\cdots+m_n)\,. \end{aligned}$$

Introduce d n-tuples of variables $\mathbf{x},\mathbf{y},\ldots,\mathbf{z}$, and extend all definitions from two sets of variables to d sets of variables accordingly.

The full multi-dimensional form of Theorem 3, applicable to any finite lattice diagram L in $\mathbf{N}^d$, is as follows. Note that we have chosen $i=f_\lambda$ in Proposition 4 in order to simplify the form of e^λ_{ij} in the results.

Theorem 4. *Let $\phi_m^{r,s,\cdots,v;\lambda}$ $(m=1,\ldots,N_L(r,s,\cdots,v;\lambda))$ be a basis of $\mathcal{H}_{r,s,\cdots,v}(\gamma^\lambda_{f_\lambda}\mathbf{M}_L)$. Then there exist unique polynomials $\psi_m^{r,s,\cdots,v;\lambda}$ $(m=1,\ldots,N_L(r,s,\cdots,v;\lambda))$ in $\mathbf{M}_L$, such that*

$$\Delta_L(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}';\cdots;\mathbf{z}+\mathbf{z}') = \sum_{r,s,\ldots,v}\sum_{\lambda\vdash n}\sum_{i=1}^{f_\lambda}\sum_{m=1}^{N_L(r,s,\cdots,v;\lambda)}\Big(\sigma^\lambda_{i,f_\lambda}\phi_m^{r,s,\cdots,v;\lambda}(\mathbf{x};\mathbf{y};\cdots;\mathbf{z})\Big)\Big(\overline{e^\lambda_{f_\lambda,i}}{}'\,\psi_m^{r,s,\cdots,v;\lambda}(\mathbf{x}';\mathbf{y}';\cdots;\mathbf{z}')\Big).$$

The ψ's have complementary degrees and conjugate characters to the ϕ's; in particular, the ψ's are a basis of $\mathcal{H}_{r',s',\ldots,v'}(\gamma_{S_1^{\lambda'}}\mathbf{M}_L)$ where $(r+r',s+s',\ldots,v+v')=\vec{\mathbf{n}}(L)$.

For a 2-dimensional partition μ, (22) and the Kostka polynomial symmetry

$$\tilde{K}_{\lambda,\mu}(q,t) = \tilde{K}_{\lambda',\mu}(q^{-1},t^{-1})t^{n(\mu)}q^{n(\mu')}$$

suggest that

$$N_\mu(n(\mu)-r,n(\mu')-s,\lambda') = N_\mu(r,s;\lambda)\,.$$

For multidimensional lattice diagrams, the ϕ's and ψ's form bases of complementary degree, conjugate character components, so this symmetry goes through in the following form.

Proposition 5. *Let L be a lattice diagram in $\mathbf{N}^d$. Then*

$$N_L(r',s',\ldots;\lambda') = N_L(r,s,\ldots;\lambda)\,, \tag{23}$$

where $(r+r',s+s',\ldots)=\vec{\mathbf{n}}(L)$.

The other symmetry of the Kostka polynomials in the two-dimensional partition case is

$$\tilde{K}_{\lambda,\mu}(q,t)=\tilde{K}_{\lambda,\mu'}(t,q)\,.$$

This can be interpreted as saying that any decomposition of $\Delta_\mu(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}')$ of the form (4) also yields a similar one for μ' simply by switching $\mathbf{x},\mathbf{x}'$ with $\mathbf{y},\mathbf{y}'$. This generalizes to multidimensional lattice diagrams in the obvious way: if L_1 is obtained from L_2 by permuting the coordinate axes, a decomposition for L_2 is obtained from any one for L_1 by permuting the variable sets representing those coordinate axes, in the same way.

Permanents. Let $f(\mathbf{x};\mathbf{y})$ be invariant under the diagonal action of S_n. On replacing equation (14) by

$$\tilde{\theta} = \sum_{\sigma\in S_n}\theta_\sigma\sigma^{-1}\,, \tag{24}$$

we have that $\theta f(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}')=\tilde{\theta}'f(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}')$. However, for a tableau T of shape λ, $N(T)$ and $P(T)$ are invariant under the transformation (24), so $\widetilde{\gamma_T}={h_T}^{-1}P(T)N(T)$, which projects into the isotypic component of type λ rather than λ' as did (14). Theorem 3 goes through when we replace the lattice determinant (1) by the *lattice permanent*

$$\Delta^L(\mathbf{x};\mathbf{y};\cdots;\mathbf{z}) \;=\; \mathrm{per}[{x_i}^{h_j}\,{y_i}^{k_j}\,\cdots\,{z_i}^{m_j}]_{i,j=1}^n\,,$$

with the exception that the characters of the ϕ's and ψ's are equal, not conjugate. Define $\mathbf{M}^L$, $N^L(r,s;\lambda)$, and so on in terms of Δ^L analogously to how their counterparts were defined for Δ_L.

Theorem 5. *Let $\phi_m^{r,s,\cdots,v;\lambda}$ $(m=1,\ldots,N^L(r,s,\cdots,v;\lambda))$ be a basis of $\mathcal{H}_{r,s,\cdots,v}(\gamma_{f_\lambda}^\lambda \mathbf{M}^L)$. Then there exist unique polynomials $\psi_m^{r,s,\cdots,v;\lambda}$ $(m=1,\ldots,N^L(r,s,\cdots,v;\lambda))$ in $\mathbf{M}^L$, such that*

$$\Delta^L(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}';\cdots;\mathbf{z}+\mathbf{z}') \;=\;$$
$$\sum_{r,s,\ldots,v}\sum_{\lambda\vdash n}\sum_{i=1}^{f_\lambda}\sum_{m=1}^{N^L(r,s,\cdots,v;\lambda)}\left(\sigma_{i,f_\lambda}^\lambda\phi_m^{r,s,\cdots,v;\lambda}(\mathbf{x};\mathbf{y};\cdots;\mathbf{z})\right)\left(\widetilde{e_{f_\lambda,i}^{\lambda}}{}'\;\psi_m^{r,s,\cdots,v;\lambda}(\mathbf{x}';\mathbf{y}';\cdots;\mathbf{z}')\right).$$

The ψ's have complementary degrees and the same characters as the ϕ's; in particular, the ψ's are a basis of are a basis of $\mathcal{H}_{r',s',\ldots,v'}(\widetilde{\gamma_{S_{f_\lambda}^\lambda}}\,\mathbf{M}^L)$ where $(r+r',s+s',\ldots,v+v')=\vec{\mathbf{n}}(L)$.

Also note that the symmetry (23) is replaced by $N^L(r',s',\ldots;\lambda) \;=\; N^L(r,s,\ldots;\lambda)$. At present there are no conjectured values of any families of N^L.

Multiple summands. We may expand

$$f(\mathbf{x}+\mathbf{x}'+\mathbf{x}'';\mathbf{y}+\mathbf{y}'+\mathbf{y}'') \;=\; \sum_{m_1=1}^N\sum_{m_2=1}^N \phi_{m_1}(\mathbf{x};\mathbf{y})\psi_{m_2}(\mathbf{x}';\mathbf{y}')\xi_{m_1,m_2}(\mathbf{x}'';\mathbf{y}'')\,,$$

where the ξ's are determined from the ϕ and ψ's. They may be 0 or have other linear dependencies, though. The number of factors is the number of summands replacing each variable; in general, if there are k summands, we choose bases of the first $k-1$ of them, and this determines the last one uniquely.

Delivation. We may generalize everything up to this point to "delivation," which generalizes differentiation and translates. The advantages of delivation are that the Taylor expansion formula can be stated in fields of prime characteristic (although the natural representation of S_n has denominators h_T that still require characteristic 0), and that we gain new indeterminates that carry combinatorial statistics on "how much differentiation of each order" was done.

Let α_k, β_k $(k\ge 1)$ be indeterminates or non-zero elements of $\mathbf{K}$. These are called the *constants of delivation.* We replace our differentiation operators ∂_{x_i} by delivation operators:

$$\partial^*_{x_i}{x_i}^k \;=\; \begin{cases}\alpha_k{x_i}^{k-1} & \text{if } k\ge 1,\\ 0 & \text{if } k=0;\end{cases} \qquad\qquad \partial^*_{y_i}{y_i}^k \;=\; \begin{cases}\beta_k{y_i}^{k-1} & \text{if } k\ge 1,\\ 0 & \text{if } k=0;\end{cases}$$

and in each case extend linearly w.r.t. all other variables. We also define

$$[k]_\alpha! \;=\; \alpha_1\alpha_2\ldots\alpha_k \qquad [k]_\beta! \;=\; \beta_1\beta_2\ldots\beta_k$$
$$\binom{k}{j}_\alpha \;=\; \frac{[k]_\alpha!}{[j]_\alpha!\,[k-j]_\alpha!} \qquad \binom{k}{j}_\beta \;=\; \frac{[k]_\beta!}{[j]_\beta!\,[k-j]_\beta!}$$
$$(a\oplus b)_\alpha^k \;=\; \sum_{j=0}^k\binom{k}{j}_\alpha a^jb^{k-j} \qquad (a\oplus b)_\beta^k \;=\; \sum_{j=0}^k\binom{k}{j}_\beta a^jb^{k-j}$$

Given any $f\in\mathbf{K}[x_1,\ldots,x_n;y_1,\ldots,y_n]$, with expansion

$$f(\mathbf{x};\mathbf{y}) \;=\; \sum_{r_1,\ldots,r_n\ge 0}\;\sum_{s_1,\ldots,s_n\ge 0} a_{r_1,\ldots,r_n;s_1,\ldots,s_n}\;{x_1}^{r_1}\cdots{x_n}^{r_n}\;{y_1}^{s_1}\cdots{y_n}^{s_n}\,,$$

we define

$$f(\mathbf{x}\oplus\mathbf{x}';\mathbf{y}\oplus\mathbf{y}') \;=\; \sum_{\mathbf{r},\mathbf{s}} a_{\mathbf{r};\mathbf{s}}\,(x_1\oplus x_1')_\alpha^{r_1}\cdots(x_n\oplus x_n')_\alpha^{r_n}\,(y_1\oplus y_1')_\beta^{s_1}\cdots(y_n\oplus y_n')_\beta^{s_n}\,.$$

On replacing all differentiations, factorials, and shifts by the delivation counterparts, everything goes through. We may expand $\Delta_L(\mathbf{x}\oplus\mathbf{x}';\mathbf{y}\oplus\mathbf{y}')$ as in (4), and the ϕ and ψ's will have coefficients that depend on the α's and β's. For suitable bases, the denominators of the ϕ and ψ's are monomials in the α_k and β_k's.

Note that Theorem 1 goes through if each variable and its prime counterpart has its own constants of delivation, but to have an S_n action, we need the same constants for all x's, and the same constants for all y's, which is why we have only two families of constants (α_k and β_k) instead of $2n$ families.

Note that if α_k and β_k are treated as indeterminates, the problems of non-zero characteristic go away for the Taylor expansion theorem, but the group algebra elements still have denominators h_T that don't go away.

Example 2. The expansion of $\Delta_{(2,1)}(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}')$ in Example 1 turns out to be the expansion of $\Delta_{(2,1)}(\mathbf{x}\oplus\mathbf{x}';\mathbf{y}\oplus\mathbf{y}')$ as well. Now we consider $\Delta_{(1,1,1)}(\mathbf{x}\oplus\mathbf{x}';\mathbf{y}\oplus\mathbf{y}')$; this is a sum of six terms, and again, the term in (4) indexed by r,s,λ,T is shown as $t^r\,q^s\,T$. We take $\Delta_{(1,1,1)}(\mathbf{x};\mathbf{y}) = \det[x_i{}^{j-1}]_{1\le i,j\le 3}$.

$t^0\,q^0 \quad \begin{bmatrix}1 & 2 & 3\end{bmatrix}$	$(1)\,\Delta_{(1,1,1)}(\mathbf{x}';\mathbf{y}')$
$t^1\,q^0 \quad \begin{bmatrix}3 & \\ 1 & 2\end{bmatrix}$	$(x_3-x_1)\,\Big((x_2'-x_1')(-(x_1'+x_2')+\frac{\alpha_2}{\alpha_1}x_3')\Big)$
$t^1\,q^0 \quad \begin{bmatrix}2 & \\ 1 & 3\end{bmatrix}$	$(x_2-x_1)\,\Big((x_3'-x_1')(-(x_1'+x_3')+\frac{\alpha_2}{\alpha_1}x_2')\Big)$
$t^2\,q^0 \quad \begin{bmatrix}3 & \\ 1 & 2\end{bmatrix}$	$\Big((x_3-x_1)(-(x_1+x_3)+\frac{\alpha_2}{\alpha_1}x_2)\Big)\,(x_2'-x_1')$
$t^2\,q^0 \quad \begin{bmatrix}2 & \\ 1 & 3\end{bmatrix}$	$\Big((x_2-x_1)(-(x_1+x_2)+\frac{\alpha_2}{\alpha_1}x_3)\Big)\,(x_3'-x_1')$
$t^3\,q^0 \quad \begin{bmatrix}3\\2\\1\end{bmatrix}$	$\Delta_{(1,1,1)}(\mathbf{x};\mathbf{y})\,(1) \;=\; (x_3-x_2)(x_3-x_1)(x_2-x_1)\,(1)$

7. Algorithm to compute a's

While studying the main equation (4), we used computer explorations to compute the values $a^{r,s;S_i^\lambda}$ of equation (19). An efficient algorithm for computing these values is as follows.

1. Let

$$H \;:=\; \Delta_\mu(\mathbf{x}+\mathbf{x}';\mathbf{y}+\mathbf{y}')\big|_{(\mathbf{x};\mathbf{y})\text{-bidegree }(r,s)}.$$

2. For each partition λ of n, do the following. The partitions may be traversed in any order.

2λ. For $i = f_\lambda, f_\lambda - 1, \ldots, 1$, let

$$\begin{array}{lrcl} \text{2a.} & a^{r,s;S_i^\lambda} & = & \gamma_i^\lambda H \\ \text{2b.} & H & := & H - a^{r,s;S_i^\lambda} \end{array}$$

Proposition 6. *This algorithm is equivalent to computing the a's by equation (19).*

Proof: Each iteration of i in step 2λ is equivalent to multiplying H on the left by $(1-\gamma_i^\lambda)$; as we iterate $i = f_\lambda, f_\lambda - 1, \ldots, 1$, we obtain all of the factors in e_{ii}^λ in equation (17) except for the two leading factors. The missing factor γ_i^λ is introduced in step 2a, and the missing factor $\sigma_{ii}^\lambda = 1$ may be omitted.

As we let λ vary in step 2, the particular order in which it varies is of no relevance, because the components subtracted off in step 2b for partitions ν different from λ are annihilated by γ_i^λ since they lie in a different isotypic component. In terms of the γ's, $\gamma_i^\lambda \gamma_j^\nu = 0$ when $\lambda \neq \nu$.

8. Multidimensional small hook

Notation. We work with $(n+1)$-celled diagrams in n dimensions. In place of $\mathbf{x} = (x_1, \ldots, x_{n+1})$, $\mathbf{y} = (y_1, \ldots, y_{n+1})$, etc., we use $\mathbf{x}_1 = (x_{11}, \ldots, x_{1,n+1})$ through $\mathbf{x}_n = (x_{n,1}, \ldots, x_{n,n+1})$.

Theorem 6. *The "small hook" H_n in $\mathbf{N}^n$ with $n+1$ cells*

$$H_n = \{(0,\ldots,0), (1,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,\ldots,0,1)\}$$

has $\dim \mathbf{M}_{H_n} = \binom{2n}{n}$ *and $(q,t,\ldots,u)$-graded Frobenius characteristic*

$$\mathrm{F\,ch}\,\mathbf{M}_{H_n} = \sum_{r=0}^{n} e_r(q,t,\cdots,u)\, s_{(n+1-r,1^r)}\,. \tag{25}$$

Further, $\Delta_\mu(\mathbf{x}_1 + \mathbf{x}'_1; \cdots; \mathbf{x}_n + \mathbf{x}'_n)$ *expands as*

$$\sum_{k=0}^{n} \sum \pm \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_{i_1,1} & x_{i_1,a_1} & \cdots & x_{i_1,a_k} \\ \vdots & & & \vdots \\ x_{i_k,1} & x_{i_k,a_1} & \cdots & x_{i_k,a_k} \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x'_{j_1,1} & x'_{j_1,b_1} & \cdots & x'_{j_1,b_{n-k}} \\ \vdots & & & \vdots \\ x'_{j_{n-k},1} & x'_{j_{n-k},b_1} & \cdots & x'_{j_{n-k},b_{n-k}} \end{vmatrix} \tag{26}$$

in which the inner sum runs over all partitions of $\{2,3,\ldots,n+1\}$ into complementary subsets $\{a_1 < \ldots < a_k\}$ and $\{b_1 < \ldots < b_{n-k}\}$ and partitions of $\{1,2,\ldots,n\}$ into complementary subsets $\{i_1 < \ldots < i_k\}$ and $\{j_1 < \ldots < j_{n-k}\}$, and $\pm$ is the product of the signs of the 1-line permutations $[a_1,\ldots,a_k,b_1,\ldots,b_{n-k}]$ and $[i_1,\ldots,i_k,j_1,\ldots,j_{n-k}]$.

Proof. Form the $(n+1)\times(n+1)$ determinant

$$\Delta = \Delta_{H_n} = \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_{1,1} & x_{1,2} & \cdots & x_{1,n+1} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n+1} \\ \vdots & \vdots & & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,n+1} \end{bmatrix}\,. \tag{27}$$

Differentiating two or more times with respect to any variable kills Δ. Differentiating with respect to two or more variables in any row or column kills Δ. Differentiating with respect to k variables, no two in the same row or column, is (up to sign) the minor obtained by deleting the k rows and k columns containing those variables. Differentiating by any variable in the first column can be replaced by a linear combination of derivatives in other columns because

$$\partial_{x_{i,1}}\Delta = -(\partial_{x_{i,2}} + \cdots + \partial_{x_{i,n+1}})\Delta\,.$$

We therefore assert that the minors

$$\det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_{i_1,1} & x_{i_1,a_1} & \cdots & x_{i_1,a_k} \\ \vdots & \vdots & & \vdots \\ x_{i_k,1} & x_{i_k,a_1} & \cdots & x_{i_k,a_k} \end{bmatrix} \qquad \begin{array}{c} 0 \le k \le n \\ 1 < a_1 < \cdots < a_k \le n+1 \\ 1 \le i_1 < \cdots < i_k \le n \end{array} \tag{28}$$

form a basis of $\mathbf{M}_{H_n}$. By the preceding paragraph, they form a spanning set. Let $b_1 < b_2 < \ldots < b_{n-k}$ be the complement of $\{a_1, \ldots, a_k\}$ in $\{2, 3, \ldots, n+1\}$. Then the matrix (28) has degree 1 in variable sets $\mathbf{x}_{i_1}, \mathbf{x}_{i_2}, \ldots, \mathbf{x}_{i_k}$ and 0 in the others, and is invariant under the Young idempotent $\gamma_T = N(T)P(T)/h_T$ where

$$T = \begin{array}{llll} a_k & & & \\ a_{k-1} & & & \\ \vdots & & & \\ a_1 & & & \\ 1 & b_1 & b_2 & \cdots b_{n-k} \end{array} .$$

The degree is clear. For invariance, we have

$$P(T) x_{i_1,a_1} \cdots x_{i_k,a_k} \;=\; (n+1-k)!\, x_{i_1,a_1} \cdots x_{i_k,a_k}$$

because variables indexed by $a_1, \ldots, a_k$ are invariant under $P(T)$; and so

$$\frac{N(T)P(T)}{h_T} x_{i_1,a_1} \cdots x_{i_k,a_k} \;=\; \frac{(n+1-k)!}{h_T} \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_{i_1,1} & x_{i_1,a_1} & \cdots & x_{i_1,a_k} \\ \vdots & \vdots & & \vdots \\ x_{i_k,1} & x_{i_k,a_1} & \cdots & x_{i_k,a_k} \end{bmatrix}$$

because the alternation from $N(T)$ gives the determinant; and finally, applying γ_T again leaves this invariant because $\gamma_T \gamma_T = \gamma_T$ in the group algebra. Now since each matrix in (28) has a pair (degree, Young idempotent) uniquely associated to it, they are all linearly independent, and hence a basis. The dimension of $\mathbf{M}_{H_n}$ is

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n}{k} = \binom{2n}{n} .$$

Note that this is smaller than $(n+1)!$ in dimensions larger than 2, so the "$n!$-conjecture" does not go through to multiple dimensions. We can refine this sum into the Frobenius characteristic (25). The isotypic component of $\mathbf{M}_{H_n}$ of character $\lambda = (n+1-k, 1^k)$ has a distribution of degree weights $e_r(q, t, \ldots, u)$, with each of these degree graded subspaces having dimension given by the hook formula

$$f_\lambda = \frac{(n+1)!}{k!(n-k)!(n+1)} = \binom{n}{k} .$$

Finally, we expand $\Delta_{H_n}(\mathbf{x}_1 + \mathbf{x}'_1; \ldots; \mathbf{x}_n + \mathbf{x}'_n)$. The determinant (27) may be rewritten

$$\Delta_{H_n} \;=\; \begin{vmatrix} 1 & 0 & \cdots & 0 \\ x_{1,1} & x_{1,2} - x_{1,1} & \cdots & x_{1,n+1} - x_{1,1} \\ x_{2,1} & x_{2,2} - x_{2,1} & \cdots & x_{2,n+1} - x_{2,1} \\ \vdots & \vdots & & \vdots \\ x_{n,1} & x_{n,2} - x_{n,1} & \cdots & x_{n,n+1} - x_{n,1} \end{vmatrix} \;=\; \begin{vmatrix} x_{1,2} - x_{1,1} & \cdots & x_{1,n+1} - x_{1,1} \\ x_{2,2} - x_{2,1} & \cdots & x_{2,n+1} - x_{2,1} \\ \vdots & & \vdots \\ x_{n,2} - x_{n,1} & \cdots & x_{n,n+1} - x_{n,1} \end{vmatrix} \tag{29}$$

The translate $\Delta_{H_n}(\mathbf{x}_1 + \mathbf{x}'_1; \ldots ; \mathbf{x}_n + \mathbf{x}'_n)$ may be written as

$$\sum_{k=0}^{n} \sum_{\substack{1 \le i_1 < \ldots < i_k \le n \\ 1 \le j_1 < \ldots < j_{n-k} \le n}} \pm \begin{vmatrix} x_{i_1,2} - x_{i_1,1} & \cdots & x_{i_1,n+1} - x_{i_1,1} \\ \vdots & & \vdots \\ x_{i_k,2} - x_{i_k,1} & \cdots & x_{i_k,n+1} - x_{i_k,1} \\ x'_{j_1,2} - x'_{j_1,1} & \cdots & x'_{j_1,n+1} - x'_{j_1,1} \\ \vdots & & \vdots \\ x'_{j_{n-k},2} - x'_{j_{n-k},1} & \cdots & x'_{j_{n-k},n+1} - x'_{j_1,1} \end{vmatrix} \tag{30}$$

where $[i_1, \ldots, i_k, j_1, \ldots, j_{n-k}]$ is a 1-line permutation and $\pm$ is its sign. Use Laplace expansion of the determinants in (30) on the first k rows to write this sum as a product of determinants in unprimed and primed variables; each factor has form similar to the rightmost determinant in (29) and hence in (27), giving (26).

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References

[1] F. Bergeron, N. Bergeron, A.M. Garsia, M. Haiman, and G. Tesler, Lattice Diagram Polynomials and Extended Pieri Rules, *Adv. Math.*, to appear.

[2] A.M. Garsia, Alfred Young's Construction of the Irreducible Representations of S_n, course notes.

[3] A.M. Garsia and M. Haiman, A graded representation model for Macdonald's polynomials, *Proc. Natl. Acad. Sci. USA* **90** (April 1993), 3607–3610.

[4] I.G. Macdonald, "Symmetric Functions and Hall Polynomials," 2nd ed., Oxford University Press, Oxford, 1995.

[5] A. Young, On quantitative subsitutional analysis I, *Proc. Lond. Math. Soc.* **33** (1901), 97–146.

[6] ———, On quantitative subsitutional analysis II, *Proc. Lond. Math. Soc.* **34** (1902), 361–397.

[7] ———, On quantitative subsitutional analysis, *J. Lond. Math. Soc.* **3** (1928), 14–19.

[8] ———, On quantitative subsitutional analysis III, *Proc. Lond. Math. Soc.* **28** (1928), 255–292.

[9] ———, On quantitative subsitutional analysis IV, *Proc. Lond. Math. Soc.* **31** (1930), 253–272.

[10] ———, "The Collected Papers of Alfred Young 1873–1940," Mathematical Expositions, no. 21, University of Toronto Press, Toronto, 1977.