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Matchings in Graphs on Non-orientable Surfaces

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P.W. Kasteleyn stated that the number of perfect matchings in a graph embedding on a surface of genus g is given by a linear combination of 4^g Pfaffians of modified adjacency matrices of the graph, but didn't actually give the matrices or the linear combination. We generalize this to enumerating the perfect matchings of a graph embedding on an arbitrary compact boundaryless 2-manifold S with a linear combination of $2^{2-\chi(S)}$ Pfaffians. Our explicit construction proves Kasteleyn's assertion, and additionally treats graphs embedding on non-orientable surfaces. If a graph embeds on the connected sum of a genus g surface with a projective plane (respectively, Klein bottle), the number of perfect matchings can be computed as a linear combination of 2^{2g+1} (respectively, 2^{2g+2}) Pfaffians. We also introduce "crossing orientations," the analogue of Kasteleyn's "admissible orientations" in our context, describing how the Pfaffian of a signed adjacency matrix of a graph gives the sign of each perfect matching according to the number of edge-crossings in the matching. Finally, we count the perfect matchings of an $m \times n$ grid on a Möbius strip.

Key Words: Kasteleyn, perfect matching, dimer, graph, Pfaffian.

1. INTRODUCTION

In [4] and [5], Kasteleyn counts the number of perfect matchings in an undirected planar graph by orienting its edges in a certain fashion and showing that the Pfaffian of a "signed" adjacency matrix of this orientation equals the number of perfect matchings. If $\vec{G}$ is an orientation of the graph G, the Pfaffian of the antisymmetric matrix with components $a_{uv} = -a_{vu} = 1$ when $(u,v) \in V(\vec{G})$, 0 otherwise, has one term equalling ± 1 for each perfect matching of G, but the signs may vary. For a specific planar embedding of a graph, Kasteleyn gives an algorithm for constructing an "admissible orientation" for which all the signs are the same, thereby giving the proper enumeration. He gives a graph-theoretic characterization of admissible orientations, independent of the embedding. He also counts the perfect matchings in an $m \times n$ grid embedding on a torus by computing a linear combination of four Pfaffians of signed adjacency matrices of various orientations of the graph. In [6], he sketches how this could be extended to a large class of graphs embedding on a surface of genus g and states that the number of perfect matchings could be computed with a linear combination of $4^g = 2^{2g}$ Pfaffians, but does not explicitly show how to construct the orientations or the coefficients of the linear combination.

We present an explicit construction of formulas to compute the number of perfect matchings in finite graphs embedding in non-planar surfaces. Our construction also leads to a further result about graphs embedding on non-orientable compact surfaces. If a graph embeds on the connected sum of genus g surface with a projective plane (respectively, Klein bottle), the number of perfect matchings can be computed as a linear combination of 2^{2g+1} (respectively, 2^{2g+2}) Pfaffians. Together with orientable surfaces of genus g, these comprise all the compact boundaryless 2-manifolds.

In Section 2, we review Kasteleyn's method of enumerating perfect matchings in planar graphs, and the characterization of admissible orientations. In Section 3, we describe the complications that arise in a non-planar graph, and describe *crossing orientations*: for an arbitrary graph drawn in the plane with crossing edges, the terms of the Pfaffian are still in one-to-one correspondence with the perfect matchings, but there is generally no edge-orientation for which they have the same sign; however, an orientation can be constructed in which the sign depends on the number of pairs of crossing edges in a matching. In Section 4, we describe how to take a graph that embeds without edge-crossings on a 2-dimensional compact boundaryless manifold S, and then draw it in the plane with edge-crossings organized according to the structure of S. We also state the rule for constructing a crossing orientation of such a drawing, but we leave the detailed proof of the construction (as well as a general construction applicable to non-planar graphs drawn in some other fashion) to Section 6. In Section 5, we give explicit formulas to enumerate the perfect matchings in a graph embedding on a surface S. In Section 7, we apply these techniques to compute the number of perfect matchings in a grid on a Möbius strip, in analogy to Kasteleyn's grids in the plane and the torus. Finally, in the Appendix, we give an algorithm for computing Pfaffians; for replacing Pfaffians by determinants when using bipartite graphs; and an alternate formula for enumerating perfect matchings, in which the linear combination of a number of Pfaffians of numeric matrices is replaced by a single Pfaffian of a matrix with symbolic entries.

Following submission of this paper, the article [2] appeared in which Galluccio and Loebl also solve the problem of enumerating perfect matchings in graphs embedding on orientable compact surfaces. The results here were discovered independently; a comparison has been added to the end of Section 5.

2. ENUMERATING WEIGHTED PERFECT MATCHINGS ON PLANAR GRAPHS

Let G be an undirected graph on the vertex set $\{1, \ldots, 2p\}$, with a finite number of edges. We allow each edge $\{u, v\}$ to have a *weight* $W_{\{u,v\}}$, such as a complex number or a variable; to work in the context of unweighted graphs, set these weights to 1 for all edges, and 0 for vertex pairs that are not edges.

Let $\vec{G}$ be an orientation of G. Let $a_{uv} = a_{vu} = 0$ when $\{u, v\}$ is not an edge, or $a_{uv} = -a_{vu} = W_{\{u,v\}}$ be the weight of the directed edge (u, v). $A = [a_{uv}]$ is the *signed adjacency matrix* of $\vec{G}$, and is related to the ordinary directed weighted adjacency matrix B of $\vec{G}$ via $A = B - B^t$. A is an antisymmetric $2p \times 2p$ matrix.

We review the *Pfaffian* of a matrix. See [7, p. 317] for further details in this context. Let A be a $2p \times 2p$ antisymmetric matrix. Let $\mathbf{m} = \{\{u_1, v_1\}, \ldots, \{u_p, v_p\}\}$ range over the partitions of $\{1, \ldots, 2p\}$ into p sets of size 2, and define the *signed weight* of $\mathbf{m}$ as

$$w_{\mathbf{m}} = \operatorname{sign}\begin{pmatrix} 1 & 2 & \cdots & 2p-1 & 2p \\ u_1 & v_1 & \cdots & u_p & v_p \end{pmatrix} \cdot a_{u_1,v_1} \cdots a_{u_p,v_p} \tag{1}$$

(where the sign is of a permutation expressed in 2-line notation). Note that reversing the order of elements in a pair to v_k, u_k does not affect the value of $w_{\mathbf{m}}$ because it negates the permutation sign and also negates a matrix element, while permuting the order of the pairs does not change the sign of the permutation or of $w_{\mathbf{m}}$; any of the $2^p\, p!$ representations of $\mathbf{m}$ will give an equivalent result. The *Pfaffian* of A is defined as

$$\operatorname{Pf} A = \sum_{\mathbf{m}} w_{\mathbf{m}} \ . \tag{2}$$

Up to sign, the Pfaffian may be computed by the formula

$$(\operatorname{Pf} A)^2 = \det A \tag{3}$$

and since determinants are efficiently computable by row reduction, Pfaffians are as well; see Appendix A.1.

A *perfect matching* of G is a partition $\mathbf{m}$ of its $2p$ vertices into p edges of G. Take A to be the signed adjacency matrix of $\vec{G}$. When $\mathbf{m}$ is a partition that is not a perfect matching, $w_{\mathbf{m}} = 0$, so the nonzero terms of (2) correspond to the perfect matchings of G. We call $w_{\mathbf{m}}$ the *signed weight* of the perfect matching $\mathbf{m}$. We may write $w_{\mathbf{m}} = \epsilon_{\mathbf{m}} W_{\mathbf{m}}$, where the *unsigned weight* of $\mathbf{m}$ is

$$W_{\mathbf{m}} = \prod_{\{u,v\} \in \mathbf{m}} W_{\{u,v\}} \ , \tag{4}$$

and the sign of $\mathbf{m}$ is

$$\epsilon_{\mathbf{m}} = \operatorname{sign}\begin{pmatrix} 1 & 2 & \cdots & 2p-1 & 2p \\ u_1 & v_1 & \cdots & u_p & v_p \end{pmatrix} \cdot (-1)^{\#\text{ edges oriented } (v_k,u_k) \text{ in } \vec{G}}. \tag{5}$$

Relabeling the vertices causes the rows and columns of A to be simultaneously permuted, and changes the sign of all perfect matchings by the sign of that permutation.

There is a combinatorial interpretation of (3) in the context of perfect matchings. Consider the expansion

$$\det A = \sum_{\pi} (\operatorname{sign} \pi)\; a_{j_1,j_2} a_{j_2,j_3} \ldots a_{j_s,j_1}\; a_{k_1,k_2} a_{k_2,k_3} \ldots a_{k_t,k_1}\; \cdots \tag{6}$$

where we use the cycle notation $\pi = (j_1, j_2, \ldots, j_s)(k_1, k_2, \ldots, k_t)\cdots$ as π runs over permutations in S_{2p}. To be canonical, we assume $j_1, k_1, \ldots$ are the largest numbers in their respective cycles, and $j_1 < k_1 < \cdots$. If any cycles of π have of odd length, let π' be the permutation obtained on reversing the first such cycle. The π and π' terms cancel because the permutations have the same sign, but an odd number of the factors a_{uv} have been negated, so overall these two terms have opposite signs. Now suppose all cycles of π have even length. Then the perfect matchings $\mathbf{m} = \{\{j_1, j_2\}, \{j_3, j_4\}, \ldots, \{j_{s-1}, j_s\}, \{k_1, k_2\}, \ldots\}$ and $\mathbf{m}' = \{\{j_2, j_3\}, \{j_4, j_5\}, \ldots, \{j_s, j_1\}, \{k_2, k_3\}, \ldots\}$ have $w_{\mathbf{m}} w_{\mathbf{m}'}$ equal to the π term of (6). This is reversible, so pairs of partitions of $\{1, \ldots, 2p\}$ into p sets of size 2 are in bijective correspondence with permutations having only even length cycles. The permutation $\pi = \pi(\mathbf{m}, \mathbf{m}')$ corresponding to $\mathbf{m}, \mathbf{m}'$ consists of *superposition cycles*, so named because its cycles are formed by superimposing the two perfect matchings. If we only count terms of nonzero weight, this correspondence is a bijection between pairs of perfect matchings and permutations consisting only of superposition cycles. (Note that "superposition cycles" refers both to the permutation cycles $(j_1, j_2, \ldots, j_s), \ldots$ and to the graph cycles along the edges through these vertices.)

Kasteleyn formulated an algorithm to orient the edges of an undirected planar graph G so that all perfect matchings have $w_{\mathbf{m}} = W_{\mathbf{m}}$ (or they all have $w_{\mathbf{m}} = -W_{\mathbf{m}}$). Since perfect matchings are all counted with the same sign, the total unsigned weight of all perfect matchings in G equals $|\operatorname{Pf} A|$. First he characterizes all orientations that have this property.

(R1) An *admissible orientation* [6, p. 92] or *Pfaffian orientation* [7, p. 319] of a graph is any orientation of its edges such that in all superposition cycles, the number of edges pointing in each direction is odd.

It is not always possible to construct such an orientation, but when it is possible, Kasteleyn proved the following.

Theorem 2.1. *If $\vec{G}$ is an admissible orientation of G, then all perfect matchings have the same sign. Conversely, if all perfect matchings have the same sign in some orientation $\vec{G}$, it is an admissible orientation.*

Kasteleyn found a rule to construct an admissible orientation of a planar graph. It is easier to use than (R1) and implies (R1).

(R2) [6, p. 93] A planar graph can be oriented so that around each face, the number of edges pointing clockwise is odd.

Note that (R1) depends only on the topology of the graph, while (R2) depends on the specific embedding of it in the plane.

An orientation (R2) $\vec{G}$ of the undirected graph G may be constructed in polynomial time by the following algorithm. Orient the edges of a spanning subgraph of G arbitrarily. Traverse the remaining edges of G in an order that forms at most one face at a time (not counting the infinite face). As a face is formed, orient the new edges so that the face has an odd number of edges pointing clockwise around it. This construction has the following property that immediately implies (R1).

Lemma 2.1. *[6, p. 93] Let G be a planar graph oriented by (R2). In every closed cycle, the number of edges oriented clockwise along the cycle is opposite in parity to the number of vertices enclosed in the region of the plane inside the cycle.*

This implies (R1) because any superposition cycle must enclose an even number of vertices, since they cannot be matched with vertices in the region outside the cycle.

3. NON-PLANAR GRAPHS

Now we consider the problem of enumerating perfect matchings in an arbitrary finite graph. Fix once and for all a drawing of the graph G in the plane. Edges may cross each other at non-vertex points. For any cycle C in the graph, and vertex v not along the cycle, the question of whether v is inside or outside C may be complicated by C having crossing edges. We say that v is *inside* C if the winding number of C around v is odd, and is *outside* if the winding number is even. Note that it does not matter in which direction we traverse C since traversing it the other way negates the winding number but does not affect its parity. Every vertex is inside, outside, or along C, and the total number of vertices inside C is denoted $\iota(C)$. See Figure 1. Here we view the plane as the complex plane

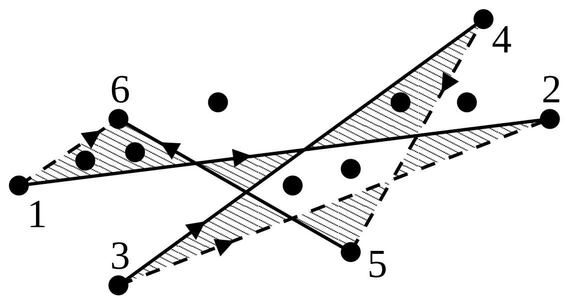

FIG. 1. Since the cycle $C = (1, \ldots, 6)$ crosses itself, it divides the plane into several regions; the dark ones are "inside" and the white ones are "outside." The cycle has 6 vertices on it, $\iota(C) = 3$ inside, and 4 outside. There are $\kappa_e(C) = 4$ monochromatic edge-crossings, and $r(C) = 2$ edges routed the wrong way.

and coordinatize the drawing of the graph accordingly; if C is drawn as a piecewise-differentiable curve, the winding number is

$$\frac{1}{2\pi i} \oint_C \frac{dz}{z - v} \tag{7}$$

and more generally, it is determined by the element that C represents in the fundamental group of the punctured complex plane $\mathbb{C} - z$ (see [8, p. 347#6a]).

In our fixed drawing of G, let $\mathbf{m}$ be a perfect matching, and $\kappa(\mathbf{m})$ be the number of times edges in it cross. If edges are drawn so as to cross themselves or to cross each other multiple times, $\kappa(\mathbf{m})$ should include this multiplicity. If k edges pass through the same non-vertex point, the separate pairs form $\binom{k}{2}$ crossings. Now consider any superposition cycle C formed by superimposing two perfect matchings $\mathbf{m}$ and $\mathbf{m}'$. Color the edges of C from $\mathbf{m}$ black, and the edges from $\mathbf{m}'$ white. Let $\kappa_e(C)$ denote the number of *monochromatic crossings* among edges of C, that is, crossings between edges an even distance apart along the cycle. Again, this should be counted with multiplicity if appropriate. Edges not in C are not included in this count. If C has length 2 (because $\mathbf{m}$ and $\mathbf{m}'$ both share a common edge), it is traversed once as a black edge $\{u, v\}$ and then once as a white edge $\{v, u\}$, so every crossing is actually counted twice in $\kappa_e(C)$, giving $\kappa_e(C) \equiv 0 \pmod 2$. Also in this case, no vertices are inside C, so $\iota(C) = 0$. See Section 6.2 for further discussion of complications in counting crossings.

The *routing number* $r(C)$ of a cycle $C = (v_1, \ldots, v_s)$ is the number of edges oriented opposite to the direction we traverse the cycle:

$$r(C) = \left| \{(v_2, v_1), (v_3, v_2), \ldots, (v_s, v_{s-1}), (v_1, v_s)\} \cap E(\vec{G}) \right| . \tag{8}$$

Although G is undirected, we need to traverse C one direction or the other; the direction is known because C comes from a permutation, or by specifying clockwise or counterclockwise, etc. For even length cycles (such as all superposition cycles), the two routings $r(v_1, \ldots, v_s)$ and $r(v_s, \ldots, v_1)$ have the same parity because the number of edges pointing one way, plus the number pointing the other way, is the even number s. A cycle is *clockwise odd* [6, p. 92] when it has an odd number of edges pointing along it when traversed clockwise. Equivalently, $r(C)$ is odd when C is traversed counterclockwise. Rule (R1) requires $r(C)$ to be odd for all superposition cycles, and by Lemma 2.1, rule (R2) requires $r(C)$ and $\iota(C)$ to have opposite parity when C is any counterclockwise cycle (including ones of odd length). Theorem 2.1 is a consequence of the following property of routing numbers.

LEMMA 3.1. *Let $\mathbf{m}$ and $\mathbf{m}'$ be perfect matchings. Let $\pi = \pi(\mathbf{m}, \mathbf{m}') = (j_1, \ldots, j_s)(k_1, \ldots, k_t) \cdots$ be the permutation formed as their superposition. Then*

$$\epsilon_{\mathbf{m}} \epsilon_{\mathbf{m}'} = \prod_{C \in \pi} (-1)^{r(C)+1} \tag{9}$$

where C runs over the cycles of π.

Proof. In (6), the π term is

$$\begin{aligned} w_{\mathbf{m}} w_{\mathbf{m}'} &= (\operatorname{sign} \pi)\, a_{j_1,j_2} a_{j_2,j_3} \ldots a_{j_s,j_1}\; a_{k_1,k_2} a_{k_2,k_3} \ldots a_{k_t,k_1} \;\cdots \\ &= \epsilon_{\mathbf{m}} \epsilon_{\mathbf{m}'} W_{\mathbf{m}} W_{\mathbf{m}'} \; . \end{aligned} \tag{10}$$

The cycles of π are superposition cycles, so they all have even length, and each contributes a factor -1 to $\operatorname{sign} \pi$. Within a cycle, say the first, the matrix elements are the negative of the edge-weight when they are oriented opposite to the direction the cycle is traversed, so

$$a_{j_1,j_2} a_{j_2,j_3} \ldots a_{j_s,j_1} = (-1)^{r(j_1,j_2,\ldots,j_s)} w_{j_1,j_2} w_{j_2,j_3} \ldots w_{j_s,j_1} \; . \tag{11}$$

Thus each cycle C contributes a sign $(-1)^{r(C)+1}$, and the total contribution from all cycles is given by (9). ∎

In a non-planar graph, the notions of clockwise or counterclockwise routing of edges in a cycle do not make sense if the cycle has crossing edges. Also, vertices inside a cycle can be matched with vertices outside the cycle, so a cycle needn't enclose an even number of vertices. We require alternatives to (R1) and (R2). The following condition (R3) generalizes (R1) to an arbitrary graph, and an algorithm (R4) in the next section generalizes (R2).

(R3) A *crossing orientation* of a graph is an orientation in which for every superposition cycle, the number of edges pointing along it in one direction has opposite parity to the number of monochromatic crossings among the edges of the cycle plus the number of vertices in the plane regions inside the cycle. Equivalently, $r(C) + \kappa_e(C) + \iota(C)$ is odd.

Unlike an admissible orientation (R1), a crossing orientation can always be constructed; this will be shown in Section 6. For planar embeddings of graphs, crossing orientations and admissible orientations are equivalent, since all superposition cycles have $\kappa_e(C) = 0$ and $\iota(C)$ even, requiring $r(C)$ to be odd for both (R1) and (R3). Unlike admissible orientations, whether an orientation is a crossing orientation depends on how the graph is drawn in the plane: deforming an edge to cross itself, or to move it past a vertex, will change the parity of $\kappa_e(C) + \iota(C)$.

We have named these crossing orientations because they permit us to determine the parity of the number of crossings in a perfect matching.

THEOREM 3.1. *(a) A graph may be oriented so that every perfect matching has sign*

$$\epsilon_{\mathbf{m}} = \epsilon_0 \cdot (-1)^{\kappa(\mathbf{m})} \tag{12}$$

where $\epsilon_0 = \pm 1$ is constant; ϵ_0 may be interpreted as the sign of a perfect matching with no crossing edges when such exists.

(b) An orientation of a graph satisfies (a) if, and only if, it is a crossing orientation.

See Section 6 for the detailed proof. Kasteleyn [6, p. 99] states that an orientation with property (a) exists, but this explicit characterization (b) is new.

4. EMBEDDING GRAPHS ON SURFACES, AND PLANE MODELS OF SURFACES

Let S be a two dimensional surface. The graph G embeds in S if it can be drawn on S without any edges crossing. Although we will enumerate perfect matchings in graphs embedding in non-planar surfaces, this will be accomplished by drawing G in the plane with the edges crossing in a systematic fashion dictated by the structure of S. We still require that only the endpoints, not the interiors, of edges be incident with vertices, but we allow edges to cross each other at non-vertex points.

Any compact boundaryless 2-dimensional surface S can be represented in the plane by a *plane model* [1, p. 32] (also called a *pasting map* [9, p. 16]). Draw a $2n$ sided polygon P (or other non-intersecting closed curve with $2n$ segments marked off), and introduce symbols $a_1, \ldots, a_n$. Form n pairs of sides p_j, p'_j, $j = 1, \ldots, n$. Paste together p_j and p'_j. Any S can be represented by a suitable polygon and pastings. If p_j and p'_j are pasted together by traversing P clockwise along both, then place the label a_j along both p_j and p'_j, and say that S is *j-nonoriented.* If they are pasted by traversing P clockwise along one and counterclockwise along the other, label the clockwise one a_j, the counterclockwise one a_j^{-1}, and say that S is *j-oriented.* Form a word σ from these $2n$ symbols by starting at any side and reading off the labels as P is traversed clockwise. We also say that σ is j-oriented or j-nonoriented. If the occurrences of a_j or a_j^{-1} are interleaved with the occurrences of a_k or a_k^{-1}, such as in $\sigma = \cdots a_j \cdots a_k^{-1} \cdots a_j \cdots a_k \cdots$, we say that σ is *j,k-alternating*; otherwise it is *j,k-nonalternating.* This is a property of σ, not of S. Pasting can also be extended to surfaces with boundaries by using polygons with more than $2n$ sides, of which only n pairs are pasted together. The unpasted sides are boundaries of S. Label each unpasted side with its own symbol. In the ensuing discussion, these labels make no contribution to the computations; we could contract each of these sides down to a point without affecting whether a graph embeds in the resulting surface, so just delete the symbols for boundary sides from σ.

The orientable compact boundaryless surfaces are classified up to homeomorphism by their genus $g = 0, 1, 2, \ldots$, while the non-orientable ones are homeomorphic to the connected sum of genus g surface with either a Klein bottle or a projective plane, for some $g \geq 0$. Possible words representing these surfaces are in Table 1. Other words are also possible, and may be used if convenient in a particular problem. Note that the number n is related to the Euler characteristic of the surface by $n \geq 2 - \chi(S)$, and the words in the table have $n = 2 - \chi(S)$.

Now take an embedding of a graph G on this surface, and draw it within this plane model of the surface. Edges wholly contained inside the polygon P do not cross, and are called 0-edges. The edges that go through sides p_j, p'_j of P are called *j-edges.* Initially, a j-edge from vertex u to v is drawn as a curve starting at u, terminating at a point u' somewhere along p_j, resuming at the point v' identified with u' on p'_j, and continuing to v, because that is what the plane model of S dictates. To extend this drawing of an embedding of G in S to a drawing of G in the plane with crossing

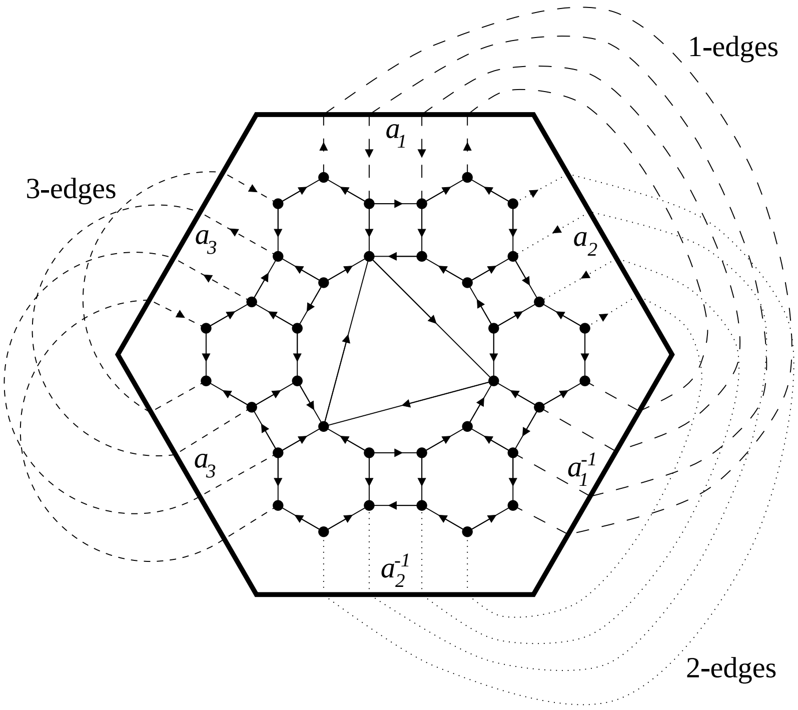

FIG. 2. A graph that embeds on the connected sum of a torus with a projective plane. The surface word $\sigma = a_1 a_2 a_1^{-1} a_2^{-1} a_3 a_3$ is $1, 2$-alternating and 3-nonoriented. The 0-edges are drawn as solid lines; the 1, 2, 3-edges cross outside the hexagon P used to represent the surface. One possible crossing orientation given by rule (R4) is shown.

Surface	word σ	n
Sphere (or plane)	null word	0
Projective plane	$a_1\, a_1$	1
Klein bottle	$a_1 a_1 a_2 a_2$ or $a_1 a_2 a_1^{-1} a_2$	2
genus g surface	$a_1 a_2 a_1^{-1} a_2^{-1}\, a_3 a_4 a_3^{-1} a_4^{-1} \cdots$	
	$a_{2g-1} a_{2g} a_{2g-1}^{-1} a_{2g}^{-1}$	$2g$
Connected sum of genus g surface with		
Projective plane	$a_1 a_2 a_1^{-1} a_2^{-1} \cdots a_{2g-1} a_{2g} a_{2g-1}^{-1} a_{2g}^{-1}$	
	$a_{2g+1} a_{2g+1}$	$2g+1$
Klein bottle	$a_1 a_2 a_1^{-1} a_2^{-1} \cdots a_{2g-1} a_{2g} a_{2g-1}^{-1} a_{2g}^{-1}$	
	$a_{2g+1} a_{2g+1} a_{2g+2} a_{2g+2}$	$2g+2$

TABLE 1.

Words representing surfaces

edges, we draw in the missing segment of the edge as a curve from u' to v' (with no self-intersections) in the region of the plane outside P. Crossings may be formed exterior to P among edges extended in this fashion. See Figure 2. It permissible for an edge to be both a j-edge and a k-edge for $j \neq k$, and to make multiple crossings of p_j and p'_j.

If S is j-oriented, the j-edges can be drawn in the plane without crossing each other (and in any case must be drawn so that each pair crosses an even number of times). If there are $N(j)$ j-edges, as we traverse p_j clockwise we encounter them in the order $1, 2, \ldots, N(j)$, and as we traverse p'_j clockwise, we encounter them in the reverse order, $N(j), \ldots, 2, 1$. Complete the drawing of the edges $1, 2, \ldots, N(j)$ by looping around the outside of P clockwise so they do not cross.

If S is j-nonoriented, the j-edges can be drawn so that every pair of j-edges crosses exactly once (and in any case must be drawn so that each pair crosses an odd number of times). As we traverse either p_j or p'_j clockwise, we encounter the edges in the same order $1, 2, \ldots, N(j)$. Complete the drawing of the edges by looping around the outside of P clockwise, so that each pair of edges crosses exactly once.

An edge that crosses p_j, p'_j multiple times should be counted in $N(j)$ with multiplicity equal to the number of crossings. It is also possible for an edge to be counted as both a j-edge and a k-edge, an so on.

We may form a crossing orientation of such a drawing of G as follows, provided the subgraph of 0-edges is connected; see Figure 2. This is proved in Section 6.4. If the subgraph of 0-edges is not connected, extra edges of weight 0 may be added to connect it. A longer method that works for all graphs, even without the crossings organized by pastings, is given in Section 6.

(R4) Orient the subgraph of 0-edges so that all its faces are clockwise odd. Orient each j-edge e ($j > 0$) as follows. Ignoring all other non 0-edges, there is a face formed by e and certain 0-edges along the boundary of the subgraph of 0-edges. Orient e so that this face is clockwise odd.

Although all surfaces we consider can be represented by pastings on a single polygon P as we've described, it may be convenient to represent a surface by pastings on a compact planar region P with a finite number of holes. The outer perimeter is a polygon P_0. The holes are polygons $P_1, \ldots, P_r$. Pastings occur between pairs of edges on each polygon, but not from one polygon to another. The pastings dictate how edges cross in the region interior to the polygons $P_1, \ldots, P_r$, and exterior to P_0. If a_j is a label placed on an edge of one polygon, and a_k is on another polygon, the j and k-edges never cross each other, so the model is j,k-nonalternating. Rule (R4) may be used as stated to orient graphs drawn in such a plane model. See Figure 10(a).

5. ENUMERATING WEIGHTED PERFECT MATCHINGS ON NON-PLANAR GRAPHS

Let A be the signed adjacency matrix for a crossing orientation (R3) of the graph G, whether constructed by (R4) or by the full method of Section 6.

Consider any perfect matching $\mathbf{m}$ in G. Let $N_{\mathbf{m}}(j)$ be the number of j-edges in $\mathbf{m}$, counted with multiplicity when appropriate. Let $C_{\mathbf{m}}(j,k)$ be the number of crossings formed by a j-edge with a k-edge. Then modulo 2, for $0 < j < k \leq n$,

$$C_{\mathbf{m}}(j,j) \equiv \begin{cases} \binom{N_{\mathbf{m}}(j)}{2} & \text{if } \sigma \text{ is } j\text{-nonoriented;} \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

$$C_{\mathbf{m}}(j,k) \equiv \begin{cases} N_{\mathbf{m}}(j) \cdot N_{\mathbf{m}}(k) & \text{if } \sigma \text{ is } j,k\text{-alternating;} \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

The total number of pairs of crossing edges is

$$C_{\mathbf{m}} = \sum_{1 \leq j \leq k \leq n} C_{\mathbf{m}}(j,k)\ . \tag{15}$$

In $\operatorname{Pf} A$, every perfect matching $\mathbf{m}$ is counted with weight

$$w_{\mathbf{m}} = \epsilon_0 (-1)^{C_{\mathbf{m}}} W_{\mathbf{m}}\ . \tag{16}$$

We will find a linear combination of Pfaffians of different weightings of the graph for which the weight of $\mathbf{m}$ is simply $\epsilon_0 W_{\mathbf{m}}$.

Introduce new variables $x_1, \ldots, x_n$. Multiply the weights of all j-edges by x_j, use the same crossing orientation as for A on the resulting weighted graph, and let $B(x_1, \ldots, x_n)$ be the signed adjacency matrix (as a function of these variables). Call this the *x-adjacency matrix.* So $b_{uv} = a_{vu}$ when (u, v) is not a j-edge for any $j \neq 0$, while $b_{uv} = a_{uv} x_j$ when $\{u, v\}$ is a j-edge. If an edge makes s crossings in either direction from p_j to p'_j, use $b_{uv} = a_{vu} {x_j}^s$. If it is both a j-edge and a k-edge, use $b_{uv} = a_{uv} x_j x_k$, and so on for similar complications.

Let $f(\omega_1, \ldots, \omega_n) \in \mathbb{C}[\omega_1, \ldots, \omega_n]/(1 - {\omega_1}^4, \ldots, 1 - {\omega_n}^4)$: in other words, all exponents are to be reduced modulo 4 to one of $0, 1, 2, 3$.

$$f = \sum_{0 \le r_1, r_2, \ldots, r_n \le 3} \alpha_{r_1, \ldots, r_n} {\omega_1}^{r_1} \cdots {\omega_n}^{r_n} \tag{17}$$

The *f-weight* of the perfect matching $\mathbf{m}$ is

$$w_{\mathbf{m}}(f) = f(i^{N_{\mathbf{m}}(1)}, \ldots, i^{N_{\mathbf{m}}(n)}) w_{\mathbf{m}} \tag{18}$$

(where $i = \sqrt{-1}$), and the f-weight of G is

$$\begin{aligned} w_G(f) &= \sum_{r_1, r_2, \ldots, r_n} \alpha_{r_1, \ldots, r_n} \operatorname{Pf} B(i^{r_1}, \ldots, i^{r_n}) && (19) \\ &= \sum_{r_1, r_2, \ldots, r_n} \alpha_{r_1, \ldots, r_n} \sum_{\mathbf{m}} w_{\mathbf{m}} \cdot i^{r_1 N_{\mathbf{m}}(1)} \cdot \ldots \cdot i^{r_n N_{\mathbf{m}}(n)} && (20) \\ &= \sum_{\mathbf{m}} w_{\mathbf{m}}(f) . && (21) \end{aligned}$$

We will find an f for which $w_{\mathbf{m}}(f) = \epsilon_0 W_{\mathbf{m}}$ for all $\mathbf{m}$.

For $0 < j < k \le n$, let

$$L_{jj} = \begin{cases} \frac{1-i}{2}\omega_j + \frac{1+i}{2}\omega_j^{-1} = \frac{1-i}{2}\left(\omega_j + i\omega_j^{-1}\right) & \text{if } \sigma \text{ is } j\text{-nonoriented;} \\ 1 & \text{otherwise.} \end{cases} \tag{22}$$

$$L_{jk} = \begin{cases} \frac{1}{2}\left(1 + {\omega_j}^2 + {\omega_k}^2 - {\omega_j}^2{\omega_k}^2\right) & \text{if } \sigma \text{ is } j,k\text{-alternating;} \\ 1 & \text{otherwise.} \end{cases} \tag{23}$$

These polynomials depend only on σ, j, and k, not on any perfect matching. On multiplying f by them, the weights of different perfect matchings change in a manner depending on the number of crossings they contain. The polynomial (23) generalizes the linear combination of Pfaffians Kasteleyn used to compute perfect matchings on a torus [4], while (22) is new, and allows computing perfect matchings on a non-orientable surface.

Lemma 5.1.

$$\begin{array}{llcl} (a) & w_{\mathbf{m}}(L_{jj}f) & = & (-1)^{C_{\mathbf{m}}(j,j)} w_{\mathbf{m}}(f) \\ (b) & w_{\mathbf{m}}(L_{jk}f) & = & (-1)^{C_{\mathbf{m}}(j,k)} w_{\mathbf{m}}(f) \end{array} \tag{24}$$

Proof. (a) If σ is j-oriented, then $C_{\mathbf{m}}(j,j) \equiv 0 \pmod 2$ and $L_{jj} = 1$, so (a) holds.

So suppose σ is j-nonoriented. Multiplying the weights of the j-edges by a number α results in the weight of this matching being multiplied by $\alpha^{N_{\mathbf{m}}(j)}$, so we have

$$w_{\mathbf{m}}({\omega_j}^r f) = (i^{r \cdot N_{\mathbf{m}}(j)}) w_{\mathbf{m}}(f) \ . \tag{25}$$

Thus,

$$w_{\mathbf{m}}(L_{jj}f) = \left(\frac{1-i}{2} (i^{N_{\mathbf{m}}(j)}) + \frac{1+i}{2} (i^{-N_{\mathbf{m}}(j)}) \right) w_{\mathbf{m}}(f) \ . \tag{26}$$

It is readily verified that

$$\frac{1-i}{2}(i^N) + \frac{1+i}{2}(i^{-N}) = (-1)^{\binom{N}{2}} = \begin{cases} 1 & \text{if } N \equiv 0 \text{ or } 1 \pmod 4; \\ -1 & \text{if } N \equiv 2 \text{ or } 3 \pmod 4, \end{cases} \tag{27}$$

so (a) holds by (13).

(b) If σ is not j,k-alternating, then $C_{\mathbf{m}}(j,k) \equiv 0$ and $L_{jk} = 1$, so (b) holds.

If σ is j,k-alternating, then

$$w_{\mathbf{m}}(L_{jk}f) = \frac{1}{2} \left(1 + (-1)^{N_{\mathbf{m}}(j)} + (-1)^{N_{\mathbf{m}}(k)} - (-1)^{N_{\mathbf{m}}(j) + N_{\mathbf{m}}(k)} \right) w_{\mathbf{m}}(f) \ . \tag{28}$$

It is readily verified that

$$\begin{aligned} &\frac{1}{2} \left(1 + (-1)^{N_{\mathbf{m}}(j)} + (-1)^{N_{\mathbf{m}}(k)} - (-1)^{N_{\mathbf{m}}(j) + N_{\mathbf{m}}(k)} \right) \\ &\quad = (-1)^{N_{\mathbf{m}}(j) \cdot N_{\mathbf{m}}(k)} = \begin{cases} -1 & \text{if } N_{\mathbf{m}}(j) \text{ and } N_{\mathbf{m}}(k) \text{ are both odd;} \\ 1 & \text{if } N_{\mathbf{m}}(j) \text{ or } N_{\mathbf{m}}(k) \text{ is even,} \end{cases} \end{aligned} \tag{29}$$

so (b) holds by (14). ∎

Theorem 5.1. *The total unsigned weight of all perfect matchings in G is*

$$\epsilon_0 w_G \left(\prod_{1 \le j \le k \le n} L_{jk} \right) \ . \tag{30}$$

Proof. We have

$$w_{\mathbf{m}}\left(\prod_{1\le j\le k\le n} L_{jk}\right) = \left(\prod_{1\le j\le k\le n} (-1)^{C_{\mathbf{m}}(j,k)}\right) w_{\mathbf{m}}(1) = (-1)^{C_{\mathbf{m}}} w_{\mathbf{m}}(1)\ .$$

But by (16), $w_{\mathbf{m}}(1) = w_{\mathbf{m}} = \epsilon_0(-1)^{C_{\mathbf{m}}} W_{\mathbf{m}}$, and on plugging this into the above equation, the signs cancel and we are left with

$$w_{\mathbf{m}}\left(\prod_{1\le j\le k\le n} L_{jk}\right) = \epsilon_0 W_{\mathbf{m}}\ . \tag{31}$$

Summing both sides of (31) over all perfect matchings and moving the sign ϵ_0 to the other side gives (30). ∎

In the product in (30), we may reduce all exponents modulo 4. If σ is j-oriented then the exponents of ω_j are always 0 or 2, and otherwise they are always 1 or 3. There are 2^n monomials in $\omega_1, \ldots, \omega_n$ with such exponents, so the expansion into Pfaffians has up to 2^n terms. On setting n for each surface according to Table 1, we have established the following theorem.

Theorem 5.2. *The number of perfect matchings in a graph may be computed as a linear combination of Pfaffians of modified signed adjacency matrices of the graph. This is achieved by evaluating equation (30), via (22), (23); (17); and (19). An upper bound on the total number of Pfaffians necessary in this expansion depends on the surface S in which the graph embeds without crossings: plane* (1)*, projective plane* (2)*, Klein bottle* (4)*, genus g surface* (4^g)*, connected sum of a projective plane with a genus g surface* (2^{2g+1})*, connected sum of a Klein bottle with a genus g surface* (2^{2g+2})*. In short, this bound is $2^{2-\chi(S)}$ Pfaffians.*

For example, consider the graph in Figure 2 that embeds on the connected sum of a torus with a projective plane. Letting $B(x_1, x_2, x_3)$ be its x-adjacency matrix, with the 0-edges having weight 1 and the j-edges having weight x_j, the number of perfect matchings is given by $w_G(f)$, where by (30),

$$\begin{aligned}
f &= L_{12}L_{33} \qquad \text{(omitting factors that equal 1)}\\
&= \frac{1}{2}(1 + {\omega_1}^2 + {\omega_2}^2 - {\omega_1}^2{\omega_2}^2)\cdot\frac{1-i}{2}(\omega_3 + i\,{\omega_3}^{-1})\\
&= \frac{1-i}{4}\Big(\omega_3 + i\,{\omega_3}^3 + {\omega_1}^2\omega_3 + i\,{\omega_1}^2{\omega_3}^3 + {\omega_2}^2\omega_3
\end{aligned}$$

$$+ i\,{\omega_2}^2{\omega_3}^3 - {\omega_1}^2{\omega_2}^2\omega_3 - i\,{\omega_1}^2{\omega_2}^2{\omega_3}^3\Big) \; .$$

Thus, by (19),

$$\begin{aligned} w_G(f) &= \frac{1-i}{4}\cdot\Big(\operatorname{Pf} B(1,1,i) + i\operatorname{Pf} B(1,1,-i) \\ &\qquad + \operatorname{Pf} B(-1,1,i) + i\operatorname{Pf} B(-1,1,-i) \\ &\qquad + \operatorname{Pf} B(1,-1,i) + i\operatorname{Pf} B(1,-1,-i) \\ &\qquad - \operatorname{Pf} B(-1,-1,i) - i\operatorname{Pf} B(-1,-1,-i)\Big) \\ &= \operatorname{Re}\frac{1-i}{2}\Big(\operatorname{Pf} B(1,1,i) + \operatorname{Pf} B(-1,1,i) \\ &\qquad\quad + \operatorname{Pf} B(1,-1,i) - \operatorname{Pf} B(-1,-1,i)\Big) \\ &= 3232. \end{aligned}$$

Remark 5.1. For non-orientable surfaces, the number of Pfaffians can actually be cut in half, provided the weights are real or are indeterminates treated as reals. If we represent the projective plane by the word $\sigma = a_1a_1$, the two Pfaffians obtained from (22) are complex conjugates, so we may compute the number of perfect matchings by taking the real part of only one Pfaffian:

$$\#\text{ perfect matchings} = \operatorname{Re} w_G\big((1-i)\omega_1\big) = \operatorname{Re}\big((1-i)\operatorname{Pf} B(i)\big)\; .$$

Similarly, using the representations of the non-orientable surfaces shown in Table 1, the subword $a_{2g+1}a_{2g+1}$ contributes one factor of form (22) that may be replaced by taking the real part of the result of using $(1-i)\omega_{2g+1}$ instead. Note that ω_{2g+1} does not appear in any other factors because $a_{2g+1}a_{2g+1}$ does not alternate with any other letters.

Remark 5.2. Galluccio and Loebl [2] have also enumerated the perfect matchings in graphs embedding on orientable surfaces of genus g, using a linear combination of 4^g Pfaffians. Their enumeration formula and edge-orientations are the ones obtained here for a plane model consisting of one $4g$ sided polygon pasted by $\sigma = a_1a_2a_1^{-1}a_2^{-1}\cdots a_{2g-1}a_{2g}a_{2g-1}^{-1}a_{2g}^{-1}$. Their analysis is similar to Theorem 3.1(a) but restricted to the case of graphs drawn this way. We treat more general plane models, and give the machinery of crossing orientations that fully generalizes Kasteleyn's "admissible orientations" to any finite graph drawn in the plane with crossing edges; this permits us to handle non-orientable surfaces concurrently with the orientable ones.

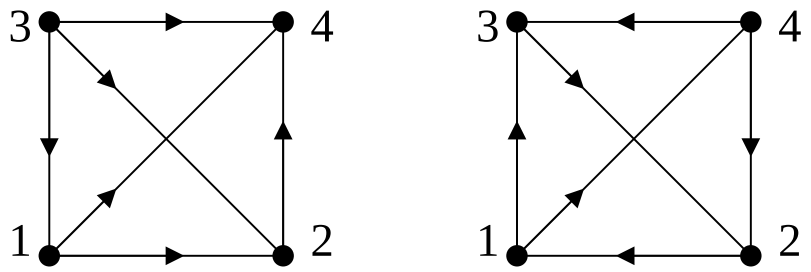

FIG. 3. A protected crossing and its possible crossing orientations, up to labeling.

6. CONSTRUCTION OF A CROSSING ORIENTATION

6.1. Orienting graphs with protected crossings

A *protected crossing* of two edges $\{1,4\}$, $\{2,3\}$ has the form shown in Figure 3. The four vertices form the complete graph K_4, and no other vertices or edges are drawn in the interior of the square 1243. Kasteleyn described how to orient graphs whose crossings are all protected; we first do this, and then we reduce the general situation to this case.

LEMMA 6.1. *[6, p. 98] A graph G whose crossings are all protected crossings can be oriented so that every perfect matching has sign $\epsilon_{\mathbf{m}} = \epsilon_0 \cdot (-1)^{\kappa(\mathbf{m})}$ for a constant $\epsilon_0 = \pm 1$.*

Proof. Let G be a graph whose N crossings are all protected crossings. Label the vertices in each $1_k, 2_k, 3_k, 4_k$ (for $k = 1, \ldots, N$) in the same form as Figure 3. Delete all edges $\{2_k, 3_k\}$ to obtain a planar graph, and orient it by (R2). Let $\epsilon_0 = \pm 1$ be the common sign of perfect matchings in this orientation. Finally, put back in the edges $\{2_k, 3_k\}$, oriented so that the cycle $(1_k, 3_k, 2_k)$ is clockwise odd, resulting in an orientation $\vec{G}$ of the original G. Note that if we deleted the edges $\{1_k, 4_k\}$ from $\vec{G}$, the resulting oriented planar graph would also satisfy (R2).

In a graph so oriented, we consider the signs of three perfect matchings that are identical but for the edges in a protected crossing. There are eight orientations possible in a protected crossing, but up to cyclic rotation of the labels, these reduce to two; see Figure 3, where the crossed edges are directed $(1,4)$ and $(2,3)$. The three other rotations of this are similar. Let $\mathbf{m}_1$ contain $\{1,4\}$, $\{2,3\}$; $\mathbf{m}_2$ contain $\{1,3\}$, $\{2,4\}$; and $\mathbf{m}_3$ contain $\{1,2\}$, $\{4,3\}$. The crossing in $\mathbf{m}_1$ has a contribution to its sign $\epsilon_{1432} = -1$

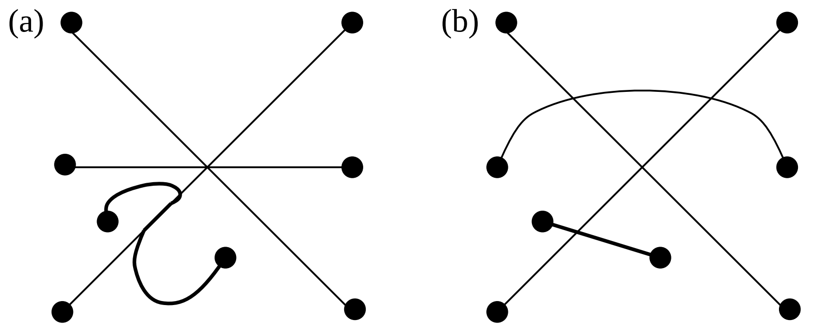

FIG. 4. Separate all crossings. Step 1: (a) has three edges through one point, and two edges sharing a segment with an infinite number of points. (b) These are deformed so as to have only two edges crossing at any point, and a finite number of crossings.

due to these vertices, while the non-crossings in $\mathbf{m}_2$ and $\mathbf{m}_3$ yield signs $\epsilon_{3124} = +1$, $\epsilon_{1234} = +1$, so $\epsilon_{\mathbf{m}_1} = -\epsilon_{\mathbf{m}_2} = -\epsilon_{\mathbf{m}_3}$.

Now let $\mathbf{m}$ be any perfect matching of G. The crossed edges are $\{1_k, 4_k\}$, $\{2_k, 3_k\}$ for $\kappa(\mathbf{m})$ values of k. These are vertex disjoint because it's a perfect matching. Replace these by $\{1_k, 2_k\}$, $\{3_k, 4_k\}$ to obtain a new perfect matching $\mathbf{m}'$ with no crossed edges. The sign is $\epsilon_0 = \epsilon_{\mathbf{m}'} = \epsilon_{\mathbf{m}} \cdot (-1)^{\kappa(\mathbf{m})}$, so $\epsilon_{\mathbf{m}} = \epsilon_0 \cdot (-1)^{\kappa(\mathbf{m})}$. ▮

6.2. Orienting graphs without protected crossings

We now prove Theorem 3.1(a). Let G be a graph with crossing edges. We will augment it to have only protected crossings; orient it by the previous section; and then remove the augmentation to form an orientation $\vec{G}$ of G.

1. Draw G with crossing edges. We augment G to a graph G' without multiplicities in edge-crossings as follows. Deform the edges if necessary (without passing any vertex through an edge) so that only two edges cross through any non-vertex point, and there are only a finite number of crossings. See Figure 4. If any edge e crosses itself, or some edges cross in multiple locations, add in 2 vertices between each crossing as in Figure 5. This breaks the edges into odd numbers of segments, with no segment intersecting itself, and no two segments intersecting multiple times. If the segments are alternately colored black and white, with the initial and ending segments in black, the crossings in G' all occur on black segments. A perfect matching of G containing e corresponds to a perfect matching in G' containing the black segments, and a perfect matching of G without e

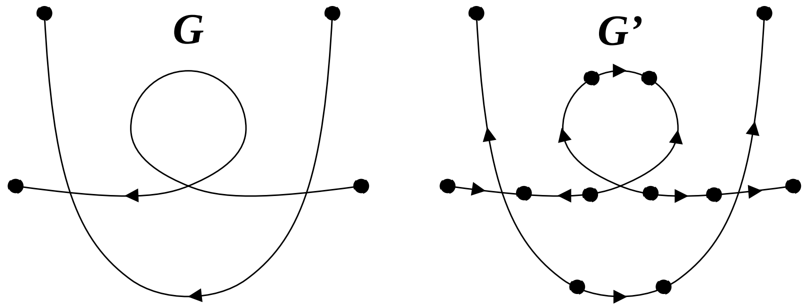

FIG. 5. **Isolate crossings.** Step 1: Form G' from G by adding pairs of vertices to split edges so that no edge has self-crossings or is in multiple crossings. Step 5: Convert the orientation $\vec{G}'$ to an orientation $\vec{G}$ according to which direction an odd number of arrows point in each split edge.

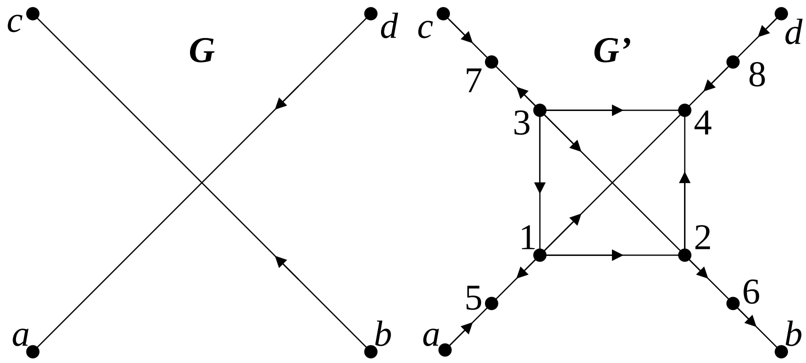

FIG. 6. **Protect crossings.** Step 2: Augment each crossing in G to form a protected crossing in G'. Step 5: Convert the orientation $\vec{G}'$ to an orientation $\vec{G}$ according to which direction an odd number of arrows point in each split edge.

corresponds to a perfect matching containing the white segments and no crossings involving them.

2. In a neighborhood of each point where two edges cross, add 8 vertices and form new edges to create the protected crossing configuration shown in Figure 6. We have finished constructing G'.

3. Form an orientation $\vec{G}'$ of G' by applying Lemma 6.1.

4. For each protected crossing of G', delete the edges $\{1,2\}$, $\{2,4\}$, $\{4,3\}$, $\{3,1\}$ by setting their weights to 0. Now, perfect matchings with edge $\{a,d\}$ in G correspond to perfect matchings with $\{a,5\}, \{1,4\}, \{8,d\}$ in G', and perfect matchings without this edge correspond to perfect matchings with $\{5,1\}, \{4,8\}$ in G'. Similarly, perfect matchings with edge $\{c,b\}$ in G correspond to perfect matchings with $\{c,7\}, \{3,2\}, \{6,b\}$ in G', and perfect matchings without $\{c,b\}$ have $\{7,3\}, \{2,6\}$.

5. Now form an orientation $\vec{G}$ of the original G by orienting each non-crossing edge as it is oriented in $\vec{G}'$. Each edge $\{u,v\}$ of G involved in crossings was split into an odd number of segments in step 1 or 2. An odd number of its segments point in one direction and an even number in the other; the edge should be oriented (u,v) in $\vec{G}$ if an odd number of the segments in $\vec{G}'$ point along the direction from u to v, and should be oriented (v,u) otherwise. See Figures 5 and 6. Any cycle C of G containing edge $\{u,v\}$, and the corresponding cycle C' of G', have $r(C) \equiv r(C') \pmod 2$ by this choice of orientation. So given two perfect matchings $\mathbf{m}_1$, $\mathbf{m}_2$ in G; the corresponding perfect matchings $\mathbf{m}_1'$, $\mathbf{m}_2'$ in G'; and the superpositions $\pi = \pi(\mathbf{m}_1, \mathbf{m}_2)$, $\pi' = \pi(\mathbf{m}_1', \mathbf{m}_2')$, Lemma 3.1 yields

$$\begin{aligned}\frac{\epsilon_{\mathbf{m}_1}}{\epsilon_{\mathbf{m}_2}} = \epsilon_{\mathbf{m}_1}\epsilon_{\mathbf{m}_2} &= \prod_{C\in\pi}(-1)^{r(C)+1} \\ &= \prod_{C'\in\pi'}(-1)^{r(C')+1} = \epsilon_{\mathbf{m}_1'}\epsilon_{\mathbf{m}_2'} = \frac{\epsilon_{\mathbf{m}_1'}}{\epsilon_{\mathbf{m}_2'}}.\end{aligned}$$

We have not changed the relative sign between any two perfect matchings. We also have $\kappa_e(C) = \kappa_e(C')$ for all superposition cycles, so on fixing $\mathbf{m}_2$ we have for all $\mathbf{m}_1$,

$$\epsilon_{\mathbf{m}_1} = \frac{\epsilon_{\mathbf{m}_2}}{\epsilon_{\mathbf{m}_2'}}\cdot\epsilon_{\mathbf{m}_1'} = \frac{\epsilon_{\mathbf{m}_2}}{\epsilon_{\mathbf{m}_2'}}\cdot\epsilon_0'(-1)^{\kappa(\mathbf{m}_1')} = \frac{\epsilon_{\mathbf{m}_2}}{\epsilon_{\mathbf{m}_2'}}\cdot\epsilon_0'(-1)^{\kappa(\mathbf{m}_1)} = \epsilon_0(-1)^{\kappa(\mathbf{m}_1)}$$

(where $\epsilon_0 = \epsilon_0'\epsilon_{\mathbf{m}_2}/\epsilon_{\mathbf{m}_2'}$). Thus (12) holds for G.

This proves Theorem 3.1(a).

Remark 6.1. In step 5, the parities of $\iota(C)$ and $\iota(C')$ are equal as well, because vertices were added to regions of the plane in pairs; for example, $\{5,1\}$ of Figure 6. Each such pair either lies on C'; inside of C'; or outside of C', so $\iota(C') - \iota(C)$ is even.

6.3. Characterizing Crossing Orientations

We now prove Theorem 3.1(b).

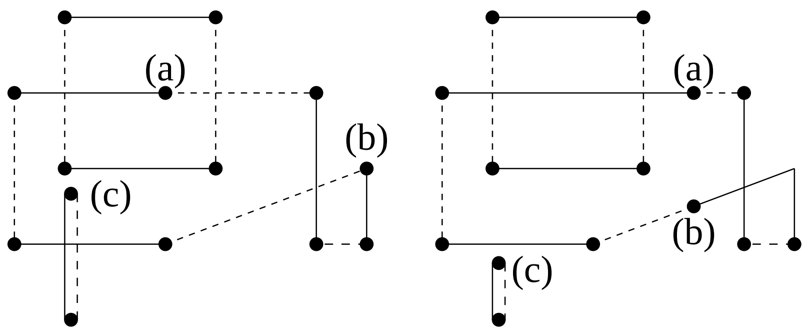

FIG. 7. The superposition cycles π of perfect matchings $\mathbf{m}$ (black) and $\mathbf{m}'$ (white, shown dashed). As we move vertices along them, the colors of crossings and the points interior to other cycles change. In (c) the edge of a 2 vertex cycle is traversed back and forth to form a cycle.

LEMMA 6.2. *Let* $\mathbf{m}$ *and* $\mathbf{m}'$ *be perfect matchings, and* $\pi = \pi(\mathbf{m}, \mathbf{m}')$ *be the permutation formed as their superposition. Then*

$$\kappa(\mathbf{m}) + \kappa(\mathbf{m}') \equiv \sum_{C \in \pi} \kappa_e(C) + \iota(C) \pmod 2 . \tag{32}$$

Proof. Color the edges of $\mathbf{m}$ black and the edges of $\mathbf{m}'$ white. We now work with the drawing of π this induces, and we disregard all other edges of G. See Figure 7 for illustrations of steps (a)–(c).

(a) Consider the drawing of a cycle C_1 of π. Choose a vertex v on it with an edge that forms a crossing with another cycle C_2. Move v along C_1 past the crossing, thereby changing the color of the portion of C_1 it moved along. We have changed the color of one edge at the crossing, thus changing either $\kappa(\mathbf{m})$ or $\kappa(\mathbf{m}')$ by ± 1 depending on the colors and whether that created or destroyed a monochromatic crossing. Simultaneously, v has moved from inside of C_2 to outside, or vice-versa, so $\iota(C_2)$ changed by ± 1, but no other $\iota(C)$ changed because we have not altered whether v is inside, outside, or along any other cycle. We have changed both sides of (32) by $1 \pmod 2$, so the equation still holds or still fails after moving v.

(b) Now consider a cycle C_1 with crossing edges. If we move a vertex along C_1 past a crossing, we change the color of one segment in the crossing, causing $\kappa_e(C_1)$ to change by ± 1. We have also changed either $\kappa(\mathbf{m})$ or $\kappa(\mathbf{m}')$ by ± 1. We have not changed any other quantity, so (32) continues to hold or continues to fail.

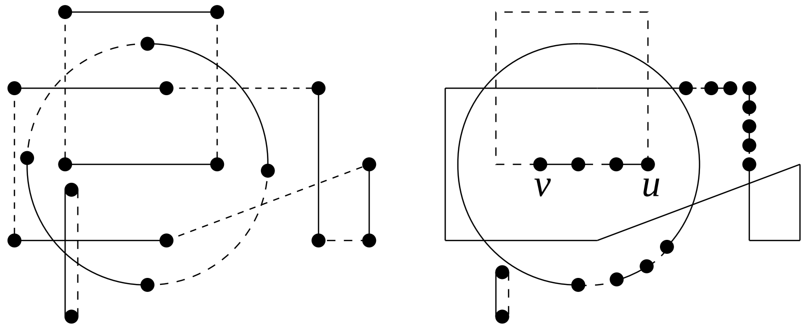

FIG. 8. (d) Move the vertices along each curve till they're all between 2 crossings; now they're also in one planar region w.r.t. all other cycles.

(c) If C_1 is a 2-cycle, its sole edge is counted in one direction as black and in the other as white. We move a vertex past a crossing by erasing the segment of the edge as we move along it. This modifies (a) and (b).

In (a), if C_2 is also 2-cycle, then C_1 and C_2 had formed a monochromatic white crossing and a monochromatic black crossing, but do no longer, so $\kappa(\mathbf{m})$ and $\kappa(\mathbf{m}')$ have each gone down by 1. No vertex is inside of C_2, so $\iota(C_2)$ has not changed.

In (a), if C_2 is not a 2-cycle, then C_1 and C_2 had formed a monochromatic crossing (either due to the white or the black edge of C_1), but do no longer, so either $\kappa(\mathbf{m})$ or $\kappa(\mathbf{m}')$ has decreased by 1. Also, the vertex has moved from inside of C_2 to outside, or vice versa, so $\iota(C)$ has changed by ± 1.

In (b), each crossing yields one black monochromatic crossing and one white, so moving the vertex reduces $\kappa(\mathbf{m})$ and $\kappa(\mathbf{m}')$ each by 1, without changing whether any vertex is interior to any cycle.

So in all cases, these vertex moves do not affect whether (32) holds or fails.

(d) We now move vertices on all the cycles of π as just described. For each cycle C forming any crossings whatsoever, use (a)–(c) to move all the vertices along the cycle until they are all congregated between a single pair of crossings; see Figure 8. (This is exactly the opposite of what we did in step 1 of the preceding section.) All 2-cycles can be deformed to form no crossings at all. It suffices to prove (32) for the graph so obtained.

The vertices of C are now all inside or all outside of any other cycle C'. This is true for all C so all $\iota(C') \equiv 0 \pmod 2$.

The two vertices u and v at either extreme of C are joined by one edge we call the 'long edge.' The long edge is entirely black or entirely white. It is drawn as a curve that forms all the intersections involving this cycle. It forms $\kappa_e(C)$ intersections with itself. With any other cycle C', it forms an even number of monochromatic intersections. To see this, suppose without loss of generality that u and v are inside C'. (They are both inside or both outside since we have moved all the vertices of C into the same region of the plane.) As we travel from u to v along the long edge, the curve alternates being inside, outside, ..., inside of C', as we pass each crossing. So altogether there is an even number of crossings. If the long edges of C and C' are the same color, we have an even contribution to $\kappa(\mathbf{m})$ or $\kappa(\mathbf{m}')$, and if they are of opposite color, we have 0 contribution to these. Either way, we do not affect their parity. Thus, the total number of crossings in this case is $\kappa(\mathbf{m}) + \kappa(\mathbf{m}') \equiv \sum_C \kappa_e(C) \pmod 2$. Since the vertices have been congregated together into groups of even size, in regions completely interior or exterior to other cycles, all $\iota(C)$ are even. So (32) holds. ▌

Proof of Theorem 3.1(b):

Let $\vec{G}$ be a crossing orientation. We are given two perfect matchings $\mathbf{m}$, $\mathbf{m}'$, and their superposition $\pi = \pi(\mathbf{m}, \mathbf{m}')$. Since $\vec{G}$ is a crossing orientation, the right side of (32) is congruent to $\sum_C (r(C) + 1) \pmod 2$; plug this into (9), obtaining

$$\begin{aligned} \epsilon_{\mathbf{m}} \epsilon_{\mathbf{m}'} &= (-1)^{\sum_C \left(\kappa_e(C) + \iota(C)\right)} \\ &= (-1)^{\kappa(\mathbf{m}) + \kappa(\mathbf{m}')} = (-1)^{\kappa(\mathbf{m})}(-1)^{\kappa(\mathbf{m}')} . \end{aligned} \tag{33}$$

On fixing any $\mathbf{m}'$ and setting $\epsilon_0 = (-1)^{\kappa(\mathbf{m}')}\epsilon_{\mathbf{m}'} = \pm 1$, we have $\epsilon_{\mathbf{m}} = \epsilon_0(-1)^{\kappa(\mathbf{m})}$ for all $\mathbf{m}$, so Theorem 3.1(a) holds. If there is a perfect matching with no crossings, we may interpret ϵ_0 as its sign.

Conversely, suppose that an orientation $\vec{G}$ satisfies Theorem 3.1(a). Let C_0 be a superposition cycle. Alternately color its edges black and white. Let $\mathbf{m}''$ be a perfect matching of the vertices of G not along C_0; this exists because C_0 is a superposition cycle, not just any even length cycle. Let $\mathbf{m}$ consist of the black edges of C_0 and all the edges of $\mathbf{m}''$. Let $\mathbf{m}'$ consist of the white edges of C_0 and all the edges of $\mathbf{m}''$. The superposition π of $\mathbf{m}$ and $\mathbf{m}'$ has the cycle C_0 and a bunch of 2-cycles, one for each edge of $\mathbf{m}''$.

Perform the construction in the preceding proof of Lemma 6.2, step (d). Now each 2-cycle C'' of π resulting from $\mathbf{m}''$ has $\kappa_e(C'') = 0$, $\iota(C'') = 0$, $r(C'') = 1$, and forms no intersections with other cycles. So by Lemma 6.2,

$$\begin{aligned} \kappa(\mathbf{m}) + \kappa(\mathbf{m}') &\equiv \sum_C (\kappa_e(C) + \iota(C)) \\ &= \kappa_e(C_0) + \iota(C_0) + \sum_{C''} (\kappa_e(C'') + \iota(C'')) \\ &\equiv \kappa_e(C_0) + \iota(C_0) , \end{aligned} \tag{34}$$

(where C'' runs over the 2-cycles induced by $\mathbf{m}''$), while

$$\begin{aligned}\sum_C (r(C)+1) &= (r(C_0)+1) + \sum_{C''}(r(C'')+1) \\ &= (r(C_0)+1) + \sum_{C''}(1+1) \\ &\equiv r(C_0)+1\ . \end{aligned} \tag{35}$$

By (9) and (35), $\epsilon_{\mathbf{m}}\epsilon_{\mathbf{m}'} = (-1)^{r(C_0)+1}$. Since the orientation satisfies Theorem 3.1(a), we also have $\epsilon_{\mathbf{m}}\epsilon_{\mathbf{m}'} = (-1)^{\kappa(\mathbf{m})+\kappa(\mathbf{m}')} = (-1)^{\kappa_e(C_0)+\iota(C_0)}$ by (34). So $r(C_0)+1 \equiv \kappa_e(C_0)+\iota(C_0) \pmod 2$, and the orientation is a crossing orientation. We have proven Theorem 3.1(b).

6.4. Proof of (R4)

We show that rule (R4) gives the same orientation as the construction of Section 6.2 when applied to graphs as we draw them in Section 4.

(i) Let G be our original graph with crossings, and G' the augmented graph from steps 1 and 2 of Section 6.2.

(ii) In step 3, we form an orientation of G' by Lemma 6.1, and then induce an orientation of G from this in step 5. The 0-edges of G are not involved in any crossings, so their final orientation is determined by (R2) in step 3. This is what (R4) says to do with the 0-edges.

(iii) Now consider any j-edge e of G ($j > 0$), a counterclockwise cycle C it forms with edges along the boundary of the 0-edges, and the corresponding cycle C' of G'. While C does not cross itself, e may have been split into many segments in G' due to other crossings. Delete the segments of G' crossing these, and delete the edges $\{1,4\}, \{2,3\}$ of all other protected crossings not involved in this j-edge. By Lemma 6.1, the resulting graph is a planar graph satisfying (R2). By Remark 6.1, the cycle C' in step 5 encloses an even number of vertices in G' (because C encloses no vertices in G), so $r(C')$ is odd by (R2). Also in step 5, $r(C) \equiv r(C') \pmod 2$, so $r(C)$ is odd too. The only edge of C not already oriented in (ii) is e, so its orientation is determined by requiring $r(C)$ to be odd, as (R4) states.

7. GRID ON THE MÖBIUS STRIP

Kasteleyn [5] enumerated the perfect matchings of an $m \times n$ grid in the plane, and of a periodic grid on a torus. We will enumerate the perfect matchings of an $m \times n$ grid on a Möbius strip. The graph and a crossing orientation (R4) are shown in Figure 9. This Möbius strip is a surface with a boundary; its word is $\sigma = a_1a_2a_1a_3$, and the boundaries do not enter in to our computation, so we use $\sigma = a_1a_1$.

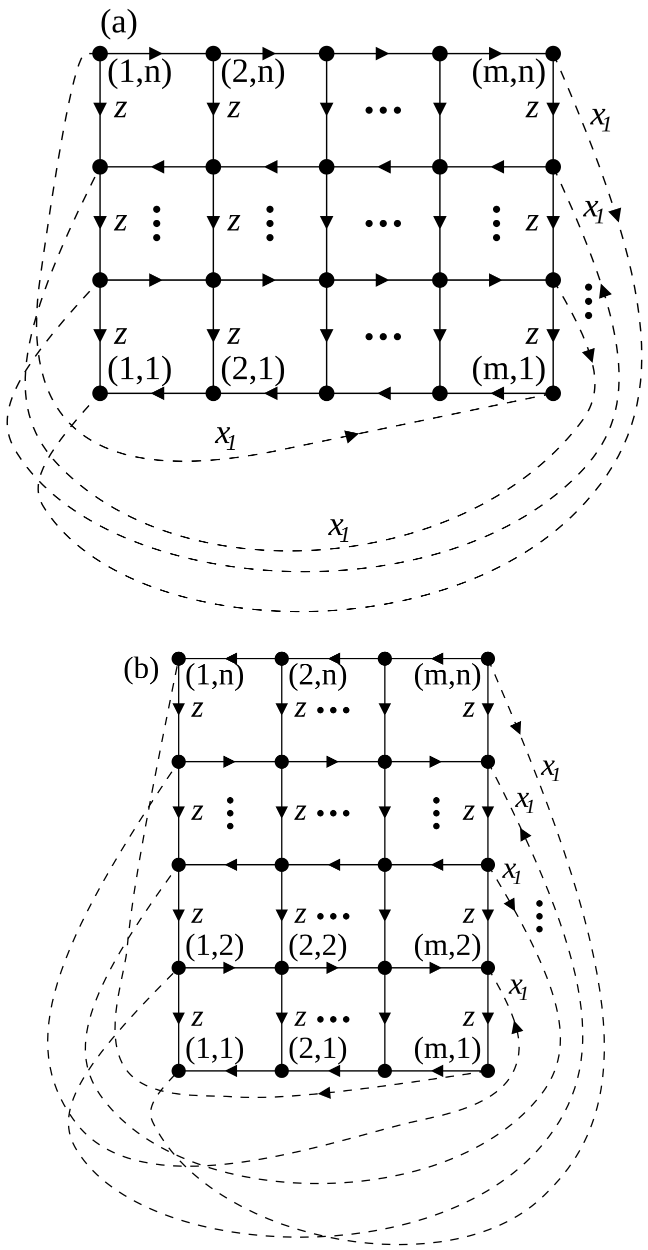

FIG. 9. An orientation (R4) of a width m, height n, grid on the Möbius strip. The pattern is shown for (a) even n and (b) odd n.

The vertical edges are weighted by z. The horizontal edges have weight 1. In the x-adjacency matrix $B(x_1)$, the crossing edges have weight x_1. The total weight of all perfect matchings is given by

$$\pm w_{m\times n}^{\text{Möbius}} = \frac{1-i}{2}\operatorname{Pf} B(i) + \frac{1+i}{2}\operatorname{Pf} B(-i) = \operatorname{Re}\big((1-i)\operatorname{Pf} B(i)\big). \quad (36)$$

We label the vertices by ordered pairs, $\{\,(j,k)\,:\,1\le j\le m,\ 1\le k\le n\,\}$. The x-adjacency matrix has components $b_{jk,j'k'}(x_1) =$

$$\begin{array}{lll} & (\delta_{j+1,j'}-\delta_{j'+1,j})\delta_{k,k'}(-1)^k & \text{(horizontal 0-edges)} \\ + & x_1\cdot\delta_{k,n+1-k'}(-1)^{n-k}(\delta_{j,m}\delta_{j',1}+(-1)^n\delta_{j',m}\delta_{j,1}) & \text{(horizontal 1-edges)} \\ + & z\cdot\delta_{j,j'}(\delta_{k+1,k'}-\delta_{k'+1,k}) & \text{(vertical 0-edges).} \end{array}$$

This can be compactly written in terms of tensor products as

$$B(x_1) = Q_m\otimes E_n + x_1 G_m^{(n)}\otimes H_n - zI_m\otimes Q_n\ , \quad (37)$$

where we have adopted certain matrices of Kasteleyn and introduced others. I_m is the $m\times m$ identity matrix. Q_n, E_n, H_n are $n\times n$, and Q_m, $G_m^{(n)}$ are $m\times m$.

$$Q_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & & \ddots & \\ 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \end{bmatrix} \quad (38)$$

$$\begin{array}{rcll} (E_n)_{k,k'} & = & \delta_{k,k'}(-1)^k & 1\le k,k'\le n \\ (H_n)_{k,k'} & = & \delta_{k+k',n+1}(-1)^{k+1} & 1\le k,k'\le n \end{array} \quad (39)$$

$$\begin{array}{c} (G_m^{(n)})_{m,1}=1, \qquad (G_m^{(n)})_{1,m}=(-1)^n, \\ (G_m^{(n)})_{k,k'}=0 \quad \text{otherwise in } 1\le k,k'\le m \end{array}$$

Define additional $n\times n$ matrices U_n, U_n^{-1} with components $1\le j,j'\le n$:

$$\begin{array}{rcl} (U_n)_{k,k'} & = & \sqrt{\frac{2}{n+1}}\ i^k\ \sin\left(\frac{kk'\pi}{n+1}\right) \\ ({U_n}^{-1})_{k,k'} & = & \sqrt{\frac{2}{n+1}}\ (-i)^k\ \sin\left(\frac{kk'\pi}{n+1}\right)\ . \end{array} \quad (40)$$

Conjugate Q_n, E_n, H_n by U_n to obtain

$$\begin{array}{rclcrcl} \widetilde{Q}_n &=& U_n^{-1} Q_n U_n & \qquad & (\widetilde{Q}_n)_{k,k'} &=& \delta_{k,k'} \cdot 2i\ \cos\frac{k\pi}{n+1} \\ \widetilde{E}_n &=& U_n^{-1} E_n U_n & & (\widetilde{E}_n)_{k,k'} &=& -\delta_{k+k',n+1} \\ \widetilde{H}_n &=& U_n^{-1} H_n U_n & & (\widetilde{H}_n)_{k,k'} &=& (-i)^{n+1}(-1)^k \delta_{k,k'}\ . \end{array} \tag{41}$$

Conjugate $B(x_1)$ by $I_m \otimes U_n$ to obtain

$$\begin{aligned} \widetilde{B}(x_1) &= (I_m \otimes U_n)^{-1}\ B(x_1)\ (I_m \otimes U_n) \\ &= Q_m \otimes \widetilde{E}_n + x_1 G_m^{(n)} \otimes \widetilde{H}_n - z I_m \otimes \widetilde{Q}_n \qquad (42) \\ &= \begin{bmatrix} \alpha_1 G - iq_1 I & 0 & 0 \ \cdots \ 0 & 0 & -Q_m \\ 0 & \alpha_2 G - iq_2 I & \cdots \ 0 \ \ 0 & -Q_m & 0 \\ 0 & 0 & \ddots \ \cdots \ \iddots & 0 & 0 \\ \vdots & \vdots & \vdots \ \cdots \ \vdots & \vdots & \vdots \\ 0 & 0 & \iddots \ \cdots \ \ddots & 0 & 0 \\ 0 & -Q_m & 0 \ \cdots \ 0 & \alpha_{n-1} G - iq_{n-1} I & 0 \\ -Q_m & 0 & 0 \ \cdots \ 0 & 0 & \alpha_n G - iq_n I \end{bmatrix} \end{aligned}$$

where we abbreviate $G = G_m^{(n)}$, $I = I_m$, and for $k = 1, \ldots, n$,

$$q_k = 2z \cos\frac{k\pi}{n+1} = -q_{n+1-k} \tag{43}$$

$$\alpha_k = x_1(-i)^{n+1}(-1)^k\ . \tag{44}$$

Although $\det B(x_1) = \det \widetilde{B}(x_1)$, we have destroyed the antisymmetry of the matrix, so we cannot directly take the Pfaffian. However, we can rearrange the rows and columns to make it antisymmetric again, resulting in a Pfaffian off by a factor ± 1 or $\pm i$. The matrices $\widetilde{H}_n$ and $\widetilde{Q}_n$ are diagonal, and $\widetilde{E}_n$ is reverse diagonal, so on rearranging the blocks, $\widetilde{B}(x_1)$ may be written as a block sum of $\lfloor n/2 \rfloor$ antisymmetric $2m \times 2m$ matrices, and an additional $m \times m$ block when n is odd. The rth block ($r = 1, \ldots, \lfloor n/2 \rfloor$), corresponding to block rows $k = r, n+1-r$ and block columns $k' = n+1-r, r$ of (42), may be expressed in terms of the $2m \times 2m$ antisymmetric matrix

$$T_m(q,\beta,\gamma) = \left[\begin{array}{cccccc|cccccc}
0 & -1 & 0 & \cdots & 0 & 0 & -q & 0 & 0 & \cdots & 0 & -\gamma \\
1 & 0 & -1 & \cdots & 0 & 0 & 0 & -q & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & -q & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & 0 & \cdots & -q & 0 \\
0 & 0 & 0 & \cdots & 1 & 0 & -\beta & 0 & 0 & \cdots & 0 & -q \\
\hline
q & 0 & 0 & \cdots & 0 & \beta & 0 & -1 & 0 & \cdots & 0 & 0 \\
0 & q & 0 & \cdots & 0 & 0 & 1 & 0 & -1 & \cdots & 0 & 0 \\
0 & 0 & q & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & q & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\
\gamma & 0 & 0 & \cdots & 0 & q & 0 & 0 & 0 & \cdots & 1 & 0
\end{array}\right] \tag{45}$$

as $T_m(iq_r, \beta_r, \gamma_r)$, where we note $-iq_{n+1-r} = iq_r$ and define

$$\beta_r = (-1)^n \alpha_{n+1-r} = -\alpha_r = -x_1 \cdot (-i)^{n+1}(-1)^r \tag{46}$$

$$\gamma_r = \alpha_{n+1-r} = (-1)^{n+1}\alpha_r = x_1 \cdot i^{n+1}(-1)^r \; . \tag{47}$$

When n is odd, the additional block is the $m \times m$ antisymmetric matrix

$$V_m(x_1) = \begin{bmatrix}
0 & -1 & 0 & 0 & \cdots & 0 & -x_1 \\
1 & 0 & -1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & -1 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & \cdots & 0 \\
0 & 0 & 0 & \ddots & & \ddots & \\
0 & 0 & 0 & \cdots & 1 & 0 & -1 \\
x_1 & 0 & 0 & \cdots & 0 & 1 & 0
\end{bmatrix} . \tag{48}$$

The block decomposition is

$$\widetilde{B}(x_1) \cong \bigoplus_{r=1}^{\lfloor n/2 \rfloor} T_m\left(2iz\ \cos\frac{r\pi}{n+1}, -x_1 \cdot (-i)^{n+1}(-1)^r, x_1 \cdot i^{n+1}(-1)^r\right)$$
$$\bigoplus V_m(x_1) \quad \text{if } n \text{ is odd.} \tag{49}$$

In computing the Pfaffian of these blocks, we will require a q-analogue of the Fibonacci numbers.

$$F_m(q) = \sum_{j=0}^{\lfloor m/2 \rfloor} \binom{m-j}{m-2j} q^{m-2j} = \sum_{j=0}^{\lfloor m/2 \rfloor} \binom{m-j}{j} q^{m-2j} \tag{50}$$

$$\widetilde{F}_m(q) = \sum_{j=0}^{\lfloor m/2 \rfloor} (-1)^j \binom{m-j}{m-2j} q^{m-2j} = \sum_{j=0}^{\lfloor m/2 \rfloor} (-1)^j \binom{m-j}{j} q^{m-2j}$$

With slight modifications, these have been studied in other contexts; see [3]. These satisfy $F_m(q) = F_{m-2}(q) + q\,F_{m-1}(q)$; $F_m(1)$ is the mth Fibonacci number; and $F_m(q) = \sum_d c_{md} q^d$ where c_{md} is the number of words in the alphabet $\{1, 2\}$ whose digits sum to m and that have exactly d 1's. These are related by

$$\widetilde{F}_m(iq) = i^m\, F_m(q). \tag{51}$$

LEMMA 7.1.

$$\begin{aligned}(-1)^m \operatorname{Pf} T_m(q,\beta,\gamma) &= (-1)^{\lfloor m/2 \rfloor}\left(\widetilde{F}_m(q) - \beta\gamma \widetilde{F}_{m-2}(q)\right) \\ &\quad + \begin{cases} 0 & \textit{if } m \textit{ is even} \\ \beta+\gamma & \textit{if } m \textit{ is odd} \end{cases}\end{aligned} \tag{52}$$

$$(-1)^{\lfloor m/2 \rfloor} \operatorname{Pf} V_m(x_1) = \begin{cases} x_1 + 1 & \textit{if } n \textit{ is odd and } m \textit{ is even} \\ 0 & \textit{if } n \textit{ is odd and } m \textit{ is odd} \end{cases} \tag{53}$$

Proof. The matrix $V_m(x_1)$ is the signed adjacency matrix of the graph shown in Figure 10(b). It is an admissible orientation of a planar graph. If m is odd, there are no perfect matchings and the Pfaffian is 0. If m is even, there are two perfect matchings:

perfect matching	weight
$\{\{1,2\},\{3,4\},\ldots,\{m-1,m\}\}$	1
$\{\{2,3\},\{4,5\},\ldots,\{m-2,m-1\},\{m,1\}\}$	x_1

Up to sign, the Pfaffian is $x_1 + 1$. Compute the sign from either matching; as listed, all edges of the first matching are against the routing, so the sign is $(-1)^{m/2}$.

The matrix $T_m(q,\beta,\gamma)$ is the signed adjacency matrix of the graph shown in Figure 10(a). It is a crossing orientation: the 0-edges $\{1, 2\}$, $\{2, 3\}$, $\ldots$, $\{2m, 1\}$ are the edges of the perimeter of the $2m$-gon, and form a clockwise odd cycle. The 1-edges $\{1, m+1\}$, $\ldots$, $\{m, 2m\}$ are oriented as by (R4), in a hole of a planar region.

We now classify the perfect matchings of Figure 10(a) according to whether they contain the edges $e_1 = \{1, 2m\}$ or $e_2 = \{m, m+1\}$.

Perfect matchings with e_1 but not e_2: Vertex m is adjacent to $m+1, m-1, 2m$, but can only be matched to $m-1$. Vertex $m+1$ is adjacent to $m, m+2, 1$, but can only be matched to $m+2$. Continuing

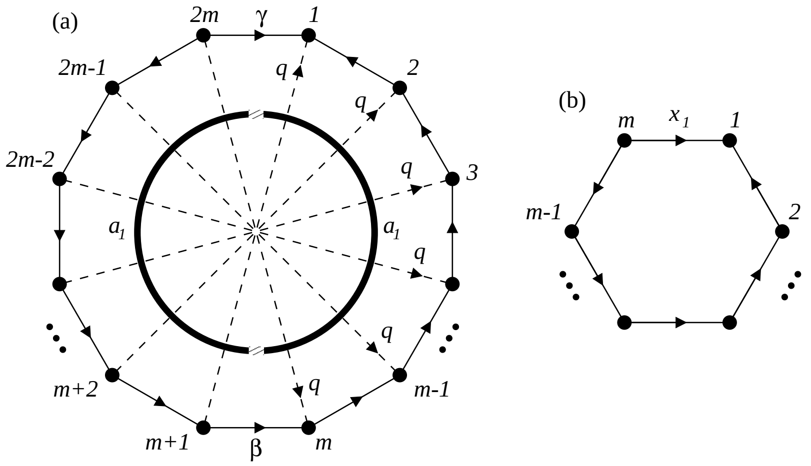

FIG. 10. (a) A graph with signed adjacency matrix $T_m(q,\beta,\gamma)$; (b) A graph with signed adjacency matrix $V_m(x_1)$.

this way, we are forced to have edges $\{1,2m\}$; $\{m,m-1\}, \{m+1,m+2\}$; $\{2,3\}, \{2m-1,2m-2\}$; …, alternating along the perimeter. If m is even, there will be two vertices remaining that cannot be matched together, so we don't form a perfect matching. If m is odd, we complete the cycle with alternating edges, forming a perfect matching of unsigned weight β. The sign of this matching is $\epsilon_0 = (-1)^m$; we place it on the left in (52).

Perfect matchings with e_2 but not e_1: If m is odd, there is one of signed weight $\epsilon_0\gamma$. If m is even, there are none.

Perfect matchings with neither e_1 nor e_2: We encode the matchings of this form by words $w = w_1w_2\cdots w_k$ comprised of digits $w_j \in \{1,2\}$, with $w_1+\cdots+w_k = m$. Look at the vertices $1,\ldots,m$ consecutively, and record a 1 if a vertex t is matched to $t+m$, and a 2 when $\{t,t+1\}$ (and hence $\{t+m,t+m+1\}$ as well) are matched. For $m=8$, the word 12212 means $\{1,9\}$, $\{2,3\}$, $\{4,5\}$, $\{6,14\}$, and $\{7,8\}$ are edges (and implies the remaining edges are $\{10,11\}$, $\{12,13\}$, and $\{15,16\}$). If there are d 1's and j 2's, with $d+j=k$, $d+2j=m$, then there are d edges that cross each other through the center, forming $\binom{d}{2}$ crossing pairs of edges. $\operatorname{Pf} T_m(q,a,c)$ counts this matching with sign $\epsilon_0(-1)^{\binom{d}{2}} = \epsilon_0(-1)^{\lfloor d/2\rfloor} = \epsilon_0(-1)^{\lfloor (m-2j)/2\rfloor} = \epsilon_0(-1)^{\lfloor m/2\rfloor}\cdot(-1)^j$, and unsigned weight $q^d = q^{m-2j}$. The number of words with d 1's and j 2's is $\binom{m-j}{j}$, so the total weight of all these perfect matchings is

$$\epsilon_0(-1)^{\lfloor m/2\rfloor}\widetilde{F}_m(q)\ . \tag{54}$$

Perfect matchings with both e_1 and e_2: We encode the matchings of this form by words whose digit sum is $m-2$, encoding how vertices $2,\dots,m-1$ are matched, in a similar fashion to the preceding case. The total signed weight is

$$\epsilon_0\beta\gamma\,(-1)^{\lfloor (m-2)/2\rfloor}\widetilde{F}_{m-2}(q) = -\epsilon_0\beta\gamma\,(-1)^{\lfloor m/2\rfloor}\widetilde{F}_{m-2}(q)\ . \tag{55}$$

Adding together the signed weights from these four cases gives (52). ∎

THEOREM 7.1. *On the $m\times n$ grid on the Möbius strip as depicted in Figure 9, the weight of all perfect matchings is*

m	n	*weight of perfect matchings*
odd	*odd*	0
odd	*even*	$\operatorname{Re}\left[(1-i)\prod_{r=1}^{n/2}\left(F_m\left(2z\ \cos\frac{r\pi}{n+1}\right) + F_{m-2}\left(2z\ \cos\frac{r\pi}{n+1}\right) + 2i(-1)^{r+n/2}\right)\right]$
even	*odd*	$2\cdot\prod_{r=1}^{(n-1)/2}\left(F_m\left(2z\ \cos\frac{r\pi}{n+1}\right)+F_{m-2}\left(2z\ \cos\frac{r\pi}{n+1}\right)\right)$
even	*even*	$\prod_{r=1}^{n/2}\left(F_m\left(2z\ \cos\frac{r\pi}{n+1}\right)+F_{m-2}\left(2z\ \cos\frac{r\pi}{n+1}\right)\right)$

Proof. When m and n are both odd, there is an odd number of vertices so there is no perfect matching; this manifests itself in our computations by a factor $\operatorname{Pf} V_m(x_1)=0$ in (53).

In all other cases,

$$\begin{aligned}\beta_r+\gamma_r &= x_1\cdot i^{n+1}\cdot(-1)^r\cdot(1+(-1)^n)\\ &= \begin{cases}2x_1\cdot i^{n+1}\cdot(-1)^r & \text{if } n \text{ is even}\\ 0 & \text{if } n \text{ is odd}\end{cases} \end{aligned}\tag{56}$$

$$\beta_r\gamma_r = -{x_1}^2(i\cdot -i)^{n+1}(-1)^{2r} = -{x_1}^2 = 1 \tag{57}$$

where in the last step, we used the fact that only $x_1 = \pm i$ are used in our computation. Plugging these and (51) into (52) yields

$$\begin{aligned}(-1)^m \operatorname{Pf} T_m(iq_r, \beta_r, \gamma_r) \\ &= (-1)^{\lfloor m/2 \rfloor} i^m \big(F_m(q_r) + F_{m-2}(q_r)\big) \\ &\quad + \begin{cases} 0 & \text{if } m \text{ is even or } n \text{ is odd} \\ 2x_1 \cdot i^{n+1} \cdot (-1)^r & \text{if } m \text{ is odd and } n \text{ is even.} \end{cases}\end{aligned} \tag{58}$$

When m is even, $(-1)^{\lfloor m/2 \rfloor} i^m = (-1)^{m/2}(-1)^{m/2} = 1$, and we have

$$(-1)^m \operatorname{Pf} T_m(iq_r, \beta_r, \gamma_r) \;=\; F_m(q_r) + F_{m-2}(q_r)\ . \tag{59}$$

On plugging in (43), the right side is a polynomial in z with nonnegative real coefficients for $r = 1, \ldots, \lfloor n/2 \rfloor$. We multiply these together over all r, obtaining a product y, independent of x_1. When n is even, we now have

$$\epsilon \cdot w_{m\times n}^{\text{Möbius}} = \tfrac{1-i}{2} y + \tfrac{1+i}{2} y = y \tag{60}$$

where $\epsilon = \pm 1$ or $\pm i$; since y and the total weight should both be polynomials in z with nonnegative real coefficients, $\epsilon = 1$.

When m is even and n is odd, we have the additional factor $\operatorname{Pf} V_m(x_1)$ evaluated in (53), giving

$$\epsilon \cdot w_{m\times n}^{\text{Möbius}} = \tfrac{1-i}{2} \cdot (1+i) y + \tfrac{1+i}{2} \cdot (1-i) y = 2y\ . \tag{61}$$

Finally, when m is odd and n is even, (58) reduces to

$$-i(-1)^m \operatorname{Pf} T_m(iq_r, \beta_r, \gamma_r) \;=\; F_m(q_r) + F_{m-2}(q_r) + 2x_1 \cdot (-1)^{r+n/2}$$

and there is no factor of $\operatorname{Pf} V_m(x_1)$. On plugging in $x_1 = \pm i$, and expressing it as a polynomial in z, all terms on the right but the constant term have nonnegative real coefficients. Multiplying the right side over r and plugging this into (36) gives the total weight up to an overall sign. The highest degree term in z has a positive real coefficient, so we've chosen the correct sign. ∎

APPENDIX: COMPUTING PFAFFIANS

A.1. COMPUTING PFAFFIANS BY ROW AND COLUMN REDUCTION

For graphs embedding in the plane, we only need to compute one Pfaffian, and can take the positive value $|\operatorname{Pf} A| = \sqrt{\det A}$ as the number of perfect matchings. For other surfaces, we compute a linear combination of Pfaffians, possibly involving complex numbers, so we must be sure to have the correct sign on each term. The Pfaffian of A can be computed by simultaneous row and column reduction.

(a) Multiplying both row j and column j by α multiplies the Pfaffian by α.

(b) Simultaneously swapping row j with row k, and column j with column k, negates the Pfaffian. Simultaneously permuting the rows and columns by the same permutation multiplies the Pfaffian by the sign of that permutation.

(c) Adding α times row j to row k, and simultaneously adding α times column j to column k, does not change the value of the Pfaffian.

Using these operations, we can reduce any antisymmetric matrix A to a block sum of 2×2 matrices,

$$C = \begin{bmatrix} 0 & a & & & \\ -a & 0 & & & \\ & & 0 & b & \\ & & -b & 0 & \\ & & & & \ddots \end{bmatrix}. \tag{A.1}$$

We have $\operatorname{Pf} C = a \cdot b \cdot \cdots$, and $\operatorname{Pf} A$ is a multiple of this that depends on the operations (a)–(c) performed in the reduction.

A.2. BIPARTITE GRAPHS

Let G be a bipartite graph. Label its vertices so that the two vertex classes are $\{1, \ldots, p\}$ and $\{p+1, \ldots, 2p\}$. The signed adjacency matrix now takes on the form

$$A = \begin{bmatrix} 0 & C \\ -C & 0 \end{bmatrix} \tag{A.2}$$

and $\operatorname{Pf} A = (-1)^{\binom{p}{2}} \det C$. So we may systematically use determinants of signed bipartite adjacency matrices instead of Pfaffians.

A.3. HAFNIANS AND PERMANENTS

The Hafnian of a $2p \times 2p$ symmetric matrix $D = [d_{ij}]$ is

$$\operatorname{Hf} D = \sum_{\mathbf{m}} d_{u_1,v_1} \cdots d_{u_p,v_p} \tag{A.3}$$

where $\mathbf{m} = \{\{u_1, v_1\}, \ldots, \{u_p, v_p\}\}$ again ranges over the partitions of $\{1, \ldots, 2p\}$ into p sets of size 2. If D is the ordinary weighted adjacency matrix of an undirected graph, $d_{ij} = W_{\{i,j\}}$, then $\operatorname{Hf} D$ is the total weight of all perfect matchings. This is not easy to compute. Computing it by the methods of this paper, we have the complete graph on $2p$ vertices, with edge-weights. For $p \geq 2$, the complete graph has Euler characteristic $\chi(K_{2p}) = 2\left[\frac{p(7-2p)}{6}\right]$ (see [1, p. 112]) so its plane model has $2n$ sides, where $n = 2 - \chi(K_{2p}) = 2\left[\frac{2p^2-7p+11}{6}\right]$. So $\operatorname{Hf} D$ can be expressed as a linear combination of 2^n Pfaffians of $2p \times 2p$ antisymmetric matrices.

The permanent of a $p \times p$ matrix $D = [d_{ij}]$ is

$$\operatorname{per} D = \sum_{\pi \in S_p} d_{1,\pi(1)} \cdots d_{p,\pi(p)} \tag{A.4}$$

summed over all permutations of p elements. If D is the ordinary (not signed) weighted adjacency matrix of a bipartite graph, then $\operatorname{per} D$ is the total weight of all perfect matchings. The best known formula to compute this in general was found by Ryser [10, p. 27], and requires about $p^2 \cdot 2^{p-1}$ operations. Using our methods, the complete bipartite graph $K_{p,p}$ has Euler characteristic $\chi(K_{p,p}) = 2\left[\frac{p(4-p)}{4}\right]$ (for $p \geq 3$) so $n = 2 - \chi(K_{p,p}) = 2\left[\frac{p^2-4p+7}{4}\right]$. Thus $\operatorname{per} D$ can be computed as a linear combination of 2^n determinants of $p \times p$ matrices, which is not as efficient as Ryser's method.

A.4. SYMBOLIC METHOD

We have exhibited a linear combination of 2^n Pfaffians of matrices with complex valued entries, which is useful for graphs with numeric weights because these Pfaffians may be computed efficiently as just described. In terms of the variables $x_1, \ldots, x_n$, the expansion

$$\operatorname{Pf} B(x_1, \ldots, x_n) = \sum_{0 \leq r_1, r_2, \ldots, r_n \leq 3} \beta_{r_1, \ldots, r_n} {x_1}^{r_1} \cdots {x_n}^{r_n} \tag{A.5}$$

is symbolic, and need not be as efficiently computable. However, if this expansion is known, we may directly compute the number of perfect matchings from it, rather than plugging it and (30) into (19).

In (A.5), $\beta_{r_1,\ldots,r_n}$ is the signed weight of all perfect matchings whose number of j-edges is congruent to $r_j \pmod 4$. We can remove the signs by taking absolute values, obtaining

$$\#\text{ perfect matchings} = \sum_{0 \leq r_1, \ldots, r_n \leq 3} \left|\beta_{r_1, \ldots, r_n}\right|. \tag{A.6}$$

We can also remove the signs by explicitly computing and cancelling them, giving the number of perfect matchings as

$$\left| \sum_{0 \le r_1, r_2, \ldots, r_n \le 3} \beta_{r_1,\ldots,r_n} (-1)^{\sum_{j=1}^{n} C_j(r_j) + \sum_{1 \le j < k \le n} C_{jk}(r_j, r_k)} \right| , \tag{A.7}$$

where the C's give the number of pairs of crossed edges:

$$\begin{aligned} C_j(r_j) &= \begin{cases} \binom{r_j}{2} & \text{if } \sigma \text{ is } j\text{-nonoriented;} \\ 0 & \text{otherwise.} \end{cases} \\ C_{jk}(r_j, r_k) &= \begin{cases} r_j \cdot r_k & \text{if } \sigma \text{ is } j, k\text{-alternating;} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

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