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Multiple Left Regular Representations Generated by Alternants

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Let $p_1 > \cdots > p_n \geq 0$, and $\Delta_p = \det \|x_i^{p_j}\|_{i,j=1}^n$. Let $\mathbf{M}_p$ be the linear span of the partial derivatives of Δ_p. Then $\mathbf{M}_p$ is a graded S_n-module. We prove that it is the direct sum of graded left regular representations of S_n. Specifically, set $\lambda_j = p_j - (n-j)$, and let $\Xi_\lambda(t)$ be the Hilbert polynomial of the span of all skew Schur functions $s_{\lambda/\mu}$ as μ varies in λ. Then the graded Frobenius characteristic of $\mathbf{M}_p$ is $\Xi_\lambda(t)\,\tilde{H}_{1^n}(x;q,t)$, a multiple of a Macdonald polynomial. Corresponding results are also given for the span of partial derivatives of an alternant over any complex reflection group.

Let (i,j) denote the lattice cell in the $i+1^{st}$ row and $j+1^{st}$ column of the positive quadrant of the plane. If L is a diagram with lattice cells $(p_1,q_1),\ldots,(p_n,q_n)$, we set $\Delta_L = \det \|x_i^{p_j} y_i^{q_j}\|_{i,j=1}^n$, and let $\mathbf{M}_L$ be the linear span of the partial derivatives of Δ_L. The bihomogeneity of Δ_L and its alternating nature under the diagonal action of S_n gives $\mathbf{M}_L$ the structure of a bigraded S_n-module. We give a family of examples and some general conjectures about the bivariate Frobenius characteristic of $\mathbf{M}_L$ for two dimensional diagrams.

Key Words: Macdonald polynomials, representations of the symmetric group, complex reflection groups, lattice diagram polynomials

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1. INTRODUCTION

We review some basic notions; the reader should consult [3] for further details. The lattice cells of the positive plane quadrant will be assigned coordinates $i, j \geq 0$ as indicated in the figure below.

$\vdots$			
$(2,0)$	$(2,1)$	$(2,2)$	
$(1,0)$	$(1,1)$	$(1,2)$	
$(0,0)$	$(0,1)$	$(0,2)$	$\cdots$

A collection of distinct lattice cells will be called a "*lattice diagram*." Given a partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0)$, its "*Ferrers diagram*" is

$$\{\,(i,j)\,:\,0 \leq i \leq k-1 \ \text{ and } \ 0 \leq j \leq \mu_{i+1} - 1\,\}\ .$$

As customary, we will use the symbol μ for the partition as well as its Ferrers diagram.

Given any sequence of lattice cells

$$L \;=\; \{(p_1, q_1)\ ,\ (p_2, q_2)\ ,\ \ldots\ ,\ (p_n, q_n)\}\ , \tag{1.1}$$

we define the "*lattice determinant*"

$$\Delta_L(x;y) \;=\; \frac{1}{p!\,q!} \det \left\| x_i^{p_j} y_i^{q_j} \right\|_{i,j=1}^{n} , \tag{1.2}$$

where $p! = p_1!\,p_2!\cdots p_n!$ and $q! = q_1!\,q_2!\cdots q_n!$. We can easily see that $\Delta_L(x;y)$ is a polynomial different from zero if and only if L consists of n distinct lattice cells. Note also that $\Delta_L(x;y)$ is bihomogeneous of degree $|p| = p_1 + \cdots + p_n$ in x and degree $|q| = q_1 + \cdots + q_n$ in y.

Given a polynomial $P(x;y)$, the vector space spanned by all the partial derivatives of P of all orders will be denoted $\mathcal{L}_\partial(P)$. We recall that the "*diagonal action*" of S_n on a polynomial

$$P(x;y) \;=\; P(x_1, \ldots, x_n; y_1, \ldots, y_n)$$

is defined by setting for a permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$

$$\sigma\ P(x;y) \;=\; P(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}; y_{\sigma_1}, y_{\sigma_2}, \ldots, y_{\sigma_n})\ .$$

It is easily seen from the definition (1.2) that Δ_L is an alternant under the diagonal action. This given, it follows that for any lattice diagram L with n cells, the vector space

$$\mathbf{M}_L \;=\; \mathcal{L}_\partial(\Delta_L)$$

is an S_n-module. Since Δ_L is bihomogeneous, this module affords a natural bigrading. Denoting by $\mathcal{H}_{r,s}[\mathbf{M}_L]$ the subspace consisting of the bihomogeneous elements of degree r in x and degree s in y, we have the direct sum decomposition

$$\mathbf{M}_L \;=\; \bigoplus_{r=0}^{|p|} \bigoplus_{s=0}^{|q|} \mathcal{H}_{r,s}[\mathbf{M}_L] \ ,$$

and the polynomial

$$F_L(q,t) \;=\; \sum_{r=0}^{|p|} \sum_{s=0}^{|q|} t^r\, q^s\, \dim \mathcal{H}_{r,s}[\mathbf{M}_L]$$

gives the "*bigraded Hilbert series*" of $\mathbf{M}_L$. In this vein, since each of the subspaces $\mathcal{H}_{r,s}[\mathbf{M}_L]$ is necessarily an S_n-submodule, we can also set

$$C_L(x;q,t) \;=\; \sum_{r=0}^{|p|} \sum_{s=0}^{|q|} t^r\, q^s\, \mathcal{F}\, \mathrm{ch}\, \mathcal{H}_{r,s}[\mathbf{M}_L] \tag{1.3}$$

where $\mathrm{ch}\,\mathcal{H}_{r,s}[\mathbf{M}_L]$ denotes the character of $\mathcal{H}_{r,s}[\mathbf{M}_L]$ and $\mathcal{F}\,\mathrm{ch}\,\mathcal{H}_{r,s}[\mathbf{M}_L]$ denotes the image of $\mathrm{ch}\,\mathcal{H}_{r,s}[\mathbf{M}_L]$ under the Frobenius map $\mathcal{F}$ which sends the irreducible character χ^λ into the Schur function s_λ. In $C_L(x;q,t)$, the "x" is only to remind us that it is a symmetric function in the infinite alphabet $x_1, x_2, x_3, \ldots$ (as customary in [8]), and we should not confuse it with the "x" appearing in $\Delta_L(x;y)$. This may be unfortunate, but it is too much of an ingrained notation to be altered at this point.

In [5], one of us (Garsia) and Haiman introduced a conjecture that would imply Macdonald's conjecture ([7, p. 163] and [8, p. 355]) that the q,t-Kostka coefficients are polynomials with nonnegative integer coefficients:

Conjecture 1.1. **($C = \tilde{H}$ Conjecture)** When μ is the Ferrers diagram of a partition, we have

$$C_\mu(x;q,t) \;=\; \tilde{H}_\mu(x;q,t)\ , \tag{1.4}$$

where $\tilde{H}_\mu(x;q,t)$ is a variant of the Macdonald polynomial given plethystically by

$$\tilde{H}_\mu(x;q,t) \;=\; J_\mu\left[\frac{X}{1-t^{-1}};q,t^{-1}\right] t^{n(\mu)} \;=\; \sum_\lambda \widetilde{K}_{\lambda,\mu}(q,t)\, s_\lambda(x) \tag{1.5}$$

where $n(\mu) = \sum_{i\geq 1}(i-1)\mu_i$, and $\widetilde{K}_{\lambda,\mu}(q,t)$ is related to the ordinary q,t-Kostka coefficient by

$$\widetilde{K}_{\lambda,\mu}(q,t) = t^{n(\mu)} K_{\lambda,\mu}(q,t^{-1}) \ .$$

Given this conjecture, $\mathbf{M}_\mu$ is a graded version of the left regular representation of S_n, and $\dim \mathbf{M}_\mu = n!$ (where $\mu \vdash n$). This we refer to as the "*$n!$ conjecture*." In [3], we and our coauthors considered a family of lattice diagrams L with the following property:

Property 1.1. *A lattice diagram L with n cells has the* "multiple left regular representation" *(MLRR) property when the module $\mathbf{M}_L$ decomposes into a direct sum of left regular representations of S_n. We then have $\dim \mathbf{M}_L = k \cdot n!$ for some integer k.*

In dimensions $d \geq 3$, there is a family of lower order ideals L of $\mathbf{N}^d$ that do not possess this property; see [12, Thm. 6]. In [3, Conj. I.1], we gave a family of two dimensional diagrams which conjecturally possessed this property. We then conjectured that all two dimensional lattice diagrams L had this property, but have since found counterexamples.

In this paper, we consider both proven and conjectural families of diagrams with the MLRR property. In Section 2, we prove that all one dimensional diagrams have the MLRR property. In fact, for any complex reflection group G and G-alternant $\Delta(x)$, the space $\mathcal{L}_\partial(\Delta)$ is a graded multiple of the left regular representation of G, and we consider examples involving the wreath product $C_m \wr S_n$ in Section 3. We provide additional properties of the one dimensional case in Section 4. In Section 5, we consider two dimensional diagrams consisting of a partition plus an external cell; this is dual to the diagrams considered in [3], which were constructed by removing a cell from the diagram of a partition. Finally, in Section 6, we conclude with additional conjectures for general diagrams that unify properties of all the cases we've considered.

2. ONE DIMENSIONAL DIAGRAMS

We consider here the special case of diagrams of the form

$$L_p \;=\; \{(p_1,0),(p_2,0),\ldots,(p_n,0)\} \tag{2.1}$$

where $p \in \mathcal{D}_n = \{\,(p_1,\ldots,p_n) \in \mathbf{N}^n \ : \ p_1 > p_2 > \cdots > p_n \geq 0\,\}$. We set

$$\Delta_p(x_1,\ldots,x_n) \;=\; \det \ \|x_i^{p_j}\|_{i,j=1}^{n} \ , \tag{2.2}$$

(normalized differently than (1.2)) and we let $\mathcal{L}_\partial(\Delta_p)$ denote the vector space spanned by all partial derivatives of Δ_p. Our goal is to show that

$\mathcal{L}_\partial(\Delta_p)$ carries a multiple of the left regular representation of S_n and obtain an explicit expression for its Frobenius characteristic. Note that when $p = (n-1, n-2, \ldots, 0)$, the polynomial Δ_p reduces to the Vandermonde determinant, which we shall denote by Δ_n. In that case it is well-known that $\mathcal{L}_\partial(\Delta_n)$ is a bigraded version of the left regular representation with graded Frobenius characteristic given by the polynomial

$$\tilde{H}_{1^n}(x;q,t) \;=\; (1-t)(1-t^2)\cdots(1-t^n)\, h_n\Big[\frac{X}{1-t}\Big]\,. \tag{2.3}$$

Note further that we may always write p in the form $p = \lambda + \rho$, where $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$ is a partition, and $\rho = \rho_n = (n-1, n-2, \ldots, 1, 0)$. Now for a partition λ of length $\leq n$, let Σ_λ denote the graded vector space spanned by the skew Schur functions $s_{\lambda/\mu}(x_1, \ldots, x_n)$ as μ varies in λ, and denote by $\Xi_\lambda(t)$ its Hilbert polynomial (which does not depend on n). Our main result can be stated as follows.

Theorem 2.1. *The graded Frobenius characteristic of the space $\mathcal{L}_\partial(\Delta_{\lambda+\rho})$ is given by the polynomial*

$$\mathcal{F}\,\mathrm{ch}\,\mathcal{L}_\partial(\Delta_{\lambda+\rho}) \;=\; \Xi_\lambda(t)\; \tilde{H}_{1^n}(x;q,t)\,. \tag{2.4}$$

Before we can give the proof, we need some auxiliary facts and observations. If $P(x_1, \ldots, x_n)$ is a polynomial, we shall denote by $P(\partial)$ the differential operator obtained on replacing x_i by ∂_{x_i}. This given, we have

Lemma 2.1. *For $p, q \in \mathcal{D}_n$, if*

$$\Delta_q(\partial)\; \Delta_p(x) \neq 0\,, \tag{2.5}$$

then

$$q_1 \leq p_1\;,\quad q_2 \leq p_2\;,\quad \cdots\quad,\; q_n \leq p_n\,. \tag{2.6}$$

In particular, the polynomials $\Delta_p(x)$ constitute a basis of the alternants in $\mathbf{Q}[x]$, and they are orthogonal with respect to the scalar product

$$\big\langle P\,,\, Q\big\rangle \;=\; P(\partial)\; Q(x)\;\big|_{x=0}\,. \tag{2.7}$$

Proof. We may write

$$\Delta_q(\partial)\,\Delta_p(x) = \sum_{\theta\in S_n} \mathrm{sign}(\theta) \sum_{\sigma\in S_n} \partial_{x_{\sigma_1}}^{q_{\theta_1}} \partial_{x_{\sigma_2}}^{q_{\theta_2}} \cdots \partial_{x_{\sigma_n}}^{q_{\theta_n}}\; x_{\sigma_1}^{p_1} x_{\sigma_2}^{p_2} \cdots x_{\sigma_n}^{p_n}\,. \tag{2.8}$$

Thus from (2.5) we deduce that there must be at least one permutation θ such that

$$q_{\theta_1} \leq p_1 \ , \quad q_{\theta_2} \leq p_2 \ , \quad \cdots \ , \quad q_{\theta_n} \leq p_n \ , \tag{2.9}$$

but if this happens, a fortiori we must have (2.6).

As for the final assertion, note that if Δ_p and Δ_q have different degrees then the orthogonality is trivial. On the other hand, if they have the same degree, then the non-vanishing of the scalar product implies (2.5) and thus also (2.6), which in this case forces $p = q$. ∎

Lemma 2.2. *If $A(x)$ is a homogeneous symmetric polynomial then we have*

$$A(\partial)\Delta_n(\partial)\Delta_p(x) \neq 0$$

if and only if

$$A(\partial)\Delta_p(x) \neq 0\,,$$

and in this case we can always find a homogeneous symmetric polynomial $A'(x)$ giving

$$A'(\partial)\, A(\partial)\, \Delta_p(x) \;=\; c\, \Delta_n(x) \qquad (\textit{with } c \neq 0)\,. \tag{2.10}$$

Proof. Set

$$f(x) \;=\; A(\partial)\Delta_p(x)\,.$$

Since f is alternating, it factors as $f(x) = \Delta_n(x)\, h(x)$ where $h(x)$ is symmetric. Now if $f \neq 0$ we have $f(\partial)f(x) \neq 0$ so

$$0 \;\neq\; f(\partial)f(x) \;=\; h(\partial)\Delta_n(\partial)f(x) \;=\; h(\partial)\, \Delta_n(\partial)\, A(\partial)\, \Delta_p(x) \;=\; h(\partial)\, g(x)\,,$$

with

$$g(x) \;=\; \Delta_n(\partial)A(\partial)\Delta_p(x) \neq 0\,;$$

thus

$$0 \neq g(\partial)\, g(x) \;=\; g(\partial)\, \Delta_n(\partial)\, A(\partial)\; \Delta_p(x)\,.$$

In particular, we have

$$g(\partial)\, A(\partial)\; \Delta_p(x) \neq 0\,. \tag{2.11}$$

Note further that $g(x)$ is symmetric and

$$\deg(g) \;=\; |p| - \deg(A) - \binom{n}{2}\,.$$

This gives

$$\deg\Big(g(\partial)A(\partial)\Delta_p(x)\Big) \;=\; |p|-\deg(A) \;-\; \Big(|p|-\deg(A)-\tbinom{n}{2}\Big) \;=\; \tbinom{n}{2}\,.$$

Since $g(\partial)A(\partial)\Delta_p(x)$ must be alternating, from (2.11) we derive that

$$g(\partial)A(\partial)\Delta_p(x) \;=\; c\,\Delta_n(x) \qquad (\textit{with } c \neq 0)\,.$$

Thus we may take $A'(x)=g(x)$ in (2.10) and our proof is complete. ∎

We are now in a position to deal with the special case $p=(k+n-1,k+n-2,\ldots,k)$, which is both interesting in its own right and useful in our further developments.

To begin with note that the orthogonal complement of our space $\mathcal{L}_\partial(\Delta_p)$ is the ideal I_p of polynomial differential operators that kill Δ_p. In symbols,

$$I_p \;=\; (\; P(x) \;:\; P(\partial)\;\Delta_p(x)=0\;)\;.$$

In particular, we also have

$$I_p^{\perp} \;=\; \mathcal{L}_\partial(\Delta_p)\,. \tag{2.12}$$

Now it develops that

Proposition 2.1. *When $p=(k+n-1,k+n-2,\ldots,k)$,*

$$I_p \;=\; \big(h_{k+1},h_{k+2},\cdots,h_{k+n}\big)\,.$$

Proof. Note that since

$$h_{k+i}(x)\Delta_n(x) \;=\; \Delta_{(n+k+i-1,n-2,\ldots,1,0)}(x)\,,$$

we deduce from Lemma 2.1 that for $i\geq 1$,

$$h_{k+i}(\partial)\Delta_n(\partial)\;\Delta_{(k+n-1,k+n-2,\ldots,k)}(x) \;=\; 0\,,$$

and Lemma 2.2 gives that

$$h_{k+i}(\partial)\;\Delta_{(k+n-1,k+n-2,\ldots,k)}(x) \;=\; 0\,.$$

Thus we must have

$$h_{k+i}(x) \;\in\; I_p \qquad (\forall\, i\geq 1)\,.$$

In particular, we deduce the inclusion of ideals

$$J_{k,n} = \big(h_{k+1}, h_{k+2}, \cdots, h_{k+n}\big) \;\subseteq\; J_k = \big(h_{k+1}, h_{k+2}, \cdots\big) \;\subseteq\; I_p\,. \quad (2.13)$$

Thus

$$\begin{aligned} \frac{1}{(1-x_1t)\cdots(1-x_nt)} &= \sum_{k=0}^{\infty} h_m(x)t^m \\ &\cong_{k,n} \sum_{m=0}^{k} h_m(x)t^m + \sum_{m\geq k+n+1} h_m(x)t^m\,, \end{aligned}$$

where the symbol $\cong_{k,n}$ represents equality modulo $J_{k,n}$. Thus for $i = 1, \ldots, n$, we must also have

$$\begin{aligned} \frac{1}{(1-x_it)\cdots(1-x_nt)} \cong_{k,n}\; & (1-x_1t)\cdots(1-x_{i-1}t) \\ & \times \left(\sum_{m=0}^{k} h_m(x)t^m + \sum_{m\geq k+n+1} h_m(x)t^m \right)\,. \end{aligned}$$

The coefficients of $t^{k+i},\ t^{k+i+1},\ \cdots,\ t^{k+n}$ on the right all vanish. Equating coefficients of t^{k+i}, we get

$$h_{k+i}(x_i, \ldots, x_n) \;\cong_{k,n}\; 0\,. \quad (2.14)$$

This may also be rewritten as

$$x_i^{k+i} \;\cong_{k,n}\; -\sum_{l=0}^{k+i-1} x_i^l\, h_{k+i-l}(x_{i+1}, \ldots, x_n)\,.$$

Now, using this relation, we can recursively express (mod $J_{k,n}$) any monomial $x_1^{m_1}x_2^{m_2}\cdots x_i^{m_i}\cdots x_n^{m_n}$ as a linear combination of monomials where x_i is raised to a power less than $k+i$ at the expense of raising the powers of the variables x_j with $j > i$. Applying this process successively for $i = 1, 2, \ldots, n$, and using the fact that, for $i = n$, (2.14) reduces to

$$x_n^{k+n} \;\cong_{k,n}\; 0\,,$$

we see that every monomial can be expanded (mod $J_{k,n}$) in terms of monomials

$$x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_n^{\epsilon_n} \qquad (\text{with } 0 \leq \epsilon_i \leq k+i-1)\,. \quad (2.15)$$

In other words, this collection constitutes a monomial spanning set for the quotient ring

$$\mathbf{Q}[x]/J_{k,n} \;=\; \mathbf{Q}[x_1,\ldots,x_n]/J_{k,n}\,.$$

Combining this fact with the inclusions in (2.13), we are led to the string of inequalities

$$\dim \mathbf{Q}[x]/I_p \le \dim \mathbf{Q}[x]/J_k \le \dim \mathbf{Q}[x]/J_{k,n} \le (k+1)(k+2)\cdots(k+n)\,. \tag{2.16}$$

On the other hand note that since for $p=(k+n-1,k+n-2,\ldots,k)$ the lexicographically leading term of Δ_p is the monomial

$$x_1^k\, x_2^{k+1}\cdots x_n^{k+n-1}\,,$$

we see that differentiating Δ_p by all the submonomials of its leading term will yield

$$(k+1)(k+2)\cdots(k+n)$$

independent elements of $\mathcal{L}_\partial(\Delta_p)$. Thus we must also have the inequality

$$(k+1)(k+2)\cdots(k+n) \;\le\; \dim\, \mathcal{L}_\partial(\Delta_p)\,. \tag{2.17}$$

But from from (2.12) we deduce that

$$\dim\, \mathbf{Q}[x]/I_p \;=\; \dim\, \mathcal{L}_\partial(\Delta_p)\,. \tag{2.18}$$

Combining (2.16) with (2.18) and (2.17), we are forced to conclude that all these inequalities must be equalities, forcing the inclusions in (2.13) to be equalities as well, proving the proposition. ∎

Note that as a byproduct of our argument, we get the following remarkable fact.

PROPOSITION 2.2. *When $p=(k+n-1,k+n-2,\ldots,k)$, a basis for the space $\mathcal{L}_\partial(\Delta_p)$ is given by the polynomials*

$$\partial_{x_1}^{\epsilon_1}\partial_{x_2}^{\epsilon_2}\cdots\partial_{x_n}^{\epsilon_n}\Delta_p(x) \qquad (\textit{with } 0\le \epsilon_i \le k+i-1)\,. \tag{2.19}$$

Proof. Any nontrivial vanishing linear combination of the polynomials in (2.19) would yield that a nontrivial linear combination of the monomials in (2.15) vanishes modulo the ideal I_p thereby contradicting that these monomials are a basis for the quotient $\mathbf{Q}[x]/I_p$. ∎

This result has the following beautiful corollary:

Theorem 2.2. *Denoting by $\mathcal{B}_p$ the collection of polynomials in (2.19), we have that every polynomial $P \in \mathbf{Q}[x]$ has an expansion of the form*

$$P(x) \;=\; \sum_{b\in\mathcal{B}_p} b(x)\, A_b(h_{k+1}, h_{k+2}, \ldots, h_{k+n})\,, \tag{2.20}$$

with the polynomials A_b uniquely determined by P.

Proof. It suffices to prove this for homogeneous P. Write $P = P^* + P^\perp$ where $P^* \in I_p$ and $P^\perp \in I_p^\perp$. By (2.12), $I_p^\perp = \mathcal{L}_\partial(\Delta_p) \cong \mathbf{Q}[x]/I_p$, so expanding P^* and $P^\perp$ by Props. 2.1 and 2.2 gives scalars c_b and polynomials $Q_i(x)$ such that

$$P(x) \;=\; \sum_{i=1}^{n} Q_i(x)\, h_{k+i}(x) \;+\; \sum_{b\in\mathcal{B}_p} c_b\, b(x)\,.$$

Projecting each term of these sums onto its homogeneous component of degree $\deg(P)$, each $c_b = 0$ when $\deg(b) \neq \deg(P)$, and each nonzero Q_i has $\deg(Q_i) = \deg(P) - (k+i)$. Iterating this result by expanding the Q_i's in the same manner as we have expanded P, we derive that the collection of polynomials

$$\left\{\, b(x)\, h_{k+1}^{m_1} h_{k+2}^{m_2} \cdots h_{k+n}^{m_n} \;:\; b \in \mathcal{B}_p \;\text{ and }\; m_i \geq 0 \,\right\} \tag{2.21}$$

spans the polynomial ring $\mathbf{Q}[x]$. However, the generating function of their degrees is given by the expression

$$F(t) \;=\; \frac{\sum_{b\in\mathcal{B}_p} t^{\deg(b)}}{(1-t^{k+1})(1-t^{k+2})\cdots(1-t^{k+n})}\,,$$

and from (2.19) we clearly have that

$$\begin{aligned}\sum_{b\in\mathcal{B}_p} t^{\deg(b)} \;&=\; t^{k+(k+1)+\cdots+(k+n-1)}\, \frac{1-t^{-k-1}}{1-t^{-1}}\,\frac{1-t^{-k-2}}{1-t^{-1}} \cdots \frac{1-t^{-k-n}}{1-t^{-1}} \\ &=\; \frac{(1-t^{k+1})(1-t^{k+2})\cdots(1-t^{k+n})}{(1-t)^n}\,.\end{aligned}$$

Thus

$$F(t) \;=\; \frac{1}{(1-t)^n}\,,$$

and since the latter is precisely the Hilbert series of the polynomial ring $\mathbf{Q}[x]$, we must conclude that the collection in (2.21) is necessarily also an independent set, proving the theorem. ∎

We are now in a position to prove the following remarkable special case of Thm. 2.1.

THEOREM 2.3. *For $p = (k+n-1, k+n-2, \ldots, k)$, the graded Frobenius characteristic of $\mathcal{L}_\partial(\Delta_p)$ is given by the polynomial*

$$\mathcal{F}\,\mathrm{ch}\,\mathcal{L}_\partial(\Delta_p) \;=\; \begin{bmatrix} k+n \\ n \end{bmatrix}_t \tilde{H}_{1^n}(x;q,t)\,. \tag{2.22}$$

Proof. The uniqueness of the expansion in (2.20) and the invariance of the coefficients under the action of S_n yields that the Frobenius characteristics of $\mathcal{L}_\partial(\Delta_p)$ and $\mathbf{Q}[x]$ are related by the identity

$$\mathcal{F}\,\mathrm{ch}\,\mathbf{Q}[x] \;=\; \frac{\mathcal{F}\,\mathrm{ch}\,\mathcal{L}_\partial(\Delta_p)}{(1-t^{k+1})(1-t^{k+2})\cdots(1-t^{k+n})}\,. \tag{2.23}$$

Since it is well-known that

$$\mathcal{F}\,\mathrm{ch}\,\mathbf{Q}(x_1,\ldots,x_n) \;=\; h_n[\tfrac{X}{1-t}]\,,$$

from (2.23) and (2.3) we deduce that

$$\mathcal{F}\,\mathrm{ch}\,\mathcal{L}_\partial(\Delta_p) \;=\; \frac{(1-t^{k+1})(1-t^{k+2})\cdots(1-t^{k+n})}{(1-t)(1-t^2)\cdots(1-t^n)}\,\tilde{H}_{1^n}(x;q,t)\,,$$

which is another way of writing (2.22). ∎

Remark 2.1. We should mention that for $p = (n-1, n-2, \ldots, 1, 0)$, all these results are known. In particular, we see that Thm. 2.2 is a generalization of the well-known classical result (see [2, pp. 39–41]) that asserts that every polynomial $P \in \mathbf{Q}[x_1,\ldots,x_n]$ has a unique expansion in the form

$$P(x) \;=\; \sum_{b\in\mathcal{B}_n} b(x)\,A_b(x)\,, \tag{2.24}$$

where $\mathcal{B}_n$ is any basis of $\mathcal{L}_\partial(\Delta_n)$ and the coefficients A_b are symmetric polynomials in $x_1,\ldots,x_n$. It develops that we only need this special case in our proof of Thm. 2.1. Thus by pure chance this generalization contributes to our development here being completely self-contained.

We are now in a position to carry out our first step in the proof of Thm. 2.1:

PROPOSITION 2.3. *For an arbitrary $p \in \mathcal{D}_n$, set*

$$J_p \;=\; (\,A(x_1,\ldots,x_n)\;:\;A \text{ is symmetric and } A(\partial)\Delta_p(x) = 0\,)\,.$$

Then every polynomial $Q(x_1,\ldots,x_n)$ in $\mathcal{L}_\partial(\Delta_p)$ has an expansion of the form

$$Q(x) \;=\; \sum_{b\in\mathcal{B}_n} A_b(\partial_x)\, b(\partial_x)\;\Delta_p(x)\,, \tag{2.25}$$

with the coefficients $A_b(x)$ symmetric polynomials in $J_p^\perp$ uniquely determined by Q.

Proof. By hypothesis, $Q(x)$ can be expressed as $P(\partial)\Delta_p(x)$ for some $P(x)$ (not unique). Expand $P(x)$ as in (2.24). For each $b\in\mathcal{B}_n$, write $A_b = A_b^* + A_b^\perp$ where $A_b^*\in J_p$ and $A_b^\perp \in J_p^\perp$. Then

$$Q(x) \;=\; P(\partial)\Delta_p(x) \;=\; \sum_{b\in\mathcal{B}_n} A_b^*(\partial)\, b(\partial)\;\Delta_p(x) + \sum_{b\in\mathcal{B}_n} A_b^\perp(\partial)\, b(\partial)\;\Delta_p(x)\,.$$

The first sum vanishes because $A_b^*(\partial)\Delta_p(x)=0$, and we are left with the second sum, of form (2.25).

To prove this expansion is unique, suppose that

$$\sum_{b\in\mathcal{B}_n} A_b(\partial)\; b(\partial)\Delta_p(x) \;=\; 0\,, \tag{2.26}$$

with A_b symmetric polynomials in $J_p^\perp$. It suffices to assume they are homogeneous and that all nonzero terms in (2.26) have the same degree. We must show that all $A_b=0$, so assume they are not all 0. Choose $A_{b_0}\neq 0$ of minimum possible degree.

By Lemma 2.2, there is a polynomial $A'(x)$ and scalar $c_{b_0}\neq 0$ for which

$$A'(\partial)A_{b_0}(\partial)\Delta_p(x) \;=\; c_{b_0}\,\Delta_n(x)\,.$$

In fact, we have

$$A'(\partial)A_b(\partial)\Delta_p(x) \;=\; c_b\,\Delta_n(x)$$

for scalars c_b for all $b\in\mathcal{B}_n$, because the expression on the left is alternating and has degree at most $\binom{n}{2}$; when it has degree equal to $\binom{n}{2}$, it must be a scalar multiple of $\Delta_n(x)$; and when it has smaller degree, it must be identically 0, and we take $c_b=0$.

Apply $A'(\partial)$ to (2.26):

$$0 \;=\; \sum_{b\in\mathcal{B}_n} A'(\partial)\, A_b(\partial)\, b(\partial)\;\Delta_p(x) \;=\; \sum_{b\in\mathcal{B}_n} c_b\, b(\partial)\,\Delta_n(x)\,.$$

The derivatives $b(\partial)\Delta_n(x)$ as b ranges over $\mathcal{B}_n$ are linearly independent, so all coefficients $c_b=0$. This violates $c_{b_0}\neq 0$, so the assumption that some $A_b \neq 0$ is false, and the expansion indeed is unique, completing our proof. ▮

PROPOSITION 2.4. *Let $A_1(x), \ldots, A_N(x)$ be a homogeneous basis of the symmetric polynomials in $J_p^{\perp}$. Let*

$$F_p(t) \;=\; \sum_{i=1}^{N} t^{|p|-\binom{n}{2}-\deg(A_i)}.$$

Then $F_p(t) = \sum_{i=1}^{N} t^{\deg(A_i)}$, and setting $\mathbf{M}_p = \mathcal{L}_\partial(\Delta_p)$, we have

$$\mathcal{F}\,\mathrm{ch}\,\mathbf{M}_p \;=\; F_p(t)\,\tilde{H}_{1^n}(x;q,t) \;=\; F_p(t)\,h_n\left[\tfrac{X}{1-t}\right]\,\prod_{j=1}^{n}(1-t^j)\,.$$

Proof. Let $A_1(x), \ldots, A_N(x)$ be a homogeneous basis of the symmetric polynomials in $J_p^{\perp}$. By (2.25), the collection

$$\{\, A_i(\partial)b(\partial)\Delta_p(x) \;:\; i = 1, \ldots, N \quad \text{and} \quad b \in \mathbf{B}_n \,\}$$

is a basis of $\mathbf{M}_p$. This decomposes $\mathbf{M}_p$ into N parts, the i^{th} part being

$$\mathbf{M}_p^{(i)} \;=\; \mathrm{span}\,\{\, A_i(\partial)b(\partial)\Delta_p(x) \;:\; b \in \mathbf{B}_n \,\}\;.$$

For each i there is a homogeneous symmetric polynomial $A_i'(x)$ for which $A_i'(\partial)A_i(\partial)\Delta_p(x) = c_i\Delta_n(x)$, with $c_i \neq 0$, by Lemma 2.2. Then

$$\mathbf{M}_p \xrightarrow{\;A_i(\partial)\;} \mathbf{M}_p^{(i)} \xrightarrow{\;A_i'(\partial)\;} \mathbf{M}_n = \mathcal{L}_\partial(\Delta_n)$$

is a composition of surjective, character-preserving homomorphisms that lower degrees by $\deg(A_i)$ and $\deg(A_i')$, respectively. Noting that $\deg(A_i) + \deg(A_i') = |p| - \binom{n}{2}$, we therefore have

$$\mathcal{F}\,\mathrm{ch}\,\mathbf{M}_p^{(i)} \;=\; t^{|p|-\binom{n}{2}-\deg(A_i)}\cdot\mathcal{F}\,\mathrm{ch}\,\mathbf{M}_n \;=\; t^{|p|-\binom{n}{2}-\deg(A_i)}\tilde{H}_{1^n}(x;q,t)\,.$$

Interchanging the roles of $A_i(x)$ and $A_i'(x)$ in this argument gives $F_p(t) = \sum_{i=1}^{N} t^{|p|-\binom{n}{2}-\deg(A_i')} = \sum_{i=1}^{N} t^{\deg(A_i)}$. ▮

Remark 2.2. Note that this reduces the problem of finding $\mathcal{F}\,\mathrm{ch}\,\mathcal{L}_\partial(\Delta_p)$ to determining the action of symmetric differential operators on $\Delta_p(x)$. In particular, we must determine a set of such operators that, when applied to $\Delta_p(x)$, yield a basis of all alternants in $\mathcal{L}_\partial(\Delta_p)$. The following result yields a preliminary step in this direction.

THEOREM 2.4. *Let λ and μ be partitions with at most n parts, and $\rho = (n-1, n-2, \ldots, 0)$. Then*

$$s_\mu(\partial_{x_1}, \ldots, \partial_{x_n}) \frac{\Delta_{\rho+\lambda}(x_1, \ldots, x_n)}{d_\lambda} = \sum_{\nu \subseteq \lambda} c^\lambda_{\mu,\nu} \frac{\Delta_{\rho+\nu}(x_1, \ldots, x_n)}{d_\nu}, \quad (2.27)$$

where $c^\lambda_{\mu,\nu}$ are the Littlewood-Richardson coefficients and for convenience we have set $d_\lambda = \langle \Delta_{\rho+\lambda}, \Delta_{\rho+\lambda} \rangle$ and $d_\nu = \langle \Delta_{\rho+\nu}, \Delta_{\rho+\nu} \rangle$.

Proof. It follows from Lemma 2.1 that we have the expansion

$$s_\mu(\partial)\Delta_{\rho+\lambda}(x) = \sum_{\nu \vdash |\lambda|-|\mu|} \langle s_\mu(\partial)\Delta_{\rho+\lambda}, \Delta_{\rho+\nu} \rangle \frac{\Delta_{\rho+\nu}(x)}{d_\nu}. \quad (2.28)$$

Since differentiation is dual to multiplication with respect to the scalar product in (2.7), we see that we can write

$$\langle s_\mu(\partial)\Delta_{\rho+\lambda}, \Delta_{\rho+\nu} \rangle = \langle \Delta_{\rho+\lambda}, s_\mu \, \Delta_{\rho+\nu} \rangle = \langle \Delta_{\rho+\lambda}, s_\mu \, s_\nu \, \Delta_n \rangle. \quad (2.29)$$

The Littlewood-Richardson rule then gives

$$\langle \Delta_{\rho+\lambda}, s_\mu \, s_\nu \, \Delta_n \rangle = \sum_{\theta \vdash |\lambda|} c^\theta_{\mu\nu} \langle \Delta_{\rho+\lambda}, s_\theta \, \Delta_n \rangle = \sum_{\theta \vdash |\lambda|} c^\theta_{\mu\nu} \langle \Delta_{\rho+\lambda}, \Delta_{\rho+\theta} \rangle,$$

and the orthogonality of our alternants reduces this to

$$\langle \Delta_{\rho+\lambda}, s_\mu \, s_\nu \, \Delta_n \rangle = c^\lambda_{\mu\nu} \, d_\lambda.$$

Combining this with (2.29) and substituting in (2.28) yields

$$s_\mu(\partial)\Delta_{\rho+\lambda}(x) = \sum_{\nu \subseteq \lambda \,;\, \nu \vdash |\lambda|-|\mu|} c^\lambda_{\mu,\nu} \, \Delta_{\rho+\nu}(x) \frac{d_\lambda}{d_\nu},$$

which gives (2.27) upon division by d_λ. ∎

We are finally in a position to give our

Proof of Thm. 2.1. Note first that the map

$$\Phi : s_\nu(x_1, \ldots, x_n) \longrightarrow \frac{\Delta_{\nu+\rho}(x_1, \ldots, x_n)}{d_\nu}$$

gives an isomorphism of the space of symmetric polynomials onto the space of alternants that increases the degree by $\binom{n}{2}$. Thus we derive from (2.27) and the Littlewood-Richardson rule for skew Schur functions

$$s_{\lambda/\mu} = \sum_{\nu\subseteq\lambda} c^{\lambda}_{\mu,\nu}\, s_\nu \tag{2.30}$$

that the dimension of the space of alternants in $\mathcal{L}_\partial(\Delta_p)$ that are homogeneous of degree $m+\binom{n}{2}$ is the same as dimension of the vector space spanned by the collection of skew Schur functions

$$\left\{\, s_{\lambda/\mu}(x_1,\ldots,x_n) \;:\; \mu \vdash |\lambda| - m \,\right\}\,,$$

and that is equal to the coefficient of t^m in the polynomial $\Xi_\lambda(t)$ that occurs in (2.4). Thus the Hilbert series of the alternants in $\mathcal{L}_\partial(\Delta_p)$ is given by

$$\Xi_\lambda(t)\, t^{\binom{n}{2}}\,.$$

On the other hand, from Prop. 2.4, it follows that for $p=\lambda+\rho$, the Hilbert series of these alternants should also be given by

$$F_p(t)\, t^{\binom{n}{2}}\,.$$

Thus we must have $F_p(t) \;=\; \Xi_\lambda(t)$, and the theorem follows by combining Prop. 2.4 with Thm. 2.4. ∎

3. DIAGRAMS ARISING FROM COMPLEX REFLECTION GROUPS

Results similar to those of the previous section hold for complex reflection groups. Let G be a finite $n\times n$ matrix group generated by reflections, acting on polynomials $P(x_1,\ldots,x_n)$ via

$$T_A\; P(x) = P(xA) \qquad \text{for } A\in G.$$

Let $R=\mathbf{C}[x_1,\ldots,x_n]$. Let R^G be the G-invariant polynomials, and $I_G = (R^G_+)$ be the ideal generated by G-invariant polynomials of positive degree. Replace the scalar product (2.7) by

$$\big\langle P\,,\,Q\big\rangle \;=\; \overline{P(\partial)}\, Q(x)\,\big|_{x=0}\,. \tag{3.1}$$

The "*harmonics*" of G are

$$\begin{aligned}\mathbf{H}_G \;=\; I_G^{\perp} \;&=\; \left\{\, P\in R \,:\, \big\langle P\,,\,Q\big\rangle = 0 \;\text{ for all }\; Q\in I_G \,\right\}\\ &=\; \left\{\, P\in R \,:\, Q(\partial)P = 0 \;\text{ for all }\; Q\in R^G \,\right\}\,.\end{aligned}$$

The discriminant $\Delta_G(x)$ is the product of reflecting hyperplanes raised to one less than their order. Steinberg [10] proved

THEOREM 3.1. *$\mathbf{H}_G = \mathcal{L}_\partial(\Delta_G(x))$, and this yields a graded version of the regular representation of G.*

A polynomial $\Delta(x)$ is a "*G-alternant*" when

$$(\det A)\, T_A\ P(x) = P(x) \qquad \text{for all } A \in G.$$

Now let $\Delta(x)$ be any G-alternant and $I_\Delta = (P : P(\partial)\Delta(x) = 0)$ be the ideal of polynomials that kill Δ. Then $I_{\Delta_G} = I_G$, and the proofs of Thm. 2.1 and Prop. 2.4 may be adapted to yield

THEOREM 3.2. *Let $\Delta(x)$ be any G-alternant. Let $A_1(x), \ldots, A_N(x)$ be a homogeneous basis of the G-invariant polynomials in $I_\Delta^\perp$. Then we have the direct sum decomposition*

$$\mathcal{L}_\partial(\Delta(x)) \;=\; \bigoplus_{i=1}^{N} \mathbf{H}_G(\partial)\ A_i(\partial)\ \Delta(x)\,.$$

Thus, the graded character of $\mathcal{L}_\partial(\Delta(x))$ is given by the formula

$$\mathrm{ch}_t\, \mathcal{L}_\partial(\Delta(x)) = F_\Delta(t)\ \mathrm{ch}_t\, \mathbf{H}_G$$

where $F_\Delta(t) = \sum_{i=1}^N t^{\deg(\Delta)-\deg(\Delta_G)-\deg(A_i)} = \sum_{i=1}^N t^{\deg(A_i)}$.

We consider the complex reflection group formed as the wreath product $G_{n,m} = C_m \wr S_n$, where C_m is the cyclic group of order m; see [9] and [11]. This group may be identified with the group of $n \times n$ "pseudopermutation" matrices, where the nonzero entries are taken from the group of m^{th} roots of unity. Its order is $n!\, m^n$. For each divisor d of m, we also consider the subgroup $G_{n,m,d}$ consisting of those matrices in $G_{n,m}$ in which the product of the nonzero entries is a dth root of unity; $G_{n,m,d}$ has order $n!\, m^{n-1}\, d$. In this notation, the hyperoctahedral group B_n is $G_{n,2} = G_{n,2,2}$; the Weyl group D_n is $G_{n,2,1}$; and $G_{n,m} = G_{n,m,m}$.

Fix n, m, d. Let $\omega = e^{2\pi i/m}$. Let $[m]_t = \frac{1-t^m}{1-t} = 1 + t + \cdots + t^{m-1}$ and $[m]_t! = [m]_t\,[m-1]_t \cdots [1]_t$. For each polynomial $f(x) = f(x_1, \ldots, x_n)$, we abbreviate $f(x^m) = f(x_1^m, \ldots, x_n^m)$. For each integer sequence $q = (q_1, \ldots, q_n) \in \mathcal{D}_n$, and any number b, we set

$$mq + b = (mq_1 + b, \ldots, mq_n + b)\,.$$

We first focus on $G_{n,m}$. The $G_{n,m}$-invariants of $\mathbf{C}[x]$ are generated by the symmetric functions $e_1(x^m), \ldots, e_n(x^m)$; the set of all invariants is $\{\, f(x^m) : f \in \Lambda \,\}$ (where Λ is the set of S_n-symmetric functions in $\mathbf{C}[x]$).

$G_{n,m}$ has 2-fold reflections through the hyperplanes $x_j - \omega^r x_k$ (for $j \neq k$ and $0 \leq r \leq m-1$), and m-fold reflections through each x_j, so the discriminant is

$$\begin{aligned}\Delta_{n,m}(x_1,\ldots,x_n) &= (x_1\cdots x_n)^{m-1}\prod_{1\le j<k\le n}\prod_{r=0}^{m-1}(x_j-\omega^r x_k)\\ &= \det\|{x_j}^{km-1}\|_{1\le j,k\le n} \;=\; \Delta_p(x_1,\ldots,x_n)\end{aligned}$$

where $p=(nm-1,(n-1)m-1,\ldots,m-1)$. A basis of the $G_{n,m}$-alternating polynomials in $\mathbf{C}[x]$ is given by

$$\Delta_{mq+m-1}(x)=(x_1\cdots x_n)^{m-1}\Delta_q(x^m) \qquad (\text{with } q\in\mathcal{D}_n).$$

PROPOSITION 3.1. *Let $m,n\ge 1$. We have the direct sum decomposition*

$$\mathcal{L}_\partial(\Delta_{n,m}(x)) = \bigoplus_\epsilon e_1(\partial)^{\epsilon_1}\,e_2(\partial)^{\epsilon_2}\,\cdots e_n(\partial)^{\epsilon_n}\,\mathbf{H}_{S_n}(\partial)\;\Delta_{n,m}(x) \tag{3.2}$$

(with $0\le\epsilon_1,\ldots,\epsilon_n\le m-1$), from which it follows that

$$\mathcal{F}\,\mathrm{ch}\,\mathcal{L}_\partial(\Delta_{n,m}(x)) \;=\; [m]_t\,[m]_{t^2}\,\cdots\,[m]_{t^n}\,\tilde{H}_{1^n}(x;q,t) \tag{3.3}$$

and

$$\Xi_{(n(m-1),(n-1)(m-1),\cdots,1(m-1))}(t) \;=\; [m]_t\,[m]_{t^2}\,\cdots\,[m]_{t^n}\,. \tag{3.4}$$

Proof. Gallo [4] gives a basis for $\mathbf{H}_{G_{n,m}}$: $\{e_1(x)^{\epsilon_1}\cdots e_n(x)^{\epsilon_n}\,b\}$, where $0\le\epsilon_i\le m-1$ and $b\in\mathcal{B}_n$ (see Remark 2.1). This gives (3.2). The symmetric polynomials within this basis are those with $b=1$; Prop. 2.4 then gives (3.3), and comparing this with (2.4) gives (3.4). ▌

THEOREM 3.3. *Let $p=\lambda+\rho\in\mathcal{D}_n$, and $q=mp+m-1=\theta+\rho$. Then the graded multiplicity of the left regular representation of $G_{n,m}$ for $\mathcal{L}_\partial(\Delta_q)$ is $\Xi_\lambda(t^m)$. Thus,*

$$\Xi_\theta(t)=\Xi_\lambda(t^m)\prod_{i=1}^n[m]_{t^i}\,. \tag{3.5}$$

Moreover,

$$I_q=(\,P(x_1^m,\ldots,x_n^m)\;:\;P(x_1,\ldots,x_n)\in I_p\,)\,. \tag{3.6}$$

Proof. Every S_n-alternant in $\mathcal{L}_\partial(\Delta_p)$ has the form

$$f(\partial)\Delta_p(x)=\sum_{u\in\mathcal{D}_n}a_u^f\,\Delta_u(x) \tag{3.7}$$

where $f \in \Lambda$. Since the Δ_u are orthogonal with respect to the scalar product (3.1), the coefficients are

$$a_u^f = \langle \Delta_u \, , \, f(\partial)\Delta_p \rangle / \langle \Delta_u \, , \, \Delta_u \rangle = \langle \Delta_u \, , \, f(\partial)\Delta_p \rangle / u!$$

with $u! = u_1! \cdots u_n!$. Likewise, the $G_{n,m}$-alternants in $\mathcal{L}_\partial(\Delta_q)$ are

$$f(\partial^m)\Delta_q(x) = \sum_{u \in \mathcal{D}_n} b_u^f \; \Delta_{mu+m-1}(x) \; ,$$

where $f \in \Lambda$ and $f(\partial^m) = f(\partial_{x_1}^m, \ldots, \partial_{x_n}^m)$. The coefficients are

$$b_u^f = \langle \Delta_{mu+m-1} \, , \, f(\partial^m)\Delta_q \rangle / (mu + m - 1)! \; .$$

The two systems of coefficients are simply related: using adjoints and termwise expansion of the scalar product (3.1), we obtain

$$\begin{aligned} \langle \Delta_{mu+m-1} \, , \, f(\partial^m) \, \Delta_{mp+m-1} \rangle &= \langle \overline{f(x^m)} \, \Delta_{mu+m-1} \, , \, \Delta_{mp+m-1} \rangle \\ &= \frac{(mp+m-1)!}{p!} \cdot \langle \overline{f} \, \Delta_u \, , \, \Delta_p \rangle \; . \end{aligned}$$

Then

$$b_u^f = \frac{q!}{p!} \frac{\langle \overline{f} \Delta_u \, , \, \Delta_p \rangle}{(mu+m-1)!} = \frac{q!}{p!} \frac{u!}{(mu+m-1)!} a_u^f \; .$$

Thus, if we let $f_1, \ldots, f_M$ be any basis of Λ up to degree $|p|$, and u range over all sequences componentwise bounded by p, the matrices $A = [a_u^f]_{f,u}$ and $B = [b_u^f]_{f,u}$ are related by $A = BD$ for an invertible diagonal matrix D. In particular, choose $f_1, \ldots, f_M$ so that

$$f_k(\partial)\Delta_p(x) \qquad (\text{with } k = 1, \ldots, N)$$

is a basis of the alternants of $\mathcal{L}_\partial(\Delta_p)$, while

$$f_k(\partial)\Delta_p(x) = 0 \qquad (\text{for } k = N+1, N+2, \ldots, M).$$

Then

$$f_k(\partial^m)\Delta_q(x) \qquad (\text{with } k = 1, \ldots, N)$$

is a basis of the alternants of $\mathcal{L}_\partial(\Delta_q)$, giving the multiplicity $\Xi_\lambda(t^m)$, while

$$f_k(\partial^m)\Delta_q(x) = 0 \qquad (k = N+1, N+2, \ldots, M),$$

giving (3.6).

Finally, the graded Frobenius characteristic of $\mathcal{L}_\partial(\Delta_q)$ is

$$\Xi_\theta(t)\ \tilde{H}_{1^n}(x;q,t) = \Xi_\lambda(t^m)\ \mathcal{F}\,\mathrm{ch}\,\mathcal{L}_\partial(\Delta_{G_{n,m}})\ ,$$

where the left side is given by Thm. 2.1 and the right side by Thm. 3.2. Evaluating $\mathcal{F}\,\mathrm{ch}\,\mathcal{L}_\partial(\Delta_{G_{n,m}})$ by (3.3) and simplifying gives (3.5). ▮

Next we consider $G_{n,m,d}$. Its polynomial invariants are generated by $e_1(x^m),\ldots,e_{n-1}(x^m)$ and $e_n(x^d)$. The set of all $G_{n,m,d}$-invariant polynomials has a basis

$$\left\{\, f(x^m)\,(e_n(x^d))^j\ :\ f\in\Lambda \ \text{ and }\ j=0,1,\ldots,c-1 \,\right\} \tag{3.8}$$

where we set $c=m/d$.

The discriminant of $G_{n,m,d}$ is

$$\Delta_{n,m,d}(x_1,\ldots,x_n) = \det\|{x_j}^{(k-1)m+d-1}\|_{1\le j,k\le n} = \Delta_p(x_1,\ldots,x_n)$$

where $p=((n-1)m+d-1,(n-2)m+d-1,\ldots,d-1)$. A basis of the $G_{n,m,d}$-alternating polynomials in $\mathbf{C}[x]$ is

$$\{\, \Delta_{mq+hd-1}(x)\ :\ q\in\mathcal{D}_n \ \text{ and }\ 1\le h\le c\,\}\ .$$

THEOREM 3.4. *Let $p=\lambda+\rho\in\mathcal{D}_n$, and $q=mp+hd-1=\theta+\rho$ with $d|m$, $c=m/d$, $h=1,\ldots,c$. Let $\tilde\lambda=(\lambda_1-1,\ldots,\lambda_n-1)$. Then the graded multiplicity of the left regular representation of $G_{n,m,d}$ for $\mathcal{L}_\partial(\Delta_q)$ is*

$$[h]_{t^{nd}}\Xi_\lambda(t^m)+t^{ndh}[c-h]_{t^{nd}}\Xi_{\tilde\lambda}(t^m) \tag{3.9}$$

(interpreting $\Xi_{\tilde\lambda}$ as 0 when $p_n=0$). Thus,

$$\Xi_\theta(t) = \left([h]_{t^{nd}}\Xi_\lambda(t^m)+t^{ndh}[c-h]_{t^{nd}}\Xi_{\tilde\lambda}(t^m)\right)[d]_{t^n}\prod_{j=1}^{n-1}[m]_{t^j}\ . \tag{3.10}$$

Proof. The proof is similar to that of Thm. 3.3.

The $G_{n,m,d}$-alternants in $\mathcal{L}_\partial(\Delta_q)$ have the form

$$e_n(\partial^d)^j\ f(\partial^m)\ \Delta_q(x) = f(\partial^m)\Delta_{mp+(h-j)d-1}(x) \tag{3.11}$$

for $f\in\Lambda$ and $0\le j\le c$. For distinct values of j, the subspaces of alternants so obtained intersect only trivially, because modulo m, the degree of (3.11) in each variable is congruent to $(h-j)d-1$.

For each S_n-alternant in $\mathcal{L}_\partial(\Delta_p)$ of form (3.7), and $0 \leq j < h$,

$$e_n(\partial^d)^j \; f(\partial^m)\; \Delta_q(x) = \sum_{u \in \mathcal{D}_n} \frac{q!}{p!} \frac{u!}{(mu + (h-j)d - 1)!} \; a_u^f \; \Delta_{mu+(h-j)d-1} \; .$$

For $h \leq j \leq c$, we have $h-j \leq 0$, so when $u_n = 0$, we have $mu+(h-j)d-1 \notin \mathcal{D}_n$ because its n^{th} component is negative; thus this expansion is invalid. When $p_n = 0$, we have no further alternants because $e_n(\partial^d)^j \; \Delta_q(x) = 0$ for $j \geq h$. For $p_n > 0$, we set $\tilde{p} = (p_1 - 1, \ldots, p_n - 1)$, so that $mp + (h-j)d - 1 = m\tilde{p} + (c+h-j)d - 1$ with $1 \leq c+h-j \leq c$. Then for $f \in \Lambda$, the expansion

$$f(\partial)\Delta_{\tilde{p}}(x) = \sum_{u \in \mathcal{D}_n} \tilde{a}_u^f \; \Delta_u(x)$$

gives

$$e_n(\partial^d)^j \; f(\partial^m)\; \Delta_q(x) = \sum_{u \in \mathcal{D}_n} \frac{q!}{\tilde{p}!} \frac{u!}{(mu + (h-j)d - 1)!} \; \tilde{a}_u^f \; \Delta_{mu+(h-j)d-1} \; .$$

The graded multiplicity of the left regular representation of $G_{n,m,d}$ for $\mathcal{L}_\partial(\Delta_q)$ is

$$\left(\sum_{j=0}^{h-1} t^{ndj} \right) \Xi_\lambda(t^m) + \left(\sum_{j=h}^{c-1} t^{ndj} \right) \Xi_{\tilde{\lambda}}(t^m) \; .$$

This equation can be rewritten as (3.9).

Next, from the degrees of the basic invariants of $G_{n,m,d}$, the Hilbert polynomial of $\mathcal{L}_\partial(\Delta_{n,m,d})$ is

$$[m]_t \, [2m]_t \, \cdots [(n-1)m]_t \, [dn]_t$$

while the Hilbert polynomial of $\mathcal{L}_\partial(\Delta_n)$ is $[n]_t!$. Since S_n is a subgroup of $G_{n,m,d}$, we have $\mathbf{H}_{G_{n,m,d}}$ is a graded multiple of $\mathbf{H}_{S_n}$; the multiplicity is

$$\frac{[m]_t \, [2m]_t \, \cdots [(n-1)m]_t \, [dn]_t}{[n]_t!} = [m]_t \cdots [m]_{t^{n-1}} \, [d]_{t^n} \; ,$$

giving (3.10). ∎

As a consequence, we have the following result.

Proposition 3.2. *Let $a \geq 0$ and $b, n \geq 1$ be integers, and*

$$\theta \;=\; (a + (n-1)(b-1), a + (n-2)(b-1), \cdots, a + 0(b-1)) \, . \quad (3.12)$$

Let division give $a = bQ + R$ with Q, R integers, $0 \le R < b$. Then

$$\Xi_\theta(t) \;=\; \frac{[a+1+(R+1)(n-1)]_t \cdot \prod_{i=1}^{n-1}[b(Q+i)]_t}{[n]_t!} \tag{3.13}$$

Proof. Applying Thm. 3.4 with $m = c = b$, $d = 1$, $h = R+1$, $p = (n-1+Q, n-2+Q, \ldots, Q)$, $q = bp + R = \theta + \rho$, gives that $\Xi_\theta(t)$ is

$$\left([R+1]_{t^n}\Xi_{(Q^n)}(t^b) + t^{n(R+1)}[b-R-1]_{t^n}\Xi_{((Q-1)^n)}(t^b)\right)[b]_t\cdots[b]_{t^{n-1}}\ .$$

By Thm. 2.3 at $k = Q$, this becomes

$$\left([R+1]_{t^n}\begin{bmatrix} Q+n \\ n \end{bmatrix}_{t^b} + t^{n(R+1)}[b-R-1]_{t^n}\begin{bmatrix} Q+n-1 \\ n \end{bmatrix}_{t^b}\right)[b]_t\cdots[b]_{t^{n-1}}\ .$$

Expanding the binomial coefficients as products reduces this to

$$\left([R+1]_{t^n}[Q+n]_{t^b} + t^{n(R+1)}[b-R-1]_{t^n}[Q]_{t^b}\right)\frac{\prod_{j=1}^{n-1}[Q+j]_{t^b}[b]_{t^j}}{[n]_{t^b}!}\ . \tag{3.14}$$

The fraction on the right may be simplified using $[Q+j]_{t^b}[b]_{t^j}/[j]_{t^b} = [b(Q+j)]_t/[j]_t$ to get

$$\frac{\prod_{j=1}^{n-1}[Q+j]_{t^b}[b]_{t^j}}{[n]_{t^b}!} = \frac{\prod_{j=1}^{n-1}[b(Q+j)]_t}{[n]_t!}\frac{[b]_t}{[b]_{t^n}}\ .$$

The parenthesized part of (3.14) equals

$$\begin{aligned}
&[R+1]_{t^n}[Q+n]_{t^b} + ([b]_{t^n} - [R+1]_{t^n})[Q]_{t^b} \\
&\qquad = [R+1]_{t^n}([Q+n]_{t^b} - [Q]_{t^b}) + [b]_{t^n}[Q]_{t^b} \\
&\qquad = t^{bQ}[R+1]_{t^n}[n]_{t^b} + [b]_{t^n}[Q]_{t^b}
\end{aligned}$$

and $[b]_t/[b]_{t^n}$ times this equals

$$\begin{aligned}
&t^{bQ}[R+1]_{t^n}\ [b]_t[n]_{t^b}/[b]_{t^n} + [b]_t[Q]_{t^b} \\
&\qquad = t^{bQ}[R+1]_{t^n}[n]_t + [bQ]_t \;=\; t^{bQ}[(R+1)n]_t + [bQ]_t \\
&\qquad = [(R+1)n + bQ]_t = [a+1+(R+1)(n-1)]_t\ .
\end{aligned}$$

Combining the parts of (3.14) back together gives (3.13). ∎

Allen [1] has constructed bases of the modules $\mathbf{M}_{\theta+\rho}$ pertaining to this proposition when $0 \leq a < b$, including a different decomposition of (3.2) than what we presented.

4. FURTHER PROPERTIES OF ONE DIMENSIONAL DIAGRAMS

Let $\Sigma_\lambda^m = \operatorname{span}\{\, s_{\lambda/\mu} : \mu \subseteq \lambda \text{ and } \mu \vdash |\lambda| - m \,\}$. We give some properties of its Hilbert polynomial, $\Xi_\lambda(t) = \sum_{m=0}^{n} (\dim \Sigma_\lambda^m)\ t^m$.

Proposition 4.1. *Let $N = |\lambda| > 0$.*

(a) $\Xi_{\lambda'}(t) = \Xi_\lambda(t)$

(b) $\dim \Sigma_\lambda^m = \operatorname{rank} \left\| c_{\mu,\nu}^{\lambda} \right\|_{\mu \vdash N-m,\ \nu \vdash m}$,
where the matrix entries are Littlewood-Richardson coefficients.

(c) $\Xi_\lambda(t) = t^N \Xi_\lambda(t^{-1})$

(d) The coefficients of t^0, t^1, t^{N-1}, and t^N in $\Xi_\lambda(t)$ are always 1.

Proof. (a) The standard involution $\omega s_\mu = s_{\mu'}$ on symmetric functions is a degree preserving isomorphism of the vector spaces Σ_λ and $\Sigma_{\lambda'}$, so they have the same Hilbert polynomial.

(b) Expanding the spanning set $\{\, s_{\lambda/\mu} : \mu \vdash |\lambda| - m \,\}$ of Σ_λ^m in terms of ordinary Schur functions via (2.30) gives the stated transition matrix. The dimension of Σ_λ^m is the rank of this matrix.

(c) The Littlewood-Richardson coefficients satisfy $c_{\mu,\nu}^\lambda = c_{\nu,\mu}^\lambda$, so the matrices evaluated in (b) for Σ_λ^m and Σ_λ^{N-m} are transpose to each other. Thus they have the same rank, so $\Xi_\lambda(t)$ is symmetric.

Note that it need not be unimodal; the smallest non-unimodal case is $\Xi_{(4,2)}(t) = \Xi_{(2,2,1,1)}(t) = t^6 + t^5 + 2t^4 + t^3 + 2t^2 + t + 1$.

(d) There is only one subdiagram μ of λ of each size 0, 1, and N, so that Σ_λ^N, Σ_λ^{N-1}, and Σ_λ^0 each have dimension 1. All subdiagrams μ of size $N-1$ have $s_{\lambda/\mu} = s_1$, so Σ_λ^1 has dimension 1 as well. ∎

Once a particular $\Xi_\lambda(t)$ is computed, the next two results give two infinite families of diagrams p for which $\mathcal{F}\,\mathrm{ch}\,\mathcal{L}_\partial(\Delta_p)$ can be computed using the same value of $\Xi_\lambda(t)$.

Proposition 4.2. *Let $p_1 > p_2 > \cdots > p_n \geq 0$ and $u_1 > u_2 > \cdots > u_r \geq 0$ be integers with $\{p_1, \ldots, p_n, K-1-u_1, \ldots, K-1-u_r\} = \{0, \ldots, K-1\}$, where $K = n + r$. Let $p = \lambda + \rho_n$ as usual. Then we have (2.4) and*

$$\mathcal{F}\,\mathrm{ch}\,\mathcal{L}_\partial(\Delta_u) = \Xi_\lambda(t)\ \tilde{H}_{1^r}(x;q,t)\,. \tag{4.1}$$

Proof. In decreasing order, $u_1 > u_2 > \cdots > u_r$ may be written

$$\begin{array}{l} K-1, K-2, \cdots, K-p_n, \\ K-p_n-2, K-p_n-3, \cdots, K-p_{n-1}, \\ K-p_{n-1}-2, K-p_{n-1}-3, \cdots, K-p_{n-2}, \\ \cdots \\ K-p_1-2, K-p_1-3, \cdots, 1, 0, \end{array}$$

and successively subtracting $r-1, r-2, \ldots, 0$ from these gives $u = \mu + \rho_r$, with

$$\begin{aligned} \mu &= (n^{p_n}, (n-1)^{p_{n-1}-p_n-1}, \ldots, 1^{p_1-p_2-1}, 0^{K-p_1-1}) \\ &= (n^{\lambda_n}, (n-1)^{\lambda_{n-1}-\lambda_n}, \ldots, 1^{\lambda_1-\lambda_2}, 0^{K-p_1-1}) = \lambda' . \end{aligned}$$

By Thm. 2.1, $\mathcal{F}\,\mathrm{ch}\,\mathcal{L}_\partial(\Delta_u) = \Xi_\mu(t)\,\tilde{H}_{1^r}(x;q,t)$; but $\Xi_\mu(t) = \Xi_{\lambda'}(t) = \Xi_\lambda(t)$ by the preceding proposition. ∎

Because $\Xi_\lambda(t)$ does not depend on n so long as $n \geq \ell(\lambda)$, the following is an immediate corollary of Thm. 2.1.

Proposition 4.3. *Let $p = \rho_n + \lambda \in \mathcal{D}_n$ and*

$$p^{(r)} = \rho_{n+r} + \lambda = (p_1 + r, p_2 + r, \cdots, p_n + r, r-1, r-2, \cdots, 1, 0).$$

Then $\mathcal{F}\,\mathrm{ch}\,\mathcal{L}_\partial(\Delta_{p^{(r)}}) = \Xi_\lambda(t)\,\tilde{H}_{1^{n+r}}(x;q,t)$.

We now evaluate $\Xi_\lambda(t)$ when λ is a hook.

Proposition 4.4. *Let $\lambda = (a+1, 1^b)$ be a hook, $a, b > 0$, and $N = a+b+1$. Let $c = \min\{a, b\} + 1$. Then*

$$\Xi_{(a+1,1^b)} \;=\; 1 + t^N + \sum_{m=1}^{c-1} m\,(t^m + t^{N-m}) \;+\; c\sum_{m=c}^{N-c} t^m \,.$$

Proof. We assume $b > a$ so that $b > N/2$. The case $a > b$ is similar, and $a = b$ will be treated separately. We consider partitions $\mu \subseteq \lambda$ of size m. The cases $m = 0, 1, N-1, N$ are special and have already been considered in Prop. 4.1. For $2 \leq m \leq N/2 < b$, the hooks that occur are $\mu = (k+1, 1^{m-1-k})$ with $k = 0, 1, \ldots, \min\{m-1, a\}$, and

$$\begin{aligned} s_{(a+1,1^b)/(k+1,1^{m-1-k})} \;&=\; h_{a-k}\; e_{b+1-m+k} \\ &= \begin{cases} s_{(a-k+1,1^{b-m+k})} + s_{(a-k,1^{b+1-m+k})} & \text{if } k \neq a \text{ and } k \neq m-b-1; \\ s_{(a-k)} & \text{if } k = m-b-1; \\ s_{(1^{b+1+m-k})} & \text{if } k = a. \end{cases} \end{aligned} \tag{4.2}$$

The dominant partition in each case is different, so all $\min\{m-1,a\}+1$ of these are independent.

By Prop. 4.1, when $m > N/2$, the coefficients can be deduced from the fact that $\Sigma_\lambda(t)$ has symmetric coefficients. Alternately, we are constrained to $\max\{0, m-1-b\} \leq k \leq \min\{a, m-1\} = a$ in order that μ fit within the leg of λ as well as the arm. As k varies, the dominant terms in (4.2) are all distinct, so there are $a - \max\{0, m-1-b\} = \min\{a, a+b+1-m\} = \min\{a, N+1-m\}$ linearly independent skew Schur functions.

Finally, in the case $a = b$, the preceding gives the coefficients whenever $m \neq a+1$. When $m = a+1$, the shapes μ with $k = 1, \ldots, a$ give distinct dominant partitions in (4.2). But $k = 0$ corresponds to $\mu = (1^a)$, giving

$$s_{(a+1,1^a)/(1^a)} \;=\; \sum_{m=1}^{a} (-1)^{m-1} s_{(a+1,1^a)/(m+1,1^{a-m})}\,,$$

so Σ_λ^m only has dimension a, not $a+1$. ▮

5. TWO DIMENSIONAL PARTITIONS PLUS ONE CELL

If μ is a partition, we denote by $\mu/[i,j]$ the lattice diagram obtained by removing the cell (i,j) from the diagram of μ, and refer to the cell (i,j) as the "*hole*" of $\mu/[i,j]$. We denote by $\mu + [i,j]$ the diagram obtained by adding the cell (i,j) to the diagram of μ, and refer to (i,j) as a "*pebble*." In [3], we and our coauthors explored the structure of the module $\mathbf{M}_{\mu/[i,j]} = \mathcal{L}_\partial(\Delta_{\mu/[i,j]})$; we now briefly give analogous results and conjectures for $\mathbf{M}_{\mu+[i,j]} = \mathcal{L}_\partial(\Delta_{\mu+[i,j]})$, where $\Delta_{\mu/[i,j]}$ and $\Delta_{\mu+[i,j]}$ are given by (1.2).

We set $\mu^\perp = \left\{\, (i,j) \in \mathbf{Z}^2 \,:\, i,j \geq 0 \;\text{ and }\; (i,j) \notin \mu \,\right\}$. We shall write $(i',j') \leq (i,j)$ to mean $i' \leq i$ and $j' \leq j$. For any $s = (i,j) \in \mu^\perp$, the collection of cells

$$\left\{\, (i',j') \in \mu^\perp \,:\, (i',j') \leq (i,j) \,\right\}$$

will be called the "*antishadow*" of (i,j) with respect to μ. It is the rotation and translation of a Ferrers diagram of a partition

$$\rho \;=\; (j+1-\nu_{i+1}, j+1-\nu_i, j+1-\nu_{i-1}, \ldots, j+1-\nu_{i+1-L}) \quad (5.1)$$

where the "*dual leg*" $L_\mu(s) = i - \lambda_{j+1}$ is the number of cells strictly south of s and outside μ, and the "*dual arm*" $A_\mu(s) = j - \lambda_{i+1}$ is the number of cells strictly west of s and outside μ. All these quantities are illustrated in Fig. 1.

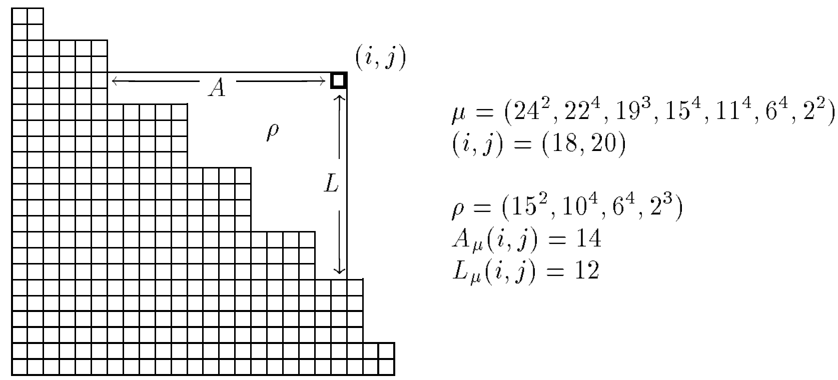


FIG. 1. Antishadow ρ of (i,j) relative to μ.

For integers $h, k \geq 0$ but not both 0, on setting

$$D_x = \sum_{r=1}^{n} \partial_{x_r} \,, \qquad D_y = \sum_{r=1}^{n} \partial_{y_r} \,, \qquad D_{hk} = \sum_{r=1}^{n} \partial_{x_r}^h \, \partial_{y_r}^k \,,$$

the following three results for pebbles follow from Prop. I.1 of [3] in the same way that the analogous results were given there for holes:

PROPOSITION 5.1. *For any partition $\mu \vdash n-1$ and $(i,j) \in \mu^\perp$, we have*

$$D_{hk} \Delta_{\mu+[i,j]}(x;y) = \begin{cases} \pm \Delta_{\mu+[i-h,j-k]}(x;y) & \text{if } (i-h, j-k) \in \mu^\perp \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 5.2. *Let $\mu \vdash n-1$ and $i, j, h, k \geq 0$. Then if (i,j), $(i-h, j-k) \in \mu^\perp$, we have*

$$D_x^h \; D_y^k \; \mathbf{M}_{\mu+[i,j]} \; = \; D_{hk} \; \mathbf{M}_{\mu+[i,j]} \; = \; \mathbf{M}_{\mu+[i-h,j-k]} \,,$$

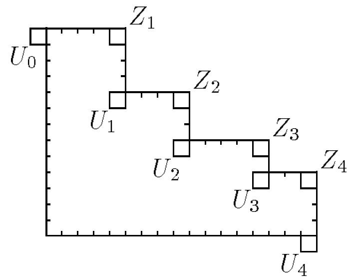
meaning that both $D_x^h D_y^k$ and D_{hk} are surjective linear maps. In particular we have the inclusion

$$\mathbf{M}_{\mu+[i',j']} \subseteq \mathbf{M}_{\mu+[i,j]}$$

for all cells (i',j') in the antishadow of (i,j) with respect to μ.

PROPOSITION 5.3. *The collection of polynomials*

$$\left\{ \, \Delta_{\mu+[i',j']}(x;y) \; : \; (i',j') \in \mu^\perp \;\; \text{and} \;\; (i',j') \leq (i,j) \, \right\}$$



$$\mu = (17^4, 14^2, 9^3, 5^4)$$

$$\begin{array}{ll} u_0 = t^{12}q^{-1} & z_1 = t^{12}q^4 \\ u_1 = t^8q^4 & z_2 = t^8q^8 \\ u_2 = t^5q^8 & z_3 = t^5q^{13} \\ u_3 = t^3q^{13} & z_4 = t^3q^{16} \\ u_4 = t^{-1}q^{16} & \end{array}$$

FIG. 2. Weights of corners of a partition.

forms a basis for the submodule of alternants of $\mathbf{M}_{\mu+[i,j]}$.

Computational evidence leads to a conjectured refinement of this:

Conjecture 5.1. For any $\mu \vdash n-1$ and any $(i,j) \in \mu^{\perp}$, the S_n-module $\mathbf{M}_{\mu+[i,j]}$ decomposes into the direct sum of k left regular representations of S_n, where k is the number of cells in the antishadow of (i,j) with respect to μ.

We will give a Frobenius characteristic formulation of this conjecture, but first we must introduce some notation. Let μ have m corners $Z_1, \ldots, Z_m$ labelled as we encounter them from upper left to lower right. The coordinates are $Z_j = (\alpha_j, \beta_j)$. Also consider the "*concave corner*" cells $U_0, \ldots, U_m$, with coordinates $U_j = (\alpha_{j+1}, \beta_j)$, where $\alpha_{m+1} = \beta_0 = -1$. The "*weight*" of Z_j is $z_j = t^{\alpha_j} q^{\beta_j}$, and the weight of U_j is $u_j = t^{\alpha_{j+1}} q^{\beta_j}$. These are illustrated in Fig. 2.

Let $\nu^{(i)}$ be the partition obtained on removing Z_i from μ. In [6, Prop. I.3], we used the Pieri rules [8, p. 340] for Macdonald polynomials to show

$$\partial_{p_1} \tilde{H}_\mu(x;q,t) \;=\; \sum_{\nu \to \mu} c_{\mu,\nu}(q,t) \tilde{H}_\nu(x;q,t) \tag{5.2}$$

where $\nu \to \mu$ means ν runs over the $\nu^{(i)}$; ∂_{p_1}, differentiation by p_1, is the Hall scalar product adjoint to multiplication by p_1; and

$$c_{\mu,\nu^{(i)}}(q,t) \;=\; \frac{1}{(1-1/q)(1-1/t)} \, \frac{1}{z_i} \, \frac{\prod_{j=0}^{m}(z_i - u_j)}{\prod_{j=1;\, j\neq i}^{m}(z_i - z_j)}\,. \tag{5.3}$$

Now let ν be a partition as depicted in Fig. 2, and let $\mu^{(i)}$ be the partition obtained by adding the cell $U_i' = (\alpha_{i+1}+1, \beta_i+1)$ to ν, for $i = 0, \ldots, m$. By a similar computation, another of Macdonald's Pieri formulas can be written

$$e_1(x)\ \tilde H_\nu(x;q,t) \ = \ \sum_{\mu\leftarrow\nu} d_{\mu,\nu}(q,t)\tilde H_\mu(x;q,t)\,, \tag{5.4}$$

where $\mu \leftarrow \nu$ means μ runs over the $\mu^{(i)}$, and

$$d_{\mu^{(i)},\nu}(q,t) \ = \ \frac{1}{qt}\ \frac{1}{u_i}\ \frac{\prod_{j=1}^m (u_i - z_j)}{\prod_{j=0;\,j\neq i}^m (u_i-u_j)}\,. \tag{5.5}$$

Computations with the modules $\mathbf{M}_{\mu+[i,j]}$ have led us to conjecture the following analogue of [3, Conj. I.3] as their Frobenius characteristic; the coefficients are given by (5.3).

Conjecture 5.2. For any $(i,j)\in\mu^\perp$, we have

$$C_{\mu+[i,j]}(x;q,t) = \sum_{(i',j')} c_{\rho,\rho/[i-i',j-j']}(q,t)\ \tilde H_{\mu+[i',j']}(x;q,t)\,, \tag{5.6}$$

where (i',j') runs over the cells $\leq (i,j)$ that can be added to μ to yield a partition.

THEOREM 5.1. *The validity of the preceding conjecture for all $(i,j)\in\mu^\perp$ is equivalent to*

(a) The four term recursion

$$\begin{aligned} C_{\mu+[i,j]} \ &= \ \frac{t^L - q^{A+1}}{t^L-q^A} C_{\mu+[i,j-1]} \ + \ \frac{q^A - t^{L+1}}{q^A - t^L} C_{\mu+[i-1,j]} \\ &\qquad - \frac{q^{A+1}-t^{L+1}}{q^A-t^L} C_{\mu+[i-1,j-1]} \end{aligned} \tag{5.7}$$

(b) together with the boundary conditions that each of the three terms of the form $C_{\mu+[i',j']}$ on the right is equal to zero when $(i',j')\notin\mu^\perp$, and to $\tilde H_{\mu+[i',j']}$ when (i',j') is an exterior corner of μ.

A representation theoretic interpretation of (5.7) comes from the following modules. For a fixed $\mu\vdash n-1$ and varying $(i,j)\in\mu^\perp$, let $\bar{\mathbf{K}}^x_{i,j}$ denote the kernel of the operator D_x as a map of $\mathbf{M}_{\mu+[i,j]}$ onto $\mathbf{M}_{\mu+[i-1,j]}$, and $\bar{\mathbf{K}}^y_{i,j}$ denote the kernel of the operator D_y as a map of $\mathbf{M}_{\mu+[i,j]}$ onto $\mathbf{M}_{\mu+[i,j-1]}$. Then

$$\bar{\mathbf{K}}^x_{i,j-1}\subseteq \bar{\mathbf{K}}^x_{i,j} \qquad \text{and} \qquad \bar{\mathbf{K}}^y_{i-1,j}\subseteq\bar{\mathbf{K}}^x_{i,j}\ .$$

All of these spaces are S_n-invariant and the quotients

$$\bar{\mathbf{A}}^x_{i,j} = \bar{\mathbf{K}}^x_{i,j}/\bar{\mathbf{K}}^x_{i,j-1} \qquad \text{and} \qquad \bar{\mathbf{A}}^y_{i,j} = \bar{\mathbf{K}}^y_{i,j}/\bar{\mathbf{K}}^y_{i-1,j}$$

are well-defined bigraded S_n-modules. Denoting the Frobenius characteristics of these by $\bar{K}^x_{i,j} = \mathcal{F}\,\mathrm{ch}\,\bar{\mathbf{K}}^x_{i,j}$, $\bar{A}^x_{i,j} = \mathcal{F}\,\mathrm{ch}\,\bar{\mathbf{A}}^x_{i,j}$, and similarly for forms with y, we have by a simple algebraic argument

Proposition 5.4.

$$\bar{K}^x_{i,j} = C_{\mu+[i,j]} - t\,C_{\mu+[i-1,j]}\,, \qquad \bar{K}^y_{i,j} = C_{\mu+[i,j]} - q\,C_{\mu+[i,j-1]}\,,$$

$$\bar{A}^x_{i,j} = \bar{K}^x_{i,j} - \bar{K}^x_{i,j-1}\,, \qquad \bar{A}^y_{i,j} = \bar{K}^y_{i,j} - \bar{K}^y_{i-1,j}\,.$$

Hence, the recurrence (5.7) is equivalent to the "crucial identity"

$$q^A\ \bar{A}^x_{i,j} = t^L\ \bar{A}^y_{i,j}\,. \tag{5.8}$$

The detailed proofs of the preceding results for pebbles are completely analogous to the corresponding proofs for holes in [3].

Next, we consider "adding a pebble at infinity," which is a possible analogue of removing the cell at the origin. To do this, we require a new operation on diagrams.

Let L be a lattice diagram that fits in an $a \times b$ box: $L \subset B_{a,b}$ where

$$B_{a,b} = \{\,(i,j)\,:\,0 \le i < a \ \text{ and } \ 0 \le j < b\,\}\,. \tag{5.9}$$

Define the complement of the rotation of L in $B_{a,b}$ to be

$$R_{a,b}(L) = \{\,(i,j) \in B_{a,b}\ :\,(a-1-i, b-1-j) \notin L\,\}\,. \tag{5.10}$$

When a, b are known within a problem, we abbreviate $L^* = R_{a,b}(L)$ for all $L \subset B_{a,b}$. When L is the diagram of a partition μ, we have

$$\mu^* = (b-\mu_a, b-\mu_{a-1}, \ldots, b-\mu_1)\,.$$

In [3, Prop. I.5], we proved $C_{\mu/[0,0]}(x;q,t) = \partial_{p_1} C_\mu(x;q,t)$, which combined with the $C = \tilde{H}$ conjecture implies $C_{\mu/[0,0]}(x;q,t) = \partial_{p_1}\tilde{H}_\mu(x;q,t)$. An analogue of this for pebbles is as follows. Let $T_\mu = q^{n(\mu)}t^{n(\mu')}$ and ∇ denote the linear operator $\nabla\,\tilde{H}_\mu = T_\mu\,\tilde{H}_\mu$. Also set $M = (1-q)(1-t)$.

Theorem 5.2. *On the validity of Conj. 5.2, we have*

$$\lim_{a,b\to\infty} \mathcal{F}\,\mathrm{ch}\,\mathbf{M}_{\nu+[a-1,b-1]}(x;q,t) \ = \ \frac{1}{M}\frac{\nabla}{T_\nu}\left(e_1\,\tilde{H}_\nu(x;q,t)\right), \tag{5.11}$$

where the limits are in the ring of Laurent series in q, t with $q^r, t^r \to 0$ as $r \to \infty$.

Proof. Take $a > \nu_1'$ and $b > \nu_1$, so that $(a-1, b-1) \notin \nu \subset B_{a,b}$. Then (5.6) becomes

$$\mathcal{F}\,\mathrm{ch}\,\mathbf{M}_{\nu+[a-1,b-1]} \;=\; \sum_{\mu\leftarrow\nu} c_{\nu^*,\mu^*}(q,t)\tilde{H}_\mu(x;q,t)\,,$$

and we will show (5.13) that we can rewrite this as

$$= \frac{1}{M}\sum_{\mu\leftarrow\nu}(q^k - t^{a-h})(t^h - q^{b-k})d_{\mu,\nu}(q,t)\tilde{H}_\mu(x;q,t)$$

where $\mu/\nu = (h,k)$. Then

$$\begin{aligned}
\mathcal{F}\,\mathrm{ch}\,\mathbf{M}_{\nu+[a-1,b-1]} &= \frac{1}{M}\sum_{\mu\leftarrow\nu}(q^k - t^{a-h})(t^h - q^{b-k})\; d_{\mu,\nu}(q,t)\;\tilde{H}_\mu(x;q,t)\\
&= \frac{1}{M}\sum_{\mu\leftarrow\nu}(q^k\,t^h - q^b - t^a + q^{b-k}t^{a-h})\; d_{\mu,\nu}(q,t)\;\tilde{H}_\mu(x;q,t)\\
&= \frac{1}{M}\sum_{\mu\leftarrow\nu}\left(\frac{\nabla}{T_\nu} - q^b - t^a + \frac{q^b\,t^a}{\nabla}\right)\; d_{\mu,\nu}(q,t)\;\tilde{H}_\mu(x;q,t)\\
&= \frac{1}{M}\left(\frac{\nabla}{T_\nu} - q^b - t^a + \frac{q^b\,t^a}{\nabla}\right)\;\left(e_1\,\tilde{H}_\nu(x;q,t)\right)\,,
\end{aligned}$$

and since $t^a, q^b \to 0$ as $a, b \to \infty$, we obtain (5.11). ∎

The computation in this proof that we delayed is as follows.

Lemma 5.1. *Fix a, b and $\nu \to \mu$ both contained in $B_{a,b}$. Let $\mu/\nu = (h,k)$. Then*

$$c_{\mu,\nu}(q,t) \;=\; \frac{(q^{b-k-1} - t^{h+1})(t^{a-h-1} - q^{k+1})}{(1-q)(1-t)}\, d_{\nu^*,\mu^*}(q,t) \tag{5.12}$$

$$d_{\mu,\nu}(q,t) \;=\; \frac{(1-q)(1-t)}{(q^k - t^{a-i})(t^h - q^{b-k})}\, c_{\nu^*,\mu^*}(q,t)\,. \tag{5.13}$$

Proof. It suffices to prove (5.12). We will assume $a > \mu_1'$ and $b > \mu_1$; the cases when $a = \mu_1'$ or $b = \mu_1$ (or both) are similar, but the labels of certain corners will be off by 1 from what we give here.

Let μ be a partition with m corners as depicted in Fig. 2. Then μ^* has $m+1$ corners, whose weights are related to μ's corners via

$$\begin{gathered} z_j^* = \frac{q^{b-2}t^{a-2}}{u_{m+1-j}} \quad \text{for} \quad j = 1, \ldots, m+1, \\ u_0^* = \frac{t^{a-1}}{q}\,, \quad u_{m+1}^* = \frac{q^{b-1}}{t}\,, \quad u_j^* = \frac{q^{b-2}t^{a-2}}{z_{m+1-j}} \quad \text{for} \quad j = 1, \ldots, m. \end{gathered} \tag{5.14}$$

For convenience we set $w = q^{b-2}t^{a-2}$, $z_0 = t^{a-1}/q$ and $z_{m+1} = q^{b-1}/t$ so that $u_j^* = w/z_{m+1-j}$ for $j = 0, \ldots, m+1$.

We have $\nu = \nu^{(i)}$ for some $i = 1, \ldots, m$; then ν^* is obtained by adding the $(m+1-i)^{th}$ corner to μ^*. Plugging (5.14) into (5.5) for this gives

$$\begin{aligned} d_{\nu^*,\mu^*}(q,t) &= \frac{1}{qt}\,\frac{1}{u_{m+1-i}^*}\,\frac{\prod_{j=0}^{m}(u_{m+1-i}^* - z_{m+1-j}^*)}{\prod_{j=0;\, j\neq i}^{m+1}(u_{m+1-i}^* - u_{m+1-j}^*)} \\ &= \frac{1}{qt}\,\frac{z_i}{w}\,\frac{\prod_{j=0}^{m}(\frac{w}{z_i} - \frac{w}{u_j})}{\prod_{j=0;\, j\neq i}^{m+1}(\frac{w}{z_i} - \frac{w}{z_j})} = \frac{F}{qt}\,\frac{\prod_{j=0}^{m}(u_j - z_i)}{\prod_{j=0;\, j\neq i}^{m+1}(z_j - z_i)} \end{aligned} \tag{5.15}$$

where

$$F = \frac{z_i}{w}\,\frac{w^{m+1}}{z_i^{m+1}\;u_0\cdots u_m}\Big/\frac{w^{m+1}z_i}{z_i^{m+1}\;z_0\cdots z_{m+1}} = \frac{z_0\cdots z_{m+1}}{w\;u_0\cdots u_m} = qt$$

follows from plugging in the definition of the u's and z's. Plugging this into (5.15), separating out the factors $j = 0, m+1$ in the denominator, and using $z_i = t^h q^k$ and z_0, z_{m+1} from above, gives

$$\begin{aligned} d_{\nu^*,\mu^*}(q,t) &= \frac{1}{(z_{m+1} - z_i)(z_0 - z_i)}\,\frac{\prod_{j=0}^{m}(u_j - z_i)}{\prod_{j=1;\, j\neq i}^{m}(z_j - z_i)} \\ &= \frac{z_i\;(M/qt)\;c_{\mu,\nu^{(i)}}(q,t)}{(z_{m+1} - z_i)(z_0 - z_i)} = \frac{M\,c_{\mu,\nu^{(i)}}(q,t)}{(q^{b-1-k} - t^{h+1})(t^{a-1-h} - q^{k+1})}\,, \end{aligned}$$

which gives (5.12). ∎

6. CONJECTURES FOR TWO DIMENSIONAL DIAGRAMS

Our formulas for special diagrams suggest several conjectures about the structure of $C_L(x;q,t)$ for two dimensional lattice diagrams L. Supporting evidence for the three conjectures will be given after they are all stated.

A diagram R is a *compression* of a diagram L when R can be obtained from L by moving squares weakly downward and leftward; equivalently, there is a numbering of the cells $L = \{(i_1, j_1), \ldots, (i_n, j_n)\}$, $R = \{(i'_1, j'_1), \ldots, (i'_n, j'_n)\}$ for which $i'_k \le i_k$ and $j'_k \le j_k$ for all k.

Conjecture 6.1. **(Compression Conjecture)** Let L be any diagram with the MLRR property. Then

$$C_L(x; q, t) \;=\; \sum_{\mu} \alpha^L_\mu(q,t) \tilde{H}_\mu(x; q, t)$$

where the sum only runs over partitions μ that are compressions of L. Further, $\alpha^L_\mu(q,t)$ is a rational function that can be expressed as a polynomial divided by $\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)$, where these are the hook products

$$\tilde{h}_\mu(q,t) = \prod_{s\in\mu}(q^{a_\mu(s)} - t^{l_\mu(s)+1}) \quad \text{and} \quad \tilde{h}'_\mu(q,t) = \prod_{s\in\mu}(t^{l_\mu(s)} - q^{a_\mu(s)+1})\,.$$

The denominators automatically follow from the fact that (1.3) can be expressed as a sum of Schur functions with $\mathbf{N}[q,t]$ coefficients, and the transition matrix between Schur functions and $\tilde{H}_\mu$ has these denominators.

Conjecture 6.2. **(Rotation Conjecture)** Fix a, b, and let $L^* = R_{a,b}(L)$ for all diagrams $L \subset B_{a,b}$, as in (5.10). If L has the MLRR property, then L^* does too, and for all partitions μ, we have $\alpha^L_\mu(q,t) = \alpha^{L^*}_{\mu^*}(q,t)$, that is,

$$C_L(x; q, t) \;=\; \sum_{\mu} \alpha_\mu(q,t)\,\tilde{H}_\mu(x; q, t) \tag{6.1}$$

$$\text{iff} \qquad C_{L^*}(x; q, t) \;=\; \sum_{\mu} \alpha_\mu(q,t)\,\tilde{H}_{\mu^*}(x; q, t)\,. \tag{6.2}$$

THEOREM 6.1. *On the validity of Conj. 6.2, if L and L^* possess the MLRR property, then* $\dim \mathbf{M}_L = k \cdot |L|!$ *iff* $\dim \mathbf{M}_{L^*} = k \cdot |L^*|!$.

Proof. Set $n = |L|$ and $n^* = |L^*| = ab - n$. The Hilbert series $d_L(q,t)$ of the trivial representation in $\mathbf{M}_L$ is the coefficient of $s_n(x)$ when $C_L(x; q, t)$ is expanded in Schur functions; by (6.1), this is

$$d_L(q,t) \;=\; C_L(x; q, t)\Big|_{s_n} \;=\; \sum_{\mu} \alpha_\mu(q,t)\tilde{H}_\mu(x; q, t)\Big|_{s_n} \;=\; \sum_{\mu} \alpha_\mu(q,t)\;.$$

Assuming (6.2), the trivial representation in $\mathbf{M}_{L^*}$ has the same Hilbert series:

$$d_{L^*}(q,t) \;=\; C_{L^*}(x; q, t)\Big|_{s_{n^*}} \;=\; \sum_{\mu} \alpha_\mu(q,t)\tilde{H}_{\mu^*}(x; q, t)\Big|_{s_{n^*}} \;=\; \sum_{\mu} \alpha_\mu(q,t)\,.$$

Assuming the MLRR property holds, $\mathbf{M}_L$ is a sum of $d_L(1,1)$ copies of the regular representation of S_n, while $\mathbf{M}_{L^*}$ is a sum of $d_{L^*}(1,1)$ copies of the regular representation of S_{n^*}; but these multiplicities are equal. ∎

Fix a partition γ and one of its concave corners with coordinates $(i-1, j-1)$. Set $a = \gamma'_j - i$ (but $a = \infty$ if $j = 0$), and $b = \gamma_i - j$ ($b = \infty$ if $i = 0$). For any finite diagram $L \subset B_{a,b}$, define

$$\tau(L) \;=\; \gamma \cup \{\, (i+p, j+q) \;:\; (p,q) \in L \,\} \;. \tag{6.3}$$

This is the union of the diagram of γ with the translation of L by the vector (i, j).

Conjecture 6.3. **(Border Conjecture)** A diagram L has the MLRR property, with

$$C_L(x;q,t) \;=\; \sum_{\mu \vdash n} \alpha_\mu(q,t)\, \tilde{H}_\mu(x;q,t) \;,$$

iff $\tau(L)$ has the MLRR property, with

$$C_{\tau(L)}(x;q,t) \;=\; \sum_{\mu \vdash n} \alpha_\mu(q,t)\, \tilde{H}_{\tau(\mu)}(x;q,t) \,. \tag{6.4}$$

This would follow from the rotation conjecture by performing a sequence of $2d$ rotation operations (5.10) (where d is the number of corners of γ) to L in larger and larger boxes, to add on the border region γ.

The following theorem is consistent with this conjecture.

THEOREM 6.2. *Setting $n' = |\tau(L)| = |\gamma| + n$, the Hilbert series of the alternants of $\mathbf{M}_L$ and $\mathbf{M}_{\tau(L)}$ are related by*

$$C_{\tau(L)}(x;q,t)\Big|_{s_{1^{n'}}} = q^{n(\gamma')}\, t^{n(\gamma)}\, (t^i\, q^j)^n \cdot C_L(x;q,t)\Big|_{s_{1^n}} \tag{6.5}$$

Proof. Let $D_{hk}^{(n)}$ denote the operator D_{hk} acting on polynomials in $x_1, \ldots, x_n; y_1, \ldots, y_n$. All alternants in $\mathbf{M}_L$ are obtained by applying polynomials in the D_{hk}'s to Δ_L. By Prop. I.1 of [3], the alternants that arise this way are linear combinations of various $\Delta_R(x;y)$, where R runs over diagrams that are compressions of L. Similar statements are true for $\tau(L)$, and the specific linear combinations that occur have the same coefficients:

$$D_{hk}^{(n)} \Delta_R(x_1, \ldots, x_n; y_1, \ldots, y_n) \;=\; \sum_S c_S \Delta_S(x_1, \ldots, x_n; y_1, \ldots, y_n)$$

$$D_{hk}^{(n')}\Delta_{\tau(R)}(x_1,\ldots,x_{n'};y_1,\ldots,y_{n'}) = \sum_S c_S \Delta_{\tau(S)}(x_1,\ldots,x_{n'};y_1,\ldots,y_{n'})$$

where R is a compression of L and S runs over diagrams that are compressions of R. The bidegrees of Δ_S and $\Delta_{\tau(S)}$ differ by a constant independent of S, giving the additional factor on the right in (6.5).

Note that there is a sign ambiguity in the definition (1.2) due to the ordering of the cells, which in [3] was resolved by listing cells in lexicographic order; for the above pair of equations, we resolve it differently. In computing Δ_S, list the cells of S lexicographically. In computing $\Delta_{\tau(S)}$, list the cells of γ lexicographically, followed by the cells of S lexicographically. ▮

The evidence in support of these three conjectures includes the following:

(a) All of these conjectures are true for one dimensional diagrams (2.1). Prop. 4.2 proves the Rotation Conjecture, and Prop. 4.3 proves the Border Conjecture. Thm. 2.1 proves the Compression Conjecture, because the only compression of (2.1) is (1^n). Similarly,

$$\mathcal{F}\,\mathrm{ch}\,\mathcal{L}_\partial(\Delta_{L^p}) = \Xi_\lambda(q)\,\tilde{H}_n(x;q,t) \quad \text{where} \quad L^p = \{(0,p_1),\ldots,(0,p_n)\}\ .$$

(b) The $C=\tilde{H}$ conjecture is consistent with all three conjectures. The only compression of a partition μ is μ itself, so (1.4) agrees with the Compression Conjecture. For the Border Conjecture, given (1.4) and a partition ν obtained from μ by an operation of the form (6.3), both (1.4) and (6.4) predict $C_\nu(x;q,t)=\tilde{H}_\nu(x;q,t)$.

For the Rotation Conjecture, take any partition μ, large enough a,b, and set $\nu=\mu^*$. Given (1.4), the Rotation Conjecture says that since $C_\mu(x;q,t)=\tilde{H}_\mu(x;q,t)$, also $C_\nu(x;q,t)=\tilde{H}_\nu(x;q,t)$. By taking $a=\mu'_1$, $b=\mu_1$, we find ν has one less corner than μ. Indeed, since $C_\emptyset(x,q,t)=\tilde{H}_\emptyset(x;q,t)=1$, the validity of Conj. 6.2 would imply that $C_\mu(x;q,t)=\tilde{H}_\mu(x;q,t)$ for all μ by performing a sequence of rotations.

(c) For diagrams obtained by adding or removing a cell from the diagram of a partition, the conjectured formula for $C_{\mu/[i,j]}(x;q,t)$ in [3, Conj. I.3] and the present paper (5.6) are clearly consistent with both the Compression and Border Conjectures, and the relation of these two formulas is exactly given by the Rotation Conjecture.

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