Plethystic Formulas
for
Macdonald q, t-Kostka Coefficients

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ABSTRACT.

This work is concerned with the Macdonald q, t-analogue of the Kostka matrix. This matrix relates the two parameter Macdonald basis $\{P_\mu(x;q,t)\}$ to the modified Schur basis $\{S_\lambda[X(1-t)]\}$. The entries in this matrix, which have come to be denoted by $K_{\lambda,\mu}(q,t)$, have been conjectured by Macdonald to be polynomials in q, t with positive integral coefficients. Our main main result here is an algorithm for the construction of explicit formulas for the $K_{\lambda,\mu}(q,t)$. It is shown that this algorithm yields expressions which are polynomials with integer coefficients. Recent work of J. Remmel shows that the resulting formulas do also yield positivity of the coefficients for a wide variety of entries in the Macdonald q, t-Kostka matrix. We also obtain in this manner new explicit expressions for the Macdonald polynomials.

Introduction

Given a partition μ we shall represent it as customary by a Ferrers diagram. We shall use the French convention here and, given that the parts of μ are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0$, we let the corresponding Ferrers diagram have μ_i lattice squares in the i^{th} row (counting from the bottom up). We shall also adopt the Macdonald convention of calling the *arm, leg, coarm* and *coleg* of a lattice square s the parameters $a_\mu(s), l_\mu(s), a'_\mu(s)$ and $l'_\mu(s)$ giving the number of cells of μ that are respectively *strictly* EAST, NORTH, WEST and SOUTH of s in μ. We recall that Macdonald in [13] defines the symmetric function basis $\{P_\mu(x;q,t)\}_\mu$ as the unique family of polynomials satisfying the following conditions:

$$a) \quad P_\lambda = S_\lambda + \sum_{\mu<\lambda} S_\mu \, \xi_{\mu\lambda}(q,t)$$

$$b) \quad \langle P_\lambda \, , \, P_\mu \rangle_{q,t} \;=\; 0 \quad for \quad \lambda \neq \mu \;,$$

where $\{S_\lambda\}_\lambda$ is the Schur basis and $\langle \, , \, \rangle_{q,t}$ denotes the scalar product of symmetric polynomials defined by setting for the power basis $\{p_\rho\}$

$$\langle p_{\rho^{(1)}} \, , \, p_{\rho^{(2)}} \rangle_{q,t} \;=\; \begin{cases} z_\rho \prod_i \frac{1-q^{\rho_i}}{1-t^{\rho_i}} & \text{if } \rho^{(1)} = \rho^{(2)} = \rho \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

with z_ρ the integer that makes $n!/z_\rho$ the number of permutations with cycle structure ρ. Macdonald shows that the basis $\{Q_\mu(x;q,t)\}_\mu$, dual to $\{P_\mu(x;q,t)\}_\mu$ with respect to this scalar product, is given by the formula

$$Q_\lambda(x;q,t) \;=\; d_\lambda(q,t) \, P_\lambda(x;q,t) \;,$$

where

$$d_\lambda(q,t) \;=\; \frac{h_\lambda(q,t)}{h'_\lambda(q,t)}$$

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and

$$h_\lambda(q,t) = \prod_{s\in\lambda}(1-q^{a_\lambda(s)}t^{l_\lambda(s)+1}) \quad , \quad h'_\lambda(q,t) = \prod_{s\in\lambda}(1-q^{a_\lambda(s)+1}t^{l_\lambda(s)}) \ . \qquad \text{I.1}$$

Macdonald sets

$$J_\mu(x;q,t) = h_\mu(q,t)\,P_\mu(x;q,t) = h'_\mu(q,t)\,Q_\mu(x;q,t) \ , \qquad \text{I.2}$$

and then defines his q,t-analogues of the Kostka coefficients by means of an expansion that in λ-ring notation may be written as

$$J_\mu(x;q,t) = \sum_\lambda S_\lambda[X(1-t)]\,K_{\lambda\mu}(q,t) \ . \qquad \text{I.3}$$

This paper is concerned with the modified basis $\{\tilde H_\mu(x;q,t)\}_\mu$ defined by letting

$$\tilde H_\mu(x;q,t) = \sum_\lambda S_\lambda(x)\,\tilde K_{\lambda\mu}(q,t) \ , \qquad \text{I.4}$$

where we have set

$$\tilde K_{\lambda\mu}(q,t) = K_{\lambda\mu}(q,1/t)t^{n(\mu)} \qquad \text{I.5}$$

with

$$n(\mu) = \sum_{s\in\mu} l_\mu(s) \ .$$

If s is a cell of μ we shall refer to the monomial $w(s) = q^{a'_\mu(s)}t^{l'_\mu(s)}$ as the *weight* of s. The sum of the weights of the cells of μ will be denoted by $B_\mu(q,t)$ and will be called the *biexponent generator of* μ. Note that we have

$$B_\mu(q,t) = \sum_{s\in\mu} q^{a'_\mu(s)}t^{l'_\mu(s)} = \sum_{i\geq 1} t^{i-1}\,\frac{1-q^{\mu_i}}{1-q} \ . \qquad \text{I.6}$$

If $\gamma \vdash k$ and $n-k \geq max(\gamma)$, the partition of n obtained by prepending a part $n-k$ to γ will be denoted by $(n-k,\gamma)$. Our main result here may be stated as follows.

Theorem I.1

There is an algorithm which for any given $\gamma \vdash k$ *constructs a symmetric polynomial* $\mathbf{k}_\gamma(x;q,t)$ *such that*

$$\tilde K_{(n-k,\gamma),\mu}(q,t) = \mathbf{k}_\gamma[\,B_\mu(q,t)\,;q,t] \qquad (\ \forall \quad \mu\vdash n \geq k+max(\gamma)\ ) \ . \qquad \text{I.7}$$

This algorithm yields that the polynomials $\mathbf{k}_\gamma(x)$ *have a Schur function expansion of the form*

$$\mathbf{k}_\gamma(x;q,t) = \sum_{|\rho|\leq k} S_\rho\,\mathbf{k}_{\rho\gamma}(q,t) \qquad \text{I.8}$$

with coefficients $\mathbf{k}_{\rho\gamma}(q,t)$ *Laurent polynomials in* q,t *with integer coefficients. Moreover, each polynomial* $\mathbf{k}_\gamma$ *is uniquely determined by I.7 and I.8 .*

Surprisingly this result shows that the coefficients $\tilde{K}_{\lambda\mu}(q,t)$ depend on μ in a relatively simple manner, allowing in particular the encapsulation of a whole family of q,t-Kostka tables into a few symmetric polynomials. For instance, in view of the identity $\tilde{K}_{\lambda,\mu}(q,t) = \tilde{K}_{\lambda',\mu}(1/q,1/t)t^{n(\mu)}q^{n(\mu')}$ we see that all of the q,t-Kostka tables up to $n = 8$ can be constructed from the 12 symmetric polynomials

$$\mathbf{k}_{211}\ ,\ \mathbf{k}_{111}\ ,\ \mathbf{k}_{32}\ ,\ \mathbf{k}_{22}\ ,\ \mathbf{k}_{31}\ ,\ \mathbf{k}_{21}\ ,\ \mathbf{k}_{11}\ ,\ \mathbf{k}_{4}\ ,\ \mathbf{k}_{3}\ ,\ \mathbf{k}_{2}\ ,\ \mathbf{k}_{1}\ ,\ k_{\emptyset}$$

that are given at the end of the paper. The 40 polynomials needed to compute the q,t-Kostka tables up to $n = 12$ can be obtained (in MAPLE input format) by anonymous FTP from macaulay.ucsd.edu.

We show in section 1 that the denominators $t^{\alpha}q^{\beta}$ that appear in the formulas giving the Schur expansions of polynomials $\mathbf{k}_{\gamma}$ do in fact disappear when any $\mathbf{k}_{\gamma}$ is plethystically evaluated at a biexponent generator. Thus Theorem I.1 in particular implies that the coefficients $\tilde{K}_{\lambda\mu}(q,t)$ are polynomials with integer coefficients. We thus obtain part of the Macdonald conjecture concerning the $\tilde{K}_{\lambda\mu}(q,t)$. Our formulas have also been used with some success in [10] to prove positivity of the coefficients in $\tilde{K}_{\lambda\mu}(q,t)$ for several classes of partitions λ.

We recall that in [3] it is conjectured that $\tilde{H}_{\mu}(x;q,t)$ is (for a given $\mu \vdash n$) the bivariate Frobenius characteristic of a certain S_n-module $\mathbf{H}_{\mu}$ yielding a bigraded version of the left regular representation of S_n. In particular this would imply that the expression

$$F_{\mu}(q,t) \;=\; \sum_{\lambda} f_{\lambda}\, \tilde{K}_{\lambda\mu}(q,t)$$

should be the Hilbert series of $\mathbf{H}_{\mu}$. Here, f_{λ} denotes the number of standard tableaux of shape λ. Since Macdonald proved that

$$K_{\lambda\mu}(1,1) \;=\; f_{\lambda}\ , \tag{I.9}$$

we see that we must necessarily have

$$F_{\mu}(1,1) \;=\; \sum_{\lambda} f_{\lambda}^{2} \;=\; n! \tag{I.10}$$

According to our conjectures in [3] the polynomial

$$\partial_{p_1}\tilde{H}_{\mu}(x;q,t)$$

should give the Frobenius characteristic of the action of S_{n-1} on $\mathbf{H}_{\mu}$.

Using the fact that the operator ∂_{p_1} is the Hall scalar product adjoint to multiplication by the elementary symmetric function e_1, we can transform one of the Pieri rules given by Macdonald in [13] into the expansion of $\partial_{p_1}\tilde{H}_{\mu}(x;q,t)$ in terms of the polynomials $\tilde{H}_{\nu}(x;q,t)$ whose index ν immediately precedes μ in the Young partial order. More precisely we obtain

$$\partial_{p_1}\tilde{H}_{\mu}(x;q,t) \;=\; \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\,\tilde{H}_{\nu}(x;q,t) \tag{I.11}$$

with

$$c_{\mu\nu}(q,t) \;=\; \prod_{s\in\mathcal{R}_{\mu/\nu}} \frac{t^{l_\mu(s)}-q^{a_\mu(s)+1}}{t^{l_\nu(s)}-q^{a_\nu(s)+1}} \prod_{s\in\mathcal{C}_{\mu/\nu}} \frac{q^{a_\mu(s)}-t^{l_\mu(s)+1}}{q^{a_\nu(s)}-t^{l_\nu(s)+1}}\ , \qquad \text{I.12}$$

where $\mathcal{R}_{\mu/\nu}$ (resp. $\mathcal{C}_{\mu/\nu}$) denotes the set of lattice squares of ν that are in the same row (resp. same column) as the square we must remove from μ to obtain ν. This given, an application of $\partial_{p_1}^{n-1}$ to both sides of I.11 yields the recursion

$$F_\mu(q,t) \;=\; \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\, F_\nu(q,t)\ , \qquad \text{I.13}$$

which together with the initial condition $F_{(1)}(q,t)=1$ permits the computation of extensive tables of $F_\mu(q,t)$. It was a close study of this recursion, by means of the probabilistic interpretation given in [8], that led the first named author to discover the existence of plethystic formulas for the $\tilde K_{\lambda\mu}(q,t)$.

The crucial tool in this discovery was the following basic result

Theorem I.2

Let , be the linear operator on symmetric polynomials defined by setting

$$,\ S_\lambda[X] \;=\; e_1\, S_\lambda + \sum_{\rho\neq\lambda,\lambda/\rho\in V} S_\rho[X]\,\Big(\frac{-1}{tq}\Big)^{|\lambda-\rho|}\ \frac{h_{|\lambda-\rho|+1}[\,(1-t)(1-q)\,X-1]}{(1-t)(1-q)}\quad (\dagger) \qquad \text{I.14}$$

Then for any symmetric polynomial $P[X]$ and any partition μ we have

$$\sum_{\nu\to\mu} c_{\mu\nu}(q,t)\ P[B_\nu(q,t)] \;=\; (,\ P)[B_\mu(q,t)]$$

Remarkably, it can be shown [10] that all the polynomials $\mathbf{k}_\gamma$ can be constructed by successive applications of the operator , alternated by successive multiplications by the elementary symmetric function e_1. However, this approach leaves the possibility that the coefficients $c_{\rho,\gamma}$ in I.8 may have denominators with factors of the form $(1-t^r)(1-q^s)$. To remove this possibility we are forced to extend Theorem I.2 to all higher order *dual Pieri* coefficients. To be precise, for an integer $r\geq 1$ let ${}^{+}\partial^{(r)}$ be the operator which is adjoint to multiplication by $h_r[X/(1-t)]$ with respect to the scalar product

$$\langle p_{\rho^{(1)}}\ ,\ p_{\rho^{(2)}}\rangle_* \;=\; \begin{cases} sign(\rho) z_\rho\ \prod_i (1-q^{\rho_i})(1-t^{\rho_i}) & \text{if } \rho^{(1)}=\rho^{(2)}=\rho \text{ and} \\ 0 & \text{otherwise} \end{cases} \qquad \text{I.15}$$

and set

$${}^{+}\partial^{(r)}\tilde H_\mu(x;q,t) \;=\; \sum_{\mu/\nu\in V_r} \tilde H_\nu(x;q,t)\ {}^{+}c^{(r)}_{\mu\nu}(q,t)\ . \qquad \text{I.16}$$

We have the following extension of Theorem I.2.

(†)Here and after the symbol $\lambda/\rho\in V$ is to represent that λ/ρ is a vertical strip. Likewise we will write $\lambda/\rho\in V_r$ to express that λ/ρ is a vertical r-strip

Theorem I.3

For each $r \geq 1$, we can construct a linear operator ${}^{+},_{r}$ on symmetric polynomials such that for any symmetric polynomial $P[X]$, we have

$$\sum_{\nu:\ \mu/\nu \in V_r} {}^{+}c^{(r)}_{\mu\nu}(q,t)\ P[B_\nu(q,t)] \;=\; ({}^{+},_{r} P)[B_\mu(q,t)]\ . \qquad (\forall \mu) \qquad \text{I.17}$$

Our construction yields ${}^{+},_{r}$ in the form

$$ {}^{+},_{r}\ S_\lambda[X] \;=\; \sum_{|\rho| \leq |\lambda|+r} S_\rho\ \phi_{\rho,\lambda}(q,t) \qquad \text{I.18}$$

with the $\phi_{\rho,\lambda}(q,t)$ Laurent polynomials in q,t with integer coefficients. Moreover, ${}^{+},_{r}$ is uniquely determined by I.17 and I.18.

The contents of this paper are divided into four sections. To make our presentation as self contained as possible, in section 1, after a brief review of λ-ring notation, we derive all of the identities involving the modified basis $\{\tilde{H}_\mu(x;q,t)\}_\mu$ that are needed in our proofs. These derivations necessarily have to take for granted identities whose proofs can be found in Chapter VI of the new edition of Macdonald's monograph. In section 2 we establish Theorem I.2 and derive from it some plethystic formulas for the $\tilde{K}_{\lambda\mu}(q,t)$. We also give there the embryonic form of the algorithm which eventually yielded Theorem I.1. In this section we follow closely the path which led to the discovery of formula I.14 even though a different path is now available by specializing our present proof of Theorem I.3 to the case $r = 1$. The reason for this is that the connections between the identities in this paper and those obtained in [8] via the q,t-hook walk lead to some interesting questions. Section 3 is dedicated to the proof of Theorem I.3 and its various ramifications. Finally, in section 4, we put everything together into the proof of Theorem I.1. We terminate this section with some comments relating the hook-walk paper [8] to the present one and some tables of the polynomials $\mathbf{k}_\gamma$.

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1. Some auxiliary identities.

In this section we need to derive a few identities for the symmetric functions $\tilde{H}_\mu(x;q,t)$. Our presentation will be greatly simplified if we make use of a formalism which has come to be referred to as "λ-ring" notation. We give here a brief informal review of the basic constructs. The reader is referred to [1], [2] and [6] for further examples.

Our main need is to be able to represent in a convenient and helpful way the operation of **substitution** of a polynomial Q with integer coefficients **into** a symmetric function P. To make sure that we remember what goes in and what stays out we denote here the result of this operation by the symbol $P[Q]$. This operation is restricted to formal series Q with integer coefficients. Clearly, we can represent such a Q in the form

$$Q \;=\; Q^+ - Q^- \;,$$

where

$$Q^{\pm} \;=\; \sum_{x^p \in M^{\pm}} x^p$$

and M^+ and M^- are two multisets of monomials. For instance, if

$$Q \;=\; 3x_1^2 y_2 + 2x_1x_2x_3 - 2x_2y_3^2 - 3y_3^2$$

then

$$\begin{aligned} M^+ &= \{x_1^2y_2, x_1^2y_2, x_1^2y_2, x_1x_2x_3, x_1x_2x_3\} \\ M^- &= \{x_2y_2^2, x_2y_3^2, y_3^2, y_3^2, y_3^2\} \;. \end{aligned}$$

When $Q^- = 0$ then $P[Q]$ has a very natural definition. Indeed, if

$$M^+ \;=\; \{m_1, m_2, \ldots, m_N\}$$

then $P[Q]$ is the polynomial obtained by substituting the monomials of M for the variables of P. More precisely, we first write P as a polynomial in N variables

$$P \;=\; P(y_1, y_2, \ldots, y_N)$$

and then let

$$P[Q] \;=\; P(y_1, y_2, \ldots, y_N)\Big|_{y_i = m_i} \;. \qquad 1.1$$

We can easily see from this that if P and Q are both symmetric in the variables $x_1, x_2, \ldots, x_n$, then $P[Q]$ will also be symmetric; moreover, $P[Q]$ will be homogeneous of degree $p \cdot q$ if P and Q themselves are homogeneous of degrees p and q respectively.

Note that when P is the power symmetric function p_s, formula 1.1 reduces to

$$p_s[Q] \;=\; \sum_{i=1}^{N} (m_i)^s \;. \qquad 1.2$$

From this, we easily deduce the two basic properties

$$\begin{aligned} (i) \quad p_s[Q_1 + Q_2] &= p_s[Q_1] + p_s[Q_2] \\ (ii) \quad p_s[Q_1 Q_2] &= p_s[Q_1]\, p_s[Q_2] \;. \end{aligned} \qquad 1.3$$

The idea is to use these properties as the starting point for extending the definition of $P[Q]$ to the general case. In other words we shall simply let our definition be a consequence of the requirement that 1.3 (i) and (ii) be valid in full generality. In particular, (i) forces

$$p_s[0] = 0 \quad \text{and} \quad p_s[-Q^-] = -p_s[Q^-] \;.$$

Thus we must take

$$p_s[Q^+ - Q^-] \;=\; p_s[Q^+] - p_s[Q^-] \qquad 1.4$$

This given, the evaluation of $P[Q]$ can be routinely carried out by expanding P in terms of the power symmetric function basis and then using 1.4.

It is best to illustrate all this with the examples we need in the sequel. For instance, let us see what we get if $P = h_n$ and $Q = (1-t)X$ with $X = x_1 + x_2 + \cdots + x_m$. Now the expansion of the homogeneous symmetric function h_n in terms of the power basis may be written as

$$h_n \;=\; \sum_{\rho \vdash n} p_\rho / z_\rho \qquad 1.5$$

with

$$z_\rho \;=\; 1^{m_1} 2^{m_2} 3^{m_3} \ldots m_1! m_2! m_3! \ldots \qquad (\text{if } \rho = 1^{m_1} 2^{m_2} 3^{m_3} \ldots) \ .$$

Thus, 1.4 gives

$$p_s\big[(1-t)X\big] \;=\; (1-t^s) \sum_{i=1}^{m} x_i^s \ ,$$

and we must have

$$h_n\big[(1-t)X\big] \;=\; \sum_{\rho \vdash n} \frac{p_\rho(x_1, \ldots, x_m)}{z_\rho} \prod_i (1 - t^{\rho_i}) \ . \qquad 1.6$$

Interpreting $X/(1-q)$ as the formal power series

$$\sum_{i=0}^{\infty} q^i X$$

we are, in the same manner, led to the expansion

$$h_n\big[\tfrac{X}{1-q}\big] \;=\; \sum_{\rho \vdash n} \frac{p_\rho(x_1, \ldots, x_n)}{z_\rho} \prod_i \frac{1}{1 - q^{\rho_i}} \ . \qquad 1.7$$

In our operation we need not restrict P itself to be a polynomial. For instance, it is customary to let Ω denote the basic symmetric function kernel

$$\Omega(x_1, \ldots, x_m) \;=\; \prod_{i=1}^{m} \frac{1}{1 - x_i} \ . \qquad 1.8$$

Note that since we may write

$$\Omega \;=\; \sum_{n \geq 0} h_n \;=\; \sum_\rho \frac{p_\rho}{z_\rho} \;=\; \exp\Big(\sum_{s \geq 1} \frac{1}{s}\, p_s \Big) \ ,$$

a simple calculation based on 1.4 leads to the direct formula

$$\Omega[Q^+ - Q^-] \;=\; \prod_{m \in M^+} \frac{1}{1-m} \prod_{m \in M^-} (1-m) \ . \qquad 1.9$$

Thus we can easily see that we must have for $X = x_1 + x_2 + \cdots + x_M$, $Y = y_1 + y_2 + \cdots + y_N$

$$\Omega\big[(1-t)X\,Y\big] \;=\; \prod_{i=1}^{M}\prod_{j=1}^{N} \frac{1-t\,x_i y_j}{1-x_i y_j} \;. \qquad 1.10$$

The latter is usually referred to as the Hall-Littlewood kernel.

Allowing both P and Q to be formal power series, the Macdonald kernel [13]

$$\Omega_{qt}(x,y) \;=\; \prod_{ij}\prod_{k\geq 0} \frac{1-t\,x_i y_j\, q^k}{1-x_i y_j\, q^k} \qquad 1.11$$

may simply be written in the form

$$\Omega_{qt} \;=\; \Omega\big[\tfrac{1-t}{1-q}X\,Y\big] \;. \qquad 1.12$$

We should also note that 1.3 (ii) allows us to extend the identity

$$\Omega(x,y) \;=\; \prod \frac{1}{1-x_i y_j} \;=\; \sum_{\lambda} S_\lambda(x) S_\lambda(y)$$

to

$$\Omega[PQ] \;=\; \sum_{\lambda} S_\lambda[P]\, S_\lambda[Q] \;,$$

which we shall here and after refer to as the *general Cauchy Identity*.

In particular, we also have

$$h_n[PQ] \;=\; \sum_{\lambda\vdash n} S_\lambda[P]\, S_\lambda[Q] \;.$$

This given, our first auxiliary result may be stated as follows.

Theorem 1.1

The polynomials $\tilde{H}_\mu(x;q,t)$ form an orthogonal basis with respect to the scalar product in I.15. More precisely, the following "Cauchy type" identity holds for all $n \geq 1$:

$$e_n\big[\tfrac{XY}{(1-t)(1-q)}\big] \;=\; \sum_{\mu\vdash n} \frac{\tilde{H}_\mu[X;q,t]\,\tilde{H}_\mu[Y;q,t]}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)} \;. \qquad 1.13$$

Equivalently we must have

$$\langle \tilde{H}_\mu \,,\, \tilde{H}_\nu\rangle_* \;=\; \begin{cases} \tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t) & \text{if } \mu=\nu \text{ and} \\ 0 & \text{otherwise ,} \end{cases} \qquad 1.14$$

where

$$\tilde h_\mu(q,t) \;=\; \prod_{s\in\mu}(q^{a_\mu(s)}-t^{l_\mu(s)+1}) \qquad \text{and} \qquad \tilde h'_\mu(q,t) \;=\; \prod_{s\in\mu}(t^{l_\mu(s)}-q^{a_\mu(s)+1}) \qquad 1.15$$

Proof

Note that since the power sum expansion of the kernel on the left of 1.13 may be written in the form

$$e_n[\tfrac{XY}{(1-t)(1-q)}] \;=\; \sum_{\rho\vdash n} sign(\rho)\frac{p_\rho[X]\,p_\rho[X]}{z_\rho p_\rho[(1-t)(1-q)]} \;, \qquad 1.16$$

we see from the definition in I.15 that 1.13 and 1.14 are equivalent. So we need only establish 1.13. Our starting point is the Macdonald "Cauchy formula" (*)

$$\Omega[XY\tfrac{1-t}{1-q}] \;=\; \sum_\mu P_\mu[X;q,t]\,Q_\mu[Y;q,t] \;. \qquad 1.17$$

Using I.2 we may rewrite this as

$$\Omega[XY\tfrac{1-t}{1-q}] \;=\; \sum_\mu \frac{J_\mu[X;q,t]\,J_\mu[Y;q,t]}{h_\mu(q,t)\,h'_\mu(q,t)} \;.$$

Making the plethystic substitutions $X\to\frac{X}{1-t}$ and $Y\to\frac{Y}{1-t}$, we obtain

$$\Omega[\tfrac{XY}{(1-t)(1-q)}] \;=\; \sum_\mu \frac{H_\mu[X;q,t]\,H_\mu[Y;q,t]}{h_\mu(q,t)\,h'_\mu(q,t)} \;, \qquad 1.18$$

where for convenience we have set

$$H_\mu[X;q,t] = J_\mu[\tfrac{X}{1-t};q,t] = \sum_\lambda S_\lambda[X]\,K_{\lambda\mu}(q,t) \;. \qquad 1.19$$

Now I.4, I.5 and 1.19 give that

$$\tilde H_\mu[X;q,t] \;=\; H_\mu[X;q,1/t]\,t^{n(\mu)} \;. \qquad 1.20$$

Replacing t by $1/t$ in 1.18 and using 1.20, we get

$$\Omega[\tfrac{XY}{(1-1/t)(1-q)}] \;=\; \sum_\mu \frac{\tilde H_\mu[X;q,t]\,\tilde H_\mu[Y;q,t]}{t^{n(\mu)}h_\mu(q,1/t)\,t^{n(\mu)}h'_\mu(q,1/t)} \;.$$

Now, we can easily verify that

$$(-1)^{|\mu|}\,t^{n(\mu)+|\mu|}\,h_\mu(q,1/t) = \tilde h_\mu(q,t) \qquad \text{and} \qquad t^{n(\mu)}\,h'_\mu(q,1/t) = \tilde h'_\mu(q,t) \;. \qquad 1.21$$

(*) [14] (4.13) p. 324

Thus, equating terms of degree $2n$ in the X,Y-variables we finally obtain that

$$h_n[\tfrac{XY}{(1-1/t)(1-q)}] \;=\; (-t)^n \sum_{\mu\vdash n} \frac{\tilde{H}_\mu[X;q,t]\,\tilde{H}_\mu[Y;q,t]}{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)} \;.$$

However, this proves 1.13, since

$$h_n[\tfrac{XY}{(1-1/t)(1-q)}] \;=\; (-t)^n\, e_n[\tfrac{XY}{(1-t)(1-q)}] \;.$$

The polynomials $\tilde{H}_\mu(x;q,t)$ have a useful special evaluation, namely

Proposition 1.1

$$\tilde{H}_\mu[1-u;q,t] \;=\; \prod_{s\in\mu}(1-q^{a'}t^{l'}\,u) \;. \qquad 1.22$$

Proof

Rewriting in λ-ring notation formula 6.17 page 338 of [14], we get

$$P_\mu[\tfrac{1-u}{1-t};q,t] \;=\; \frac{\prod_{s\in\mu}(t^{l'}-q^{a'}\,u)}{\prod_{s\in\mu}(1-q^{a}t^{l+1})} \;. \qquad 1.23$$

Multiplying both sides by $h_\mu(q,t)$ and using I.2 and I.3 gives

$$H_\mu[1-u;q,t] \;=\; J_\mu[\tfrac{1-u}{1-t};q,t] \;=\; \prod_{s\in\mu}(t^{l'}-q^{a'}\,u) \;.$$

Replacing t by $1/t$ and multiplying by $t^{n(\mu)}$, from 1.20 we get

$$\tilde{H}_\mu[1-u;q,t] \;=\; t^{n(\mu)}\prod_{s\in\mu}(t^{-l'}-q^{a'}\,u) \;=\; \prod_{s\in\mu}(1-t^{l'}q^{a'}\,u) \;,$$

as desired.

An immediate consequence of 1.22 is the special case $\gamma = 1^k$ of Theorem I.1

Theorem 1.2

For any $\mu\vdash n$, we have

$$\tilde{K}_{(n-k,1^k),\mu}(q,t) \;=\; e_k[B_\mu-1] \;; \qquad 1.24$$

equivalently,

$$\mathbf{k}_{1^k}(x) \;=\; \sum_{s=0}^{k}(-1)^{k-s}\,e_s(x) \;. \qquad 1.25$$

Proof

Using I.4, 1.22 may be rewritten as

$$\sum_{\lambda} S_\lambda[1-u]\,\tilde{K}_{\lambda\mu}(q,t) \;=\; \prod_{s\in\mu}(1-t^{l'}q^{a'}\,u)\ . \qquad 1.26$$

However, it is well known and easy to show that

$$S_\lambda[1-u] \;=\; \begin{cases}(-u)^r\,(1-u) & \text{if } \lambda=(n-r,1^r)\\ 0 & \text{otherwise .}\end{cases} \qquad 1.27$$

Thus 1.26 reduces to

$$\sum_{r=0}^{n-1}(-u)^r(1-u)\,\tilde{K}_{(n-r,1^r),\mu}(q,t) \;=\; \prod_{s\in\mu}(1-t^{l'}q^{a'}\,u)\ .$$

Dividing both sides by $1-u$ and changing the sign of u gives

$$\sum_{r=0}^{n-1}u^r\,\tilde{K}_{(n-r,1^r),\mu}(q,t) \;=\; \prod_{(l',a')\neq(o,o)}(1+t^{l'}q^{a'}\,u)\ ,$$

and 1.24 follows by equating the coefficients of u^k.

Note that the λ-ring version of the addition formula for e_k gives

$$e_k[X-Y] \;=\; \sum_{s=o}^{k} e_s[X]\,(-1)^{k-s}\,h_{k-s}[Y]\ . \qquad 1.28$$

Thus 1.25 follows from 1.24 by setting $X=B_\mu(q,t)$ and $Y=1$.

A crucial role in the next section will be played by the linear operator Δ_1, defined by setting for every symmetric polynomial $P(x)$

$$\Delta_1\,P(x) \;=\; P[X]-P[X+\tfrac{(1-q)(1-t)}{z}]\;\Omega[-zX]\,\big|_{z^o}\ .\ (\dagger) \qquad 1.29$$

This is due to the fact (already noted in [14]) that Δ_1 has the basis $\{\tilde{H}_\mu(x;q,t)\}_\mu$ as its complete system of eigenfunctions.

More precisely we have

Theorem 1.3

The polynomial $\tilde{H}_\mu(x;q,t)$ is uniquely determined by the two identities

$$\begin{aligned} &a)\quad \Delta_1\,\tilde{H}_\mu(x;q,t) \;=\; (1-t)(1-q)B_\mu(q,t)\,\tilde{H}_\mu(x;q,t)\\ &b)\quad \tilde{H}_\mu(x;q,t)\,\big|_{S_n(x)} \;=\; 1\ . \end{aligned} \qquad 1.30$$

($\dagger$) Here and after the symbol "|" is to denote the operation of taking a coefficient. In particular $|_{z^o}$ denotes the operation of taking a constant term

Proof

We follow very closely the proof of 1.30 a) given in [6]. We include this here for sake of completion. The starting point is a plethystic rewriting of the corresponding Macdonald operator. We recall that the Macdonald operator D_n^1 acts on polynomials in $x_1, \ldots, x_n$ according to the formula (*)

$$D_n^1 \, P(x) \;=\; \sum_{i=1}^{n} \Big(\prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} \Big) \, P[X + (q-1)x_i] \;, \qquad 1.31$$

where $X = x_1 + x_2 + \cdots + x_n$. Our first step is to show that we can also write

$$D_n^1 \, P(x) \;=\; \frac{1}{1-t} \; P(x) \;+\; \frac{t^n}{t-1} \, P[X + \tfrac{q-1}{tz}] \; \Omega[zX(t-1)] \, |_{z^o} \;. \qquad 1.32$$

Using the Cauchy kernel $\Omega[XY]$ as the the generating function of the Schur basis, formula 1.32 will necessarily follow if we verify that for an arbitrary alphabet Y we have

$$D_n^1 \, \Omega[XY] \;=\; \frac{1}{1-t}\Omega[XY] \;-\; \frac{t^n}{1-t}\Omega[(X + \tfrac{q-1}{tz})Y] \; \Omega[z \, X(t-1)] \, |_{z^o} \;.$$

Or, equivalently that

$$\frac{D_n^1 \, \Omega[XY]}{\Omega[XY]} \;=\; \frac{1}{1-t} \;-\; \frac{t^n}{1-t}\Omega[\tfrac{q-1}{tz} \, Y] \; \Omega[z \, X(t-1)] \, |_{z^o} \;. \qquad 1.33$$

Applying D_n^1 to $\Omega[XY]$ according to the definition 1.31, we get

$$\frac{D_n^1 \, \Omega[XY]}{\Omega[XY]} \;=\; \sum_{i=1}^{n} \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} \, \Omega[(q-1)x_i Y] \;. \qquad 1.34$$

Now, expanding the kernel $\Omega[(q-1)x_iY]$ into powers of x_i, we obtain

$$\Omega[(q-1)x_iY] \;=\; 1 \;+\; \sum_{p \geq 1} h_p[(q-1)Y] \, x_i^p \;.$$

Substituting this in 1.34 and setting for convenience

$$A_i(x,t) \;=\; \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} \;,$$

we get that

$$\frac{D_n^1 \, \Omega[XY]}{\Omega[XY]} \;=\; \sum_{i=1}^{n} A_i(x,t) \;+\; \sum_{p \geq 1} h_p[(q-1)Y] \sum_{i=1}^{n} A_i(x,t) \, x_i^p \;. \qquad 1.35$$

Now from the partial fraction expansion

$$\Omega[zX(t-1)] \;=\; \prod_{i=1}^{n} \frac{1 - zx_i}{1 - ztx_i} \;=\; \frac{1}{t^n} \;+\; \frac{t-1}{t^n} \sum_{i=1}^{n} A_i(x,t) \, \frac{1}{1 - tzx_i} \;, \qquad 1.36$$

(*) (3.4) p. 315 of [14]

we easily derive that

$$\sum_{i=1}^{n} A_i(x,t) \;=\; \frac{t^n-1}{t-1}$$

and for $p \geq 1$:

$$\sum_{i=1}^{n} A_i(x,t)\, x_i^p \;=\; \frac{1}{t^p}\,\frac{t^n}{t-1}\;\Omega[zX(t-1)]\,\big|_{z^p}$$

where the symbol "$|_{z^p}$" denotes the operation of taking the coefficient of z^p in the preceding expression. Substituting these two identities in 1.35 we finally obtain that

$$\begin{aligned}
\frac{D_n^1\,\Omega[XY]}{\Omega[XY]} &= \frac{t^n-1}{t-1} + \sum_{p\geq 1}\frac{h_p[(q-1)Y]}{t^p}\,\frac{t^n}{t-1}\,\Omega[zX(t-1)]\,\big|_{z^p}\\
&= \frac{t^n-1}{t-1} + \frac{t^n}{t-1}\sum_{p\geq 1}\frac{h_p[(q-1)Y]}{(tz)^p}\,\Omega[zX(t-1)]\,\big|_{z^o}\\
&= \frac{t^n-1}{t-1} + \frac{t^n}{t-1}\,\big(\Omega[\tfrac{q-1}{tz}Y]-1\big)\,\Omega[zX(t-1)]\,\big|_{z^o}
\end{aligned}$$

and this is 1.33.

Recall that in [14] ((5.15) p 3.24) Macdonald proves that

$$D_n^1 P_\mu[X;q,t] \;=\; \Big(\sum_{i=1}^{n} t^{n-i}q^{\mu_i}\Big)\, P_\mu[X;q,t]$$

Using 1.32 with $P = P_\mu$, we may rewrite this as

$$\tfrac{1}{1-t}\;P_\mu[X;q,t] \;+\; \frac{t^n}{t-1}\,P_\mu[X+\tfrac{q-1}{tz};q,t]\;\;\Omega[zX(t-1)]\,\big|_{z^o} \;=\; \Big(\sum_{i=1}^{n} t^{n-i}q^{\mu_i}\Big)\,P_\mu[X;q,t]\;.$$

Multiplying both sides by $h_\mu(q,t)$ and making the replacement $X \rightarrow \frac{X}{1-t}$, formulas I.2 and 1.19 give

$$\tfrac{1}{1-t}\;H_\mu[X;q,t] \;+\; \frac{t^n}{t-1}\,H_\mu[X+\tfrac{(1/t-1)(q-1)}{z};q,t]\;\;\Omega[-zX]\,\big|_{z^o} \;=\; \Big(\sum_{i=1}^{n} t^{n-i}q^{\mu_i}\Big)\,H_\mu[X;q,t]\;.$$

Changing t into $1/t$ and multiplying both sides by $t^{n-1+n(\mu)}(1-t)$ brings us to

$$-t^n\,\tilde H_\mu[X;q,t] + \tilde H_\mu[X+\tfrac{(t-1)(q-1)}{z};q,t]\Omega[-zX]\,\big|_{z^o} = \Big((1-t)\sum_{i=1}^{n} t^{i-1}q^{\mu_i}\Big)\tilde H_\mu[X;q,t]\;. \qquad 1.37$$

Now, assuming $\mu_i = 0$ for $i > n$, the definition in I.6 gives that

$$(1-t)(1-q)\,B_\mu(q,t) \;=\; (1-t)\sum_{i=1}^{n} t^{i-1} \;-\; (1-t)\sum_{i=1}^{n} t^{i-1}q^{\mu_i}$$

or equivalently

$$(1-t)\sum_{i=1}^{n} t^{i-1}q^{\mu_i} \;=\; 1-t^n \;-\; (1-t)(1-q)\,B_\mu(q,t)\ . \tag{1.38}$$

Substituting 1.38 into 1.37 gives 1.30 a) with Δ_1 given by 1.29. We have thus established that the $\tilde{H}_\mu(x;q,t)$ form a complete system of eigenfunctions for the operator Δ_1. Since the polynomials $(1-t)(1-q)\,B_\mu(q,t)$ are distinct for q and t generic, we can see that the $\tilde{H}_\mu(x;q,t)$ are uniquely determined by any one of the coefficients in their Schur function expansion. Now to obtain 1.30 b) we need only observe that I.4 gives $\tilde{H}_\mu(x;q,t)|_{S_n} = \tilde{K}_{(n),\mu}(q,t)$ and 1.24 (for $k=0$) gives $\tilde{K}_{(n),\mu}(q,t)=1$. This completes our proof.

Our next task is to show that the coefficients $c_{\mu\nu}(q,t)$ occurring in I.11 are given by I.12. To do this we need two auxiliary results.

Proposition 1.2

The adjoint of the operator ∂_{p_1} with respect to the scalar product $\langle\ ,\ \rangle_*$ is multiplication by $e_1[\frac{X}{(1-t)(1-q)}]$.

Proof

Note first that the scalar product introduced in I.15 can be rewritten in terms of the customary Hall scalar product $\langle\ ,\ \rangle$. More precisely, for any two homogeneous symmetric polynomials P,Q of degree n, we have

$$\langle\, P[X]\,,\, Q[X]\,\rangle_* \;=\; (-1)^n\,\langle\, P[X]\,,\, Q[(t-1)(1-q)X]\,\rangle \tag{1.39}$$

Now it is well known and easy to show that ∂_{p_1} is the Hall scalar product adjoint of multiplication by e_1. Thus if P and Q are symmetric and homogeneous of degrees n and $n-1$ respectively, we have (using 1.39 twice)

$$\begin{aligned}\langle\, \partial_{p_1} P[X]\,,\, Q[X]\,\rangle_* &= (-1)^{n-1}\,\langle\, \partial_{p_1} P[X]\,,\, Q[(t-1)(1-q)X]\,\rangle\\ &= (-1)^{n-1}\,\langle\, P[X]\,,\, e_1\, Q[(t-1)(1-q)X]\,\rangle\\ &= \langle\, P[X]\,,\, e_1[\tfrac{X}{(1-t)(1-q)}]\, Q[X]\,\rangle_*\ ,\end{aligned} \tag{1.40}$$

and this is what we wanted to show.

Proposition 1.3

The coefficients $c_{\mu\nu}(q,t)$ and $d_{\mu\nu}(q,t)$ defined by the equations

$$\begin{aligned} a)\quad \partial_{p_1}\tilde{H}_\mu(x;q,t) &= \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\,\tilde{H}_\nu(x;q,t)\\ b)\quad e_1(x)\,\tilde{H}_\nu(x;q,t) &= \sum_{\mu\leftarrow\nu} \tilde{H}_\mu(x;q,t)\,d_{\mu\nu}(q,t)\end{aligned} \tag{1.41}$$

are related by the identity

$$c_{\mu\nu} \;=\; \frac{d_{\mu\nu}}{(1-t)(1-q)}\,\frac{\tilde{h}_\mu\tilde{h}'_\mu}{\tilde{h}_\nu\tilde{h}'_\nu}\ . \tag{1.42}$$

Proof

Taking the $*$-scalar product of both sides of 1.41 a) with $\tilde{H}_\nu(x;q,t)$ and using 1.14, we get

$$\langle\, \partial_{p_1} \tilde{H}_\mu \,,\, \tilde{H}_\nu \,\rangle_* \;=\; c_{\mu\nu}\;\; \tilde{h}_\nu \tilde{h}'_\nu$$

Similarly, taking the $*$-scalar product of both sides of 1.41 b) with $\tilde{H}_\mu(x;q,t)$ we get

$$\langle\, \tilde{H}_\mu \,,\, e_1 \tilde{H}_\nu \,\rangle_* \;=\; d_{\mu\nu}\;\; \tilde{h}_\mu \tilde{h}'_\mu \;,$$

and 1.42 follows from 1.40 since it implies that

$$\langle\, \partial_{p_1}\, \tilde{H}_\mu \,,\, \tilde{H}_\nu \,\rangle_* \;=\; \langle\, \tilde{H}_\mu \,,\, e_1[\tfrac{X}{(1-t)(1-q)}]\, \tilde{H}_\nu \,\rangle_* \;=\; \frac{1}{(1-t)(1-q)}\,\langle\, \tilde{H}_\mu \,,\, e_1\, \tilde{H}_\nu \,\rangle_* \;.$$

Theorem 1.4

$$d_{\mu\nu}(q,t) \;=\; \prod_{s\in R_{\mu/\nu}} \frac{q^{a_\nu(s)} - t^{l_\nu(s)+1}}{q^{a_\mu(s)} - t^{l_\mu(s)+1}} \prod_{s\in C_{\mu/\nu}} \frac{t^{l_\nu(s)} - q^{a_\nu(s)+1}}{t^{l_\mu(s)} - q^{a_\mu(s)+1}} \;, \qquad 1.43$$

$$c_{\mu\nu}(q,t) \;=\; \prod_{s\in R_{\mu/\nu}} \frac{t^{l_\mu(s)} - q^{a_\mu(s)+1}}{t^{l_\nu(s)} - q^{a_\nu(s)+1}} \prod_{s\in C_{\mu/\nu}} \frac{q^{a_\mu(s)} - t^{l_\mu(s)+1}}{q^{a_\nu(s)} - t^{l_\nu(s)+1}} \;. \qquad 1.44$$

Proof

It is sufficient to prove 1.43 since 1.44 then follows easily from 1.42.

Our point of departure is the Pieri rule 6.24 (iv) (given in page 340 of [14]), which in the particular case $r = 1$ may be written in the form

$$e_1\; P_\nu(x;q,t) \;=\; \sum_{\mu\leftarrow\nu} \psi'_{\mu/\nu}(q,t)\; P_\mu(x;q,t) \qquad 1.45$$

with

$$\psi'_{\mu/\nu}(q,t) \;=\; \prod_{s\in C_{\mu/\nu}} \frac{h_\mu(s)}{h'_\mu(s)} \frac{h'_\nu(s)}{h_\nu(s)} \qquad 1.46$$

where for a cell s of a partition λ we set

$$h_\lambda(s) \;=\; (1 - q^{a_\lambda(s)} t^{l_\lambda(s)+1}) \quad \text{and} \quad h'_\lambda(s) \;=\; (1 - t^{l_\lambda(s)} q^{a_\lambda(s)+1}) \;. \qquad 1.47$$

Now using I.2, we may rewrite 1.45 as

$$e_1\; J_\nu(x;q,t) \;=\; \sum_{\mu\leftarrow\nu} \psi'_{\mu/\nu}(q,t)\, \frac{h_\nu(q,t)}{h_\mu(q,t)}\; J_\mu(x;q,t) \;. \qquad 1.48$$

Now (taking account of the corner square of μ that is not in ν) we can easily derive from 1.46 that

$$(1-t)\;\; \psi'_{\mu/\nu}(q,t)\, \frac{h_\nu(q,t)}{h_\mu(q,t)} \;=\; \prod_{s\in R_{\mu/\nu}} \frac{h_\nu(s)}{h_\mu(s)} \prod_{s\in C_{\mu/\nu}} \frac{h'_\nu(s)}{h'_\mu(s)} \;. \qquad 1.49$$

Calling the right-hand side of this equation $\Phi_{\mu/\nu}(q,t)$ for a moment , we can write

$$e_1(x)\, J_\nu(x;q,t) \;=\; \frac{1}{1-t} \sum_{\mu\leftarrow\nu} \Phi_{\mu/\nu}(q,t)\, J_\mu(x;q,t) \;.$$

Making the plethystic replacement $X \to \frac{X}{1-t}$, cancelling from both sides the resulting common factor $1-t$ and using 1.19, we derive that

$$e_1(x)\, H_\nu(x;q,t) \;=\; \sum_{\mu\leftarrow\nu} \Phi_{\mu/\nu}(q,t)\, H_\mu(x;q,t) \;.$$

Making the replacement $t \to 1/t$ and using 1.20, we get

$$e_1(x)\, \tilde H_\nu(x;q,t)\, t^{-n(\nu)} \;=\; \sum_{\mu\leftarrow\nu} \Phi_{\mu/\nu}(q,1/t)\, \tilde H_\mu(x;q,t)\, t^{-n(\mu)} \;. \qquad 1.50$$

Now note that from 1.47 and 1.49 we get that

$$\Phi_{\mu/\nu}(q,t) \;=\; \prod_{s\in R_{\mu/\nu}} \frac{1-q^{a_\nu(s)}t^{l_\nu(s)+1}}{1-q^{a_\mu(s)}t^{l_\mu(s)+1}} \prod_{s\in C_{\mu/\nu}} \frac{1-t^{l_\nu(s)}q^{a_\nu(s)+1}}{1-t^{l_\mu(s)}q^{a_\mu(s)+1}} \;.$$

This given, straightforward manipulations yield that

$$\Phi_{\mu/\nu}(q,1/t) \;=\; t^{n(\mu)-n(\nu)} \prod_{s\in R_{\mu/\nu}} \frac{q^{a_\nu(s)}-t^{l_\nu(s)+1}}{q^{a_\mu(s)}-t^{l_\mu(s)+1}} \prod_{s\in C_{\mu/\nu}} \frac{t^{l_\nu(s)}-q^{a_\nu(s)+1}}{t^{l_\mu(s)}-q^{a_\mu(s)+1}} \;.$$

Substituting this in 1.50 gives 1.43 as desired.

The next two theorems express useful symmetries of the polynomials $\tilde H_\mu(x;,q,t)$.

Theorem 1.5

$$\tilde H_\mu(x;q,t) \;=\; t^{n(\mu)} q^{n(\mu')}\, \omega\, \tilde H_\mu(x;1/q,1/t) \qquad 1.51$$

Proof

Our point of departure is the first formula in (4.14) (iv) p. 324 of [14], namely

$$P_\mu(x;1/q,1/t) \;=\; P_\mu(x;q,t) \;. \qquad 1.52$$

Now, from the definitions I.1 we can easily derive that

$$h_\mu(1/q,1/t) \;=\; \frac{(-1)^n}{q^{n(\mu')}t^{n(\mu)+n}}\, h_\mu(q,t) \;. \qquad 1.53$$

Thus, multiplying both sides of 1.52 by $h_\mu(1/q,1/t)$ and using I.2, we deduce that

$$J_\mu(x;1/q,1/t) \;=\; \frac{(-1)^n}{q^{n(\mu')}t^{n(\mu)+n}}\, J_\mu(x;q,t) \;.$$

Making the plethystic substitution $X \to \frac{X}{1-1/t}$, 1.19 gives

$$\begin{aligned} H_\mu(x;1/q,1/t) &= \frac{(-1)^n}{q^{n(\mu')}t^{n(\mu)+n}}\, J_\mu[\tfrac{X}{1-1/t};q,t] \\ &= \frac{1}{q^{n(\mu')}t^{n(\mu)}}\, \omega H_\mu(x;q,t)\ . \end{aligned} \qquad 1.54$$

Multiplying both sides by $q^{n(\mu')}t^{n(\mu)}$ and using 1.20 gives

$$q^{n(\mu')}\, \tilde{H}_\mu(x;1/q,t) \ = \ \omega H_\mu(x;q,t)\ .$$

Replacing t by $1/t$, multiplying by $t^{n(\mu)}$ and using 1.20 again we get

$$t^{n(\mu)}q^{n(\mu')}\, \tilde{H}_\mu(x;1/q,1/t) \ = \ \omega\, \tilde{H}_\mu(x;q,t)\ ,$$

which is another way of writing 1.51.

Theorem 1.6

$$\tilde{H}_\mu(x;t,q) \ = \ \tilde{H}_{\mu'}(x;q,t)\ . \qquad 1.55$$

Proof

We start here with the second "duality" formula in (5.1) p. 327 of [14]. Using λ-ring notation, this may be written as

$$P_\mu(x;t,q) \ = \ \omega Q_{\mu'}[X\tfrac{1-q}{1-t};q,t]\ . \qquad 1.56$$

Now it is easily verified from the definitions in I.1 that

$$h_\mu(t,q) \ = \ h'_{\mu'}(q,t)\ . \qquad 1.57$$

Thus, multiplying both sides of 1.56 by $h_\mu(t,q)$ and using I.2, we get

$$J_\mu(x;t,q) \ = \ \omega J_{\mu'}[X\tfrac{1-q}{1-t};q,t]\ .$$

Making the plethystic substitution $X \to \frac{X}{1-q}$ gives

$$J_\mu[\tfrac{X}{1-q};t,q] \ = \ \omega J_{\mu'}[\tfrac{X}{1-t};q,t]\ ,$$

which may also be written as

$$H_\mu(x;t,q) \ = \ \omega H_{\mu'}(x;q,t)\ .$$

But now the identity in 1.54 gives that

$$H_\mu(x;t,q) \ = \ q^{n(\mu)}t^{n(\mu')}\, H_{\mu'}(x;1/q,1/t)\ .$$

Replacing q by $1/q$, we are finally led to

$$H_\mu(x;t,1/q) \ = \ q^{-n(\mu)}t^{n(\mu')}\, H_{\mu'}(x;q,1/t)\ ,$$

which, by means of 1.20, can be immediately transformed into 1.55.

2. The algorithm

The mechanism that constructs of our plethystic formulas will be better understood if we follow closely the developments that led us to its discovery. To present it we need some notation.

Since the monomial $t^{n(\mu)}q^{n(\mu')}$ will appear quite often in our treatment it will be helpful to have a special symbol for it. We shall denote it by T_μ. It will also be convenient to let $T_{\mu/\nu} = T_\mu/T_\nu$. Of course when $\nu\rightarrow\mu$ then $T_{\mu/\nu}$ is simply the "weight" of the corner cell that we must remove from μ to get ν. For any given symmetric polynomial $P[X]$ it will also be convenient to let P^* denote the polynomial

$$P^*[X] \;=\; P[\tfrac{X}{(1-t)(1-q)}] \ .$$

This given, the q,t-Kostka coefficients can be computed from the following basic identity.

Proposition 2.1

$$\tilde{K}_{\lambda\mu}(q,t) \;=\; \langle\, S^*_{\lambda'} \,,\, \tilde{H}_\mu \,\rangle_* \tag{2.1}$$

Proof

From the classical Cauchy identity we can easily derive that for any integer $n \geq 1$ we have

$$e_n[\tfrac{XY}{(1-t)(1-q)}] \;=\; \sum_{\lambda\vdash n} S_\lambda[X]\, S_{\lambda'}[\tfrac{Y}{(1-t)(1-q)}] \ . \tag{2.2}$$

Now, in view of 1.17, this says that the bases $\{S_\lambda\}_{\lambda\vdash n}$ and $\{S^*_{\lambda'}\}_{\lambda\vdash n}$ are "dual" with respect to the $*$-scalar product given by I.15. Thus 2.1 follows immediately upon $*$-scalar multiplication by $S^*_{\lambda'}$ on both sides of the relation in I.4, which is

$$\tilde{H}_\mu(x;q,t) \;=\; \sum_{\lambda} S_\lambda(x)\, \tilde{K}_{\lambda\mu}(q,t) \ . \tag{2.3}$$

As a corollary we obtain the following basic recursion.

Theorem 2.1

For any pair of partitions $\rho\vdash n-1$ and $\mu\vdash n$ we have

$$\sum_{\lambda\leftarrow\rho} \tilde{K}_{\lambda,\mu}(q,t) \;=\; \sum_{\nu\rightarrow\mu} c_{\mu\nu}(q,t)\, \tilde{K}_{\rho,\nu} \ . \tag{2.4}$$

Proof

Taking the $*$-scalar product by $S^*_{\rho'}$ on both sides of the relation in I.11 and using 2.1 gives

$$\langle\, \partial_{p_1}\tilde{H}_\mu \,,\, S^*_{\rho'} \,\rangle_* \;=\; \sum_{\nu\rightarrow\mu} c_{\mu\nu}(q,t)\, \langle\, \tilde{H}_\nu \,,\, S^*_{\rho'} \,\rangle_* \;=\; \sum_{\nu\rightarrow\mu} c_{\mu\nu}(q,t)\, \tilde{K}_{\rho,\nu} \ . \tag{2.5}$$

On the other hand, from Proposition 1.2 we derive that

$$\langle\, \partial_{p_1}\tilde{H}_\mu \,,\, S^*_{\rho'} \,\rangle_* \;=\; \langle\, \tilde{H}_\mu \,,\, e_1^* S^*_{\rho'} \,\rangle_*$$

and 2.4 then follows from 2.1 and the standard Pieri rule

$$e_1^* \, S^*_{\rho'} \;=\; \sum_{\lambda \leftarrow \rho} S^*_{\lambda'} \;\;.$$

Before going any further we should note that even the trivial case $\rho = (n-1)$ of 2.4 yields something surprising. In this case it reduces to

$$\tilde K_{(n-1,1),\mu} \;+\; \tilde K_{(n),\mu} \;=\; \sum_{\nu \to \mu} c_{\mu\nu} \, \tilde K_{(n-1),\nu} \;\;, \qquad 2.6$$

and since 1.24 for $k = 0$ and $k = 1$ gives that for any μ and ν we have

$$\tilde K_{(n),\mu} = \tilde K_{(n-1),\nu} = 1 \quad \text{and} \quad \tilde K_{(n-1,1),\mu}(q,t) = B_\mu(q,t) - 1 \;\;,$$

we see that 2.6 yields the remarkable fact that

$$\sum_{\nu \to \mu} c_{\mu\nu}(q,t) = B_\mu(q,t) \;\;. \qquad 2.7$$

Efforts to give a combinatorial proof of this identity led to the discovery in [8] of a q,t-analogue of the hook walk introduced by C. Greene, Nijenhuis and Wilf in ([11],[12]). More remarkably, computer explorations involving the q,t-hook walk led to the conjecture that 2.7 is but a special case of a general identity asserting that for each integer $k \geq 0$, there is a symmetric polynomial g_k such that for any partition μ we have

$$\sum_{\nu \to \mu} c_{\mu\nu}(q,t) \, T^k_{\mu/\nu} \;=\; g_k[B_\mu(q,t)] \;\;. \qquad 2.8$$

It develops that the validity of this conjecture allows us to recursively transform formulas expressing the $\tilde K_{\rho\nu}(q,t)$ as plethysms with B_ν into formulas expressing the $\tilde K_{\lambda\mu}(q,t)$ as plethysms with B_μ. To see how all this comes about it is best to have a close look at some examples. Now the next simplest case of 2.4 is obtained by choosing ρ to be a hook shape. This yields a formula for $\tilde K_{\lambda\mu}(q,t)$ for μ arbitrary and λ an augmented hook. More precisely we get

Proposition 2.2

For any $k \geq 1$ and $\mu \vdash n$ with $n-k-1 \geq 2$ we have

$$\tilde K_{(n-k-1,2,1^{k-1}),\mu}(q,t) \;=\; \sum_{\nu \to \mu} c_{\mu\nu}(q,t) \, e_k[B_\nu - 1] \;-\; e_{k+1}[B_\mu] \;\;. \qquad 2.9$$

Proof

Setting $\rho = (n-k-1, 1^k)$ in 2.4 we get

$$\tilde K_{(n-k-1,1^{k+1}),\mu} \;+\; \tilde K_{(n-k-1,2,1^{k-1}),\mu} \;+\; \tilde K_{(n-k,1^k),\mu} \;=\; \sum_{\nu \to \mu} c_{\mu\nu} \, \tilde K_{(n-k-1,1^k),\nu} \;\;.$$

Using 1.24 we can rewrite this as

$$\tilde{K}_{(n-k-1,2,1^{k-1}),\mu} \;=\; \sum_{\nu\to\mu} c_{\mu\nu}\, e_k[B_\nu - 1] \;-\; e_{k+1}[B_\mu - 1] \;-\; e_k[B_\mu - 1]$$

and this is 2.9 because

$$e_{k+1}[B_\mu] \;=\; e_{k+1}[B_\mu - 1] \;+\; e_k[B_\mu - 1]\ .$$

Note next that since we can write

$$B_\nu - 1 \;=\; B_\mu - 1 - T_{\mu/\nu}\ , \qquad 2.10$$

we have the expansion

$$e_k[B_\nu - 1] \;=\; \sum_{s=0}^{k} e_{k-s}[B_\mu - 1]\,(-T_{\mu/\nu})^s\ .$$

Substituting this into 2.9 we get

$$\tilde{K}_{(n-k-1,2,1^{k-1}),\mu}(q,t) \;=\; -e_{k+1}[B_\mu] \;+\; \sum_{s=0}^{k} e_{k-s}[B_\mu - 1]\,(-1)^s \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\, T^s_{\mu/\nu}\ . \qquad 2.11$$

Note further, that using 2.8 we can rewrite this in the form

$$\tilde{K}_{(n-k-1,2,1^{k-1}),\mu}(q,t) \;=\; -e_{k+1}[B_\mu] \;+\; \sum_{s=0}^{k} e_{k-s}[B_\mu - 1]\,(-1)^s\, g_s[B_\mu]\ . \qquad 2.12$$

In other words, by means of this mechanism, we can convert our plethystic formula for $\tilde{K}_{\lambda\mu}$ for λ a hook into a plethystic formula for λ an augmented hook. This is but the embryo of a procedure that can be used to construct plethystic formulas for all $\tilde{K}_{\lambda\mu}$. We can thus see the importance of proving the existence of a polynomial g_k giving 2.8. Remarkably, we can not only prove that g_k exists but we can give it a surprisingly simple explicit formula. Namely,

Theorem 2.2

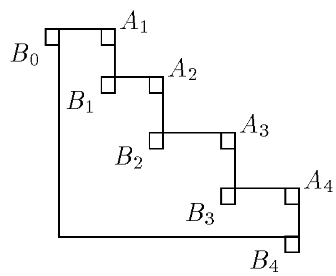
For all $k \geq 1$ we have

$$\sum_{\nu\to\mu} c_{\mu\nu}\, T^k_{\mu/\nu} \;=\; \frac{1}{(1-t)(1-q)t^k q^k}\, h_{k+1}[(1-t)(1-q)B_\mu - 1]\ . \qquad 2.13$$

In other words, in view of 2.7, we can set

$$g_k[X] \;=\; \begin{cases} \frac{1}{(1-t)(1-q)t^k q^k}\, h_{k+1}[(1-t)(1-q)X - 1] & \text{for } k > 0 \\ h_1[X] & \text{for } k = 0. \end{cases} \qquad 2.14$$

To prove 2.13 we need to establish three auxiliary results which should be of independent interest. To state them we need some notation. Let μ have m corners $A_1, A_2, \ldots, A_m$ labelled as we encounter them from left to right. For convenience let us set $A_i = (\alpha_i, \beta_i)$ with α_i and β_i respectively giving the coleg and coarm of A_i in μ. Similarly, for $i = 1, 2, \ldots, m-1$, let $B_i = (\alpha_{i+1}, \beta_i)$ denote the cell of μ with coleg α_{i+1} and coarm β_i. Finally, set $\alpha_{m+1} = \beta_o = -1$ and let $B_o = (\alpha_1, \beta_o)$, $B_m = (\alpha_{m+1}, \beta_m)$ denote the cells that are respectively immediately to the left of the highest row of μ and immediately below the last column of μ. In the figure below we illustrate the case of a 4-corner partition and the location of the cells A_1, A_2, A_3, A_4 and B_o, B_1, B_2, B_3, B_4.



This given, setting $x_i = t^{\alpha_i} q^{\beta_i}$ and $u_i = t^{\alpha_{i+1}} q^{\beta_i}$, we have the following curious identity.

Proposition 2.3

$$x_1 + x_2 + \cdots + x_m \;-\; u_o - u_1 - \cdots - u_m \;=\; (1-1/t)(1-1/q)\, B_\mu(q,t) \;-\; \frac{1}{tq} \qquad 2.15$$

Proof

If μ has k parts we may rewrite I.6 in the form

$$(1-t)(1-q)\, B_\mu(q,t) \;=\; (1-t)\sum_{i=1}^{k} t^{i-1}\,(1-q^{\mu_i}) \;=\; 1-t^k \;-\; (1-t)\sum_{i=1}^{k} t^{i-1} q^{\mu_i} \;.$$

Dividing bt tq and reorganizing terms gives

$$\begin{aligned}(1-1/t)(1-1/q)\, B_\mu(q,t) - \frac{1}{tq} &= -t^{k-1}q^{-1} - t^{-1}q^{\mu_1-1} - \sum_{i=2}^{k} t^{i-2}q^{\mu_i-1} + \sum_{i=1}^{k} t^{i-1}q^{\mu_i-1} \\ &= -t^{k-1}q^{-1} - t^{-1}q^{\mu_1-1} + \sum_{i=1}^{k-1} t^{i-1}(q^{\mu_i-1} - q^{\mu_{i+1}-1}) + t^{k-1}q^{\mu_k-1} \;.\end{aligned}$$

Clearly, the only terms that contribute to the sum on the right are those corresponding to the rows which contain a corner of μ, since for all the other terms we have $\mu_i = \mu_{i+1}$. Since the row containing A_i contributes $x_i - u_{i-1}$, rewriting all the other terms in terms of weights, we are reduced to

$$(1-1/t)(1-1/q)\, B_\mu(q,t) - \frac{1}{tq} \;=\; -u_o - u_m \;+\; \sum_{i=2}^{m}(x_i - u_{i-1}) \;+\; x_1 \;,$$

and this is a rearrangement of 2.15.

The next result gives an expression for the coefficient $c_{\mu\nu}(q,t)$ in terms of the weights $x_1, x_2, \ldots, x_m,\ u_o, u_1, \ldots, u_m$.

Proposition 2.4

If $\nu^{(i)}$ (for $i=1,..,m$) denotes the partition obtained by removing from μ the cell A_i then

$$c_{\mu\nu^{(i)}} \;=\; \frac{1}{(1-1/t)(1-1/q)}\,\frac{1}{x_i}\,\frac{\prod_{s=0}^{m}(u_s-x_i)}{\prod_{s=1\ ,\ s\neq i}^{m}(x_s-x_i)}\ . \qquad 2.16$$

Proof

For convenience let R_i (resp. C_i) denote the row (column) of cells of $\nu^{(i)}$ that are directly west (south) of A_i. Note that since

$$l_{\nu^{(i)}}(s)=\begin{cases} l_\mu(s) & \text{for } s\in R_i\\ l_\mu(s)-1 & \text{for } s\in C_i\end{cases} \qquad \text{and} \qquad a_{\nu^{(i)}}(s)=\begin{cases} a_\mu(s)-1 & \text{for } s\in R_i\\ a_\mu(s) & \text{for } s\in C_i\end{cases}$$

we can rewrite our formula 1.44 in the form

$$c_{\mu\nu^{(i)}} \;=\; \prod_{s\in R_i}\frac{t^{l_\mu(s)}-q^{a_\mu(s)+1}}{t^{l_\mu(s)}-q^{a_\mu(s)}}\prod_{s\in C_i}\frac{q^{a_\mu(s)}-t^{l_\mu(s)+1}}{q^{a_\mu(s)}-t^{l_\mu(s)}}\ . \qquad 2.17$$

It is easy to see from this that the horizontal pieces of the boundary of μ that lie above R_i produce massive cancellations in the first product in 2.17. Similarly, the vertical pieces of the boundary that are to the right of C_i cause massive cancelations in the second product. To be consistent with our way of representing the corners A_i, it is best to keep using the convention of representing a cell s by a pair (α,β) consisting of the coleg and coarm of s with respect to μ. This given, to carry out these cancellations, we only need to compute these products by grouping their factors according to the following decompositions.

$$R_i+\{A_i\} \;=\; \sum_{j=1}^{i} R_i(j) \quad \text{with} \quad R_i(j)=\{(\alpha_i,\beta)\ :\ \beta_{j-1}<\beta\le\beta_j\}\ ,$$

$$C_i+\{A_i\} \;=\; \sum_{j=i}^{m} C_i(j) \quad \text{with} \quad C_i(j)=\{(\alpha,\beta_i)\ :\ \alpha_{j+1}<\alpha\le\alpha_j\}\ .$$

Now we see that for $j<i$

$$\prod_{s\in R_i(j)}\frac{t^{l_\mu(s)}-q^{a_\mu(s)+1}}{t^{l_\mu(s)}-q^{a_\mu(s)}}=\prod_{\beta_{j-1}<\beta\le\beta_j}\frac{t^{\alpha_j-\alpha_i}-q^{\beta_i-\beta+1}}{t^{\alpha_j-\alpha_i}-q^{\beta_i-\beta}} \;=\; \frac{t^{\alpha_j-\alpha_i}-q^{\beta_i-\beta_{j-1}}}{t^{\alpha_j-\alpha_i}-q^{\beta_i-\beta_j}}$$

and for $j>i$

$$\prod_{s\in C_i(j)}\frac{q^{a_\mu(s)}-t^{l_\mu(s)+1}}{q^{a_\mu(s)}-t^{l_\mu(s)}}=\prod_{\beta_{j-1}<\beta\le\beta_j}\frac{q^{\beta_j-\beta_i}-t^{\alpha_i-\alpha+1}}{q^{\beta_j-\beta_i}-t^{\alpha_i-\alpha}} \;=\; \frac{q^{\beta_j-\beta_i}-t^{\alpha_i-\alpha_{j+1}}}{q^{\beta_j-\beta_i}-t^{\alpha_i-\alpha_j}}\ ,$$

while

$$\prod_{s\in R_i(i)-\{A_i\}}\frac{t^{l_\mu(s)}-q^{a_\mu(s)+1}}{t^{l_\mu(s)}-q^{a_\mu(s)}}=\frac{1-q^{\beta_i-\beta_{i-1}}}{1-q}\ ,\qquad \prod_{s\in C_i(i)-\{A_i\}}\frac{q^{a_\mu(s)}-t^{l_\mu(s)+1}}{q^{a_\mu(s)}-t^{l_\mu(s)}}=\frac{1-t^{\alpha_i-\alpha_{i+1}}}{1-t}\ .$$

Putting all this together, we obtain that

$$c_{\mu\nu^{(i)}} = \prod_{j=1}^{i-1} \frac{t^{\alpha_j-\alpha_i} - q^{\beta_i-\beta_{j-1}}}{t^{\alpha_j-\alpha_i} - q^{\beta_i-\beta_j}} \times \frac{1-q^{\beta_i-\beta_{i+1}}}{1-q} \times \frac{1-t^{\alpha_i-\alpha_{i+1}}}{1-t} \times \prod_{j=i+1}^{m} \frac{q^{\beta_j-\beta_i} - t^{\alpha_i-\alpha_{j+1}}}{q^{\beta_j-\beta_i} - t^{\alpha_i-\alpha_j}} \ .$$

Regrouping the factors we can write

$$(1-t)(1-q)\ c_{\mu\nu^{(i)}} = \frac{\prod_{j=1}^{i}(t^{\alpha_j-\alpha_i} - q^{\beta_i-\beta_{j-1}})}{\prod_{j=1}^{i-1}(t^{\alpha_j-\alpha_i} - q^{\beta_i-\beta_j})} \ \frac{\prod_{j=i}^{m}(q^{\beta_j-\beta_i} - t^{\alpha_i-\alpha_{j+1}})}{\prod_{j=i+1}^{m}(q^{\beta_j-\beta_i} - t^{\alpha_i-\alpha_j})} \ ,$$

and getting rid of the negative exponents, we finally derive that

$$(1-t)(1-q)\ c_{\mu\nu^{(i)}} = \frac{\prod_{j=1}^{i}(t^{\alpha_j}q^{\beta_{j-1}} - t^{\alpha_i}q^{\beta_i})}{\prod_{j=1}^{i-1}(t^{\alpha_j}q^{\beta_j} - t^{\alpha_i}q^{\beta_i})} \times \frac{\prod_{j=i}^{m}(t^{\alpha_{j+1}}q^{\beta_j} - t^{\alpha_i}q^{\beta_i})}{\prod_{j=i+1}^{m}(t^{\alpha_j}q^{\beta_j} - t^{\alpha_i}q^{\beta_i})} \times \frac{1}{t^{\alpha_i}q^{\beta_i}q^{\beta_o}t^{\alpha_{m+1}}}$$

Rewriting this in terms of the weights x_i, u_j we get

$$(1-t)(1-q)\ c_{\mu\nu^{(i)}} = \frac{\prod_{j=1}^{i}(u_{j-1}-x_i)}{\prod_{j=1}^{i-1}(x_j-x_i)} \times \frac{\prod_{j=i}^{m}(u_j-x_i)}{\prod_{j=i+1}^{m}(x_j-x_i)} \times \frac{1}{x_i} \times \frac{1}{q^{-1}t^{-1}}$$

which is another way of writing 2.16. This completes our proof.

Proposition 2.5

For any two alphabets $X = x_1 + x_2 + \cdots + x_m$ and $U = u_o + u_1 + \cdots + u_m$, we have

$$\sum_{m\geq 0} t^m\ h_m[X-U] \ = \ 1 \ + \ \sum_{i=1}^{m} \frac{1}{x_i} \frac{\prod_{s=0}^{m}(x_i-u_s)}{\prod_{s=1,s\neq i}^{m}(x_i-x_s)} \times \frac{t}{1-tx_i} \ - \ \frac{u_o u_1 \cdots u_m}{x_1 x_2 \cdots x_m}\, t \qquad 2.18$$

Proof

Note that since

$$\sum_{m\geq 0} t^m\ h_m[X-U] \ = \ \frac{\prod_{s=0}^{m}(1-tu_s)}{\prod_{s=1}^{m}(1-tx_s)} \ , \qquad 2.19$$

we need only determine the unknown coefficients in the partial fraction decomposition

$$\frac{\prod_{s=0}^{m}(1-tu_s)}{\prod_{s=1}^{m}(1-tx_s)} \ = \ \sum_{i=1}^{m} A_i\, \frac{1}{1-tx_i} \ + \ c_o \ + \ c_1\, t \ . \qquad 2.20$$

Multiplying 2.20 by $1-tx_i$ and setting $t = 1/x_i$ yields that

$$A_i \ = \ \frac{1}{x_i^2} \frac{\prod_{s=0}^{m}(x_i-u_s)}{\prod_{s=1\ ,\ \neq i}^{m}(x_i-x_s)} \ . \qquad 2.21$$

Setting $t = 0$ in 2.20 gives

$$1 \ = \ \sum_{i=1}^{m} A_i \ + \ c_o \ , \qquad 2.22$$

while dividing by t and letting $t\to\infty$ gives

$$c_1 \;=\; -\,\frac{u_o u_1\cdots u_m}{x_1x_2\cdots x_m}\;. \tag{2.23}$$

Now substituting 2.21 in 2.20, we obtain

$$\frac{\prod_{s=0}^{m}(1-tu_s)}{\prod_{s=1}^{m}(1-tx_s)} \;=\; 1 \;+\; \sum_{i=1}^{m} A_i\,\frac{tx_i}{1-tx_i} \;+\; c_1\,t\;,$$

and 2.18 follows from 2.19, 2.21 and 2.23.

Proof of Theorem 2.2

Equating coefficients of t^{k+1} in 2.18 gives that for any distinct nonvanishing values of $x_1,x_2,\ldots,x_m$, we have

$$\sum_{i=1}^{m}\frac{1}{x_i}\frac{\prod_{s=0}^{m}(x_i-u_s)}{\prod_{\substack{s=1\\ s\neq i}}^{m}(x_i-x_s)}\,x_i^k \;=\; \begin{cases} h_{k+1}[x_1+\cdots+x_m \;-\; u_o-\cdots-u_m] & \text{for } k\geq 1\,,\\[2ex] x_1+\cdots+x_m \;-\; u_o-\cdots-u_m \;+\; \frac{u_o\cdots u_m}{x_1\cdots x_m} & \text{for } k=0\,. \end{cases} \tag{2.24}$$

Now, combining 2.15 and 2.16 with 2.24 (interpreting the x_i, u_j as weights of the cells A_i, B_j) we derive that for $k\geq 1$

$$\begin{aligned}\sum_{\nu\to\mu} c_{\mu\nu}\,(T_{\mu/\nu})^k \;&=\; \frac{1}{(1-1/t)(1-1/q)}\;\sum_{i=1}^{m}\frac{1}{x_i}\frac{\prod_{s=0}^{m}(x_i-u_s)}{\prod_{\substack{s=1\\ s\neq i}}^{m}(x_i-x_s)}\,x_i^k\\ &=\; \frac{1}{(1-1/t)(1-1/q)}\;h_{k+1}\Big[(1-1/t)(1-1/q)B_\mu-\frac{1}{tq}\Big]\end{aligned}$$

and this is easily changed into 2.13.

We should note that from 2.24 we can also obtain another proof of 2.7. In fact, it is easily derived from our definition of the weights x_i, u_j that

$$\frac{u_o u_1\cdots u_m}{x_1x_2\cdots x_m} \;=\; \frac{1}{tq}\;.$$

So the case $k=0$ of 2.24 gives

$$\sum_{\nu\to\mu} c_{\mu\nu} \;=\; \frac{x_1+x_2+\cdots+x_m-u_o-u_1-\cdots-u_m}{(1-1/t)(1-1/q)} \;+\; \frac{1}{(1-1/t)(1-1/q)}\frac{1}{tq}\;,$$

which reduces to 2.7 by means of 2.15.

An immediate corollary of Theorem 2.2 is an explicit plethystic formula for $\tilde{K}_{\lambda\mu}(q,t)$ when λ is an extended hook.

Theorem 2.3

For any $k\geq 1$ and $\mu\vdash n$ with $n-k-1\geq 2$, we have

$$\begin{aligned}\tilde{K}_{(n-k-1,2,1^{k-1}),\mu}(q,t) \;&=\; -e_{k+1}[B_\mu]\\ &\quad+\; \tfrac{tq}{(1-t)(1-q)}e_{k+1}[B_\mu+\tfrac{1-tq}{tq}]\\ &\quad-\; \tfrac{tq}{(1-t)(1-q)}e_{k+1}[\tfrac{t+q-1}{tq}B_\mu+\tfrac{1-tq}{tq}]\;.\end{aligned} \tag{2.25}$$

Proof

Using 2.14 in 2.12 gives

$$\begin{aligned}\tilde{K}_{(n-k-1,2,1^{k-1}),\mu}(q,t) &= -e_{k+1}[B_\mu] + e_k[B_\mu - 1]\, B_\mu \;+ \\ &+\; \tfrac{tq}{(1-t)(1-q)} \sum_{s=1}^{k} e_{k-s}[B_\mu - 1](-1)^s h_{s+1}[\tfrac{(1-t)(1-q)}{tq} B_\mu - \tfrac{1}{tq}] \;.\end{aligned} \qquad 2.26$$

Working on this sum we successively obtain

$$\begin{aligned} -\sum_{s=2}^{k+1} e_{k+1-s}[B_\mu - 1](-1)^s h_s[\tfrac{(1-t)(1-q)}{tq} B_\mu - \tfrac{1}{tq}] \;= \\ = \;-\sum_{s=2}^{k+1} e_{k+1-s}[B_\mu - 1] e_s[-\tfrac{(1-t)(1-q)}{tq} B_\mu + \tfrac{1}{tq}] \\ = \;-\; e_{k+1}[B_\mu - 1 - \tfrac{(1-t)(1-q)}{tq} B_\mu + \tfrac{1}{tq}] \;+\; e_{k+1}[B_\mu - 1] \;+ \\ +\; e_k[B_\mu - 1](-\tfrac{(1-t)(1-q)}{tq} B_\mu + \tfrac{1}{tq}) \;. \end{aligned}$$

Using the final expression in 2.26, our desired formula 2.25 can be derived with a few routine manipulations.

Remark 2.1

We should note that Theorem 2.3 establishes the particular case $\gamma = (2, 1^{k-1})$ of Theorem I.1. Indeed, formula 2.25 simply asserts that we may take (for any $k \geq 1$)

$$\mathbf{k}_{2,1^{k-1}}(x; q, t) = -e_{k+1}(x) + \tfrac{tq}{(1-t)(1-q)} \Big(e_{k+1}[X + \tfrac{1-tq}{tq}] - e_{k+1}[\tfrac{t+q-1}{tq} X + \tfrac{1-tq}{tq}] \Big) \qquad 2.27$$

We should also mention that regardless of the presence of minus signs in 2.27, J. Remmel (see [10]) was able to show that the plethystic evaluation of $\mathbf{k}_{2,1^{k-1}}(x)$ at $B_\mu(q,t)$ always yields a polynomial in q, t with positive integer coefficients for any $\mu \vdash n \geq k+3$.

Remark 2.2

We should note that the only denominators that are introduced by uses of formula 2.13 are powers of qt. In fact, using the addition formula for h_{k+1}, we may rewrite 2.13 as

$$\sum_{\nu \to \mu} c_{\mu\nu}\, T^k_{\mu/\nu} \;=\; \tfrac{1}{(1-t)(1-q)t^k q^k} \Big(h_{k+1}[(1-t)(1-q)B_\mu] \;-\; h_k[(1-t)(1-q)B_\mu] \Big) \;,$$

and it can be easily seen (by passing to a power sum expansion) that the factor $(1-t)(1-q)$ in the denominator is cancelled out by multiples of it produced by the numerator.

The experience that was acquired in the study of the Kostka-Foulkes polynomials $\tilde{K}_{\lambda\mu}(t)$ (see for instance [9] and [5]) strongly suggests that to obtain a proof of Theorem 2.1 in full generality we should make use of higher order Pieri rules than the one expressed by 1.41. And this is precisely what we shall do in the next section. Nevertheless, we have overwhelming evidence that things are quite different in the q,t-case and that in fact, one single Pieri rule should be sufficient. What is fundamentally different in the q,t case is that the Macdonald operator, as well as the operator Δ_1 given in 1.29, have distinct eigenvalues for q and t generic. This allows the construction of formulas for the coefficients $\tilde{K}_{\lambda\mu}(q,t)$ by judiciously combined uses of Theorems 1.3 and 2.2. The point we try to get across here will be better understood after we examine some further examples.

But before we do that we shall study the nature of the rational expressions for the $\tilde{K}_{\lambda\mu}(q,t)$ that may be derived from one of the originally available algorithms. To this end let us recall that Macdonald established the following

Pieri Rule ((6.13) p. 336 of [14])

Let ν and μ be partitions and let $\mu = \nu + 1^k$, for some integer $k \geq 1$. Then we have

$$e_k(x)P_\nu(x;q,t) \;=\; P_\mu(x;q,t) \;+\; \sum_{\substack{\lambda<\mu \\ \lambda/\nu \in V_k}} \psi'_{\lambda/\nu}(q,t)\, P_\lambda(x;q,t) \;,$$

where the inequality sign "<" in $\lambda < \mu$ represents dominance and

$$\psi'_{\lambda/\nu}(q,t) = \prod_{\substack{i<j \\ \lambda_i=\nu_i \;,\; \lambda_j=\nu_j+1}} \frac{(1-q^{\nu_i-\nu_j}t^{j-i-1})(1-q^{\lambda_i-\lambda_j}t^{j-i+1})}{(1-q^{\nu_i-\nu_j}t^{j-i})(1-q^{\lambda_i-\lambda_j}t^{j-i})} \tag{2.29}$$

This rule has two immediate corollaries that are important for us here.

Proposition 2.6

The coefficients $\xi_{\lambda\mu}(q,t)$ in the Schur function expansion

$$P_\mu(x;q,t) \;=\; \sum_{\lambda\leq\mu} S_\lambda(x)\; \xi_{\lambda\mu}(q,t) \tag{2.30}$$

have rational expressions with denominators containing only factors of the form

$$(1-q^r t^s) \qquad (\; with \quad r+s\geq 1 \;) \;. \tag{2.31}$$

In particular, we may construct rational expressions for the $\tilde{K}_{\lambda\mu}(q,t)$ having in the denominators only factors of the form

$$(q^r - t^s) \qquad (\; with \quad r+s\geq 1 \;) \;. \tag{2.32}$$

Proof

Note that we may rewrite 2.28 in the form

$$P_\mu(x;q,t) \;=\; e_k(x)P_\nu(x;q,t) - \sum_{\substack{\lambda:\ \lambda<\mu \\ \lambda/\nu \in V_k}} \psi'_{\lambda/\nu}(q,t)\, P_\lambda(x;q,t) \;. \tag{2.33}$$

Thus, having computed all the P_ν with $|\nu| < |\mu|$ and all P_λ with $\lambda < \mu$, this formula may be used to compute P_μ. This gives us a fast algorithm for a recursive construction of the Macdonald polynomials in any total order that is compatible with degree and dominance. Now, at any given degree m we must start with $\mu = 1^m$, and it is shown in [13] that

$$P_{1^m}(x;q,t) \;=\; e_m(x) \;.$$

This given, we see that successive applications of 2.33 will yield a Jacobi-Trudi-like expansion for P_μ of the form

$$P_\mu(x;q,t) \;=\; \sum_{\lambda\leq\mu} e_{\lambda'}(x)\,\eta_{\lambda\mu}(q,t) \qquad 2.34$$

with the coefficients $\eta_{\lambda\mu}(q,t)$ integral polynomials in q,t divided by products of factors as given in 2.31. Thus the assertion concerning the coefficients $\xi_{\lambda\mu}(q,t)$ in 2.30 is obtained by replacing the $e_{\lambda'}(x)$'s in 2.34 by their Schur function expansions.

To transform the Schur function expansion of $P_\mu(q,t)$ into the Schur function expansion of $\tilde H_\mu(x;q,t)$, we need to go through the following sequence of steps:

$$\begin{array}{llcll}
(1) & P_\mu(x;q,t) & \rightarrow & h_\mu(q,t)P_\mu(x;q,t) & = \; J_\mu(x;q,t)\ , \\
(2) & J_\mu(x;q,t) & \rightarrow & J_\mu[\tfrac{X}{1-t};q,t] & = \; H_\mu(x;q,t)\ , \\
(3) & H_\mu(x;q,t) & \rightarrow & H_\mu(x;q,1/t)\,t^{n(\mu)} & = \; \tilde H_\mu(x;q,t)\ .
\end{array}$$

Now starting with denominator factors $(1-q^rt^s)$ the first step should, if anything, cancel some of them. The second step could at the worst produce some additional denominator factors $(1-t^r)$ in the Schur function expansion of $H_\mu(x;q,t)$. Finally, the last step changes the denominator factors $1-q^rt^s$ into factors of the form t^s-q^r and at the worst could also introduce denominator factors t^k. In summary, by this process, the coefficient $\tilde K_{\lambda\mu}(q,t)$ in the Schur function expansion of $\tilde H_\mu(x;q,t)$ may be given a tentative first expression as an integral polynomials in q,t divided by products of factors of the form

$$t^k \quad (\text{ with } \quad k\geq 1\ ) \qquad \text{and} \qquad (q^r-t^s) \quad (\text{ with } \quad r+s\geq 1\ )\ . \qquad 2.35$$

However, the identity in 1.55 yields that

$$\tilde K_{\lambda\mu}(q,t) \;=\; \tilde K_{\lambda\mu'}(t,q)\ . \qquad 2.36$$

But then, applying our results to $\tilde K_{\lambda\mu'}(t,q)$, we derive from 2.36 that $\tilde K_{\lambda\mu}(q,t)$ may also be given a second expression as an integral polynomial in q,t with denominator factors

$$q^k \quad (\text{ with } \quad k\geq 1\ ) \qquad \text{and} \qquad (t^r-q^s) \quad (\text{ with } \quad r+s\geq 1\ )\ . \qquad 2.37$$

Comparing 2.35 and 2.37 we come to the conclusion that the denominator factors t^k in the first expression and the denominator factors q^k in the second expression must cancel out when we reduce those expressions to their normal form. This leaves as the only possible denominator factors for $\tilde K_{\lambda\mu}(q,t)$ those appearing in the list in 2.32.

The Macdonald Pieri rule yields a Pieri rule for the polynomial $\tilde H_\mu(x;q,t)$ which may be stated as follows.

Proposition 2.7

For any $k \geq 0$ we have

$$h_k[\tfrac{X}{1-t}]\, \tilde{H}_\nu[X;q,t] \;=\; \sum_{\lambda/\nu \in V_k} {}^{+}d_{\lambda\nu}(q,t)\, \tilde{H}_\lambda[X;q,t] \;, \qquad 2.38$$

with

$$^{+}d_{\lambda\nu}(q,t) \;=\; \frac{\psi'_{\lambda/\nu}(q,1/t)\, \tilde{h}_\nu(q,t)}{\tilde{h}_\lambda(q,t)} \;. \qquad 2.39$$

Proof

Multiplying both sides of 2.28 by $h_\nu(q,t)$ and setting for convenience $\psi'_{\lambda/\mu}(q,t)=1$, we can rewrite 2.28 as

$$e_k[X]J_\nu[X;q,t] \;=\; \sum_{\lambda/\nu \in V_k} \frac{\psi'_{\lambda/\nu}(q,t)h_\nu(q,t)}{h_\lambda(q,t)}\, J_\lambda[X;q,t] \;.$$

Making the plethystic substitution $X \to \frac{X}{1-t}$ gives

$$e_k[\tfrac{X}{1-t}]H_\nu[X;q,t] \;=\; \sum_{\lambda/\nu \in V_k} \frac{\psi'_{\lambda/\nu}(q,t)h_\nu(q,t)}{h_\lambda(q,t)}\, H_\lambda[X;q,t] \;.$$

Replacing t by $1/t$ and multiplying both sides by $t^{n(\nu)}$ we get

$$e_k[\tfrac{-tX}{1-t}]\, \tilde{H}_\nu[X;q,t] \;=\; \sum_{\lambda/\nu \in V_k} \frac{\psi'_{\lambda/\nu}(q,1/t)\, h_\nu(q,1/t)\, t^{n(\nu)}}{h_\lambda(q,1/t)\, t^{n(\lambda)}}\, \tilde{H}_\lambda[X;q,t] \;.$$

Making the replacement $e_k[\frac{-tX}{1-t}] \to (-t)^k h_k[\frac{X}{1-t}]$ and using the first equality in 1.21 yields

$$h_k[\tfrac{X}{1-t}]\, \tilde{H}_\nu[X;q,t] \;=\; \sum_{\lambda/\nu \in V_k} \frac{\psi'_{\lambda/\nu}(q,1/t)\, \tilde{h}_\nu(q,t)}{\tilde{h}_\lambda(q,t)}\, \tilde{H}_\lambda[X;q,t] \;,$$

and this is what we wanted to show.

The basic result which plays a key role in our algorithm for computing the coefficients $\tilde{K}_{\lambda\mu}(q,t)$ may be stated as follows.

Theorem 2.4

The multiplication of a polynomial $\tilde{H}_\nu(x;q,t)$ by $e^*_k(x)$ may be expressed in the form

$$e^*_k(x)\tilde{H}_\nu(x;q,t) \;=\; \sum_{\mu \supseteq_k \nu} {}^{*}d^{(k)}_{\mu\nu}(q,t)\, \tilde{H}_\mu(x;q,t) \;, \qquad 2.40$$

where the symbol "$\mu \supseteq_k \nu$" is to mean that the sum runs over partitions μ whose diagram contains the diagram of ν and differs from it by exactly k cells. Moreover, the coefficients $^{*}d^{(k)}_{\mu\nu}(q,t)$ may be computed by successive applications of 2.38 via the formula

$$e_k[\tfrac{X}{(1-t)(1-q)}] \;=\; \sum_{\mu \vdash k} h_\mu[\tfrac{X}{1-t}]\, f_\mu[\tfrac{1}{1-q}] \;. \quad (\dagger) \qquad 2.41$$

(†) *As is customary, f_μ denotes the forgotten basis element corresponding to μ*

Finally, denoting by ${}^*\partial^{(k)}$ *the adjoint of multiplication by* e_k^* *with respect to the* $*$*-scalar product, we have*

$$ {}^*\partial^{(k)}\ \tilde H_\mu(x;q,t) \;=\; \sum_{\nu\subseteq_k\mu} {}^*c^{(k)}_{\mu\nu}(q,t)\ \tilde H_\nu(x;q,t) \tag{2.42}$$

with

$$ {}^*c^{(k)}_{\mu\nu} \;=\; \frac{1}{(1-t)(1-q)}\,\frac{\tilde h_\mu \tilde h'_\mu}{\tilde h_\nu \tilde h'_\nu}\ . \tag{2.43}$$

Proof

We need only observe that each time we use 2.38 on a polynomial $\tilde H_\nu$, we produce polynomials $\tilde H_\lambda$ indexed by partitions whose diagram contains the diagram of ν. As for 2.43, we can derive it from 2.41 in exactly the same manner 1.42 was derived from 1.41 a).

Here and after, by an *integral* polynomial we mean a polynomial with integer coefficients. The ratio of two integral polynomials in q,t will be simply referred to as a *rational* function of q,t We shall say that a rational function $f(q,t)$ is *Laurent* if it is given by an integral polynomial in $q,t,1/q,1/t$ We shall say that $f(q,t)$ is *pure-Laurent* if it is equal to an integral polynomial in $1/q,1/t$. A function $f_\mu[q,t]$ will be called *m-rational* if it evaluates to a rational function of q and t, for each $\mu\vdash n\ \geq m$. An m-rational function is said to be *plethystic* if it is of the form $f_\mu[q,t]=P[B_\mu(q,t);q,t]$ with $P(x;q,t)$ a symmetric polynomial with coefficients rational functions of q and t. In this case we shall say that P *generates* f. If the coefficients of $P(x;q,t)$ are Laurent or pure-Laurent we shall call $f_\mu[q,t]=P[B_\mu(q,t);q,t]$ *plethystic Laurent* or *plethystic pure-Laurent* as the case may be. The collections of m-rational functions will be denoted by $\mathcal{R}_{\geq m}$.

For $f\in\mathcal{R}_{\geq m}$ and for a given $\mu\vdash n\geq m+k$, set

$$ g_\mu(q,t) \;=\; \sum_{\nu\subseteq_k\mu} {}^*c^{(k)}_{\mu\nu}(q,t)\ f_\nu(q,t)\ . \tag{2.44}$$

It will be convenient to express this relation by writing

$$ g \;=\; \mathbf{C}_k f \tag{2.45}$$

and viewing $\mathbf{C}_k$ as a linear operator mapping $\mathcal{R}_{\geq m}$ into $\mathcal{R}_{\geq m+k}$. Of course these operators may composed in any order, and to express that a certain $g\in\mathcal{R}_{\geq k_1+k_2+\cdots+k_r}$ is obtained by applying successively $\mathbf{C}_{k_1},\mathbf{C}_{k_2},\ldots,\mathbf{C}_{k_r}$ to a given $f\in\mathcal{R}_{\geq m}$, we write

$$ g \;=\; \mathbf{C}_{k_r}\cdots\mathbf{C}_{k_2}\mathbf{C}_{k_1}\,f\ .$$

To make explicit the dependence on μ, we shall write

$$ g_\mu \;=\; \mathbf{C}_{k_r}\cdots\mathbf{C}_{k_2}\mathbf{C}_{k_1}\,f\ \big|_{[\mu]}\ .$$

The relevancy of this apparatus to the computation of the Kostka-Macdonald coefficients is due to the fact that each $\tilde K_{\lambda\mu}$ may be simply expressed as a polynomial in the operators $\mathbf{C}_k$ applied to the function $f \equiv 1$. Before we state the precise result, it will be good to look at some particular cases. To this end, suppose we want $\tilde{K}_{(n-7,3,2,2),\mu}(q,t)$ for a given $\mu \vdash n$. Our starting point is formula 2.1, which in this case gives

$$\tilde{K}_{(n-6,3,2,1),\mu} \;=\; \langle\, S^*_{(n-6,3,2,1)'} \,,\, \tilde{H}_\mu \,\rangle_* \;. \tag{2.46}$$

Now, expanding $S^*_{(n-6,3,2,1)'}$ by Jacobi-Trudi, we get

$$\begin{aligned} S^*_{(n-6,3,2,1)'} \;=\; & e^*_{n-6}e^*_3e^*_2e^*_1 - e^*_{n-6}e^{*2}_3 - e^*_{n-6}e^*_4e^{*2}_1 + e^*_{n-6}e^*_5e^*_1 \\ & - e^*_{n-5}e^{*2}_2e^*_1 + e^*_{n-5}e^*_3e^*_2 + e^*_{n-4}e^*_2e^{*2}_1 - e^*_{n-3}e^*_2e^*_1 \\ & + e^*_{n-5}e^*_4e^*_1 - e^*_{n-5}e^*_5 - e^*_{n-4}e^*_3e^*_1 + e^*_{n-3}e^*_3 \;. \end{aligned} \tag{2.47}$$

Before we substitute this into 2.46, note that we can write

$$\langle\, e^*_{n-6}e^*_3e^*_2e^*_1 \,,\, \tilde{H}_\mu \,\rangle_* \;=\; \langle\, e^*_{n-6} \,,\; {}^*\partial^{(3)}\, {}^*\partial^{(2)}\, {}^*\partial^{(1)}\, \tilde{H}_\mu \,\rangle \;. \tag{2.48}$$

But

$${}^*\partial^{(1)}\, \tilde{H}_\mu \;=\; \sum_{\alpha \subseteq_1 \mu} {}^*c^{(1)}_{\mu\alpha}\, \tilde{H}_\alpha \;,$$

so

$${}^*\partial^{(2)}\, {}^*\partial^{(1)}\, \tilde{H}_\mu \;=\; \sum_{\alpha \subseteq_1 \mu} {}^*c^{(1)}_{\mu\alpha} \sum_{\beta \subseteq_2 \alpha} {}^*c^{(2)}_{\alpha\beta}\, \tilde{H}_\beta \;,$$

and finally,

$${}^*\partial^{(3)}\, {}^*\partial^{(2)}\, {}^*\partial^{(1)}\, \tilde{H}_\mu \;=\; \sum_{\alpha \subseteq_1 \mu} {}^*c^{(1)}_{\mu\alpha} \sum_{\beta \subseteq_2 \alpha} {}^*c^{(2)}_{\alpha\beta} \sum_{\gamma \subseteq_3 \beta} {}^*c^{(2)}_{\beta\gamma}\, \tilde{H}_\gamma \;.$$

Substituting this in 2.48 gives

$$\langle\, e^*_{n-6}e^*_3e^*_2e^*_1 \,,\, \tilde{H}_\mu \,\rangle_* \;=\; \sum_{\alpha \subseteq_1 \mu} {}^*c^{(1)}_{\mu\alpha} \sum_{\beta \subseteq_2 \alpha} {}^*c^{(2)}_{\alpha\beta} \sum_{\gamma \subseteq_3 \beta} {}^*c^{(3)}_{\beta\gamma} \quad \langle\, e^*_{n-6} \,,\, \tilde{H}_\gamma \,\rangle_* \;.$$

Now since γ is necessarily a partition of $n-6$, from 1.13 and 1.14 we derive that for any γ we have

$$\langle\, e^*_{n-6} \,,\, \tilde{H}_\gamma \,\rangle_* \;=\; 1 \;.$$

Thus, in view of the definition of our operators $\mathbf{C}_k$, we may write

$$\langle\, e^*_{n-6}e^*_3e^*_2e^*_1 \,,\, \tilde{H}_\mu \,\rangle_* \;=\; \sum_{\alpha \subseteq_1 \mu} {}^*c^{(1)}_{\mu\alpha} \sum_{\beta \subseteq_2 \alpha} {}^*c^{(2)}_{\alpha\beta} \;. \sum_{\gamma \subseteq_3 \beta} {}^*c^{(3)}_{\beta\gamma} \;=\; \mathbf{C}_1\mathbf{C}_2\mathbf{C}_3\, 1\, |_{[\mu]} \;. \tag{2.50}$$

We should note that this identity reveals the non-trivial fact that these operators do commute when applied to the function that is identically 1.

Thus, rewriting in this manner each of the terms that result from substituting 2.47 into 2.46 we derive that for any $\mu \vdash n \geq 9$ we have

$$\tilde{K}_{(n-6,3,2,1),\mu} = \Big[\mathbf{C}_1\mathbf{C}_2\mathbf{C}_3 - \mathbf{C}_3\mathbf{C}_3 - \mathbf{C}_1\mathbf{C}_1\mathbf{C}_4 + \mathbf{C}_1\mathbf{C}_5 - \mathbf{C}_1\mathbf{C}_2\mathbf{C}_2 + \mathbf{C}_2\mathbf{C}_3 + \mathbf{C}_1\mathbf{C}_1\mathbf{C}_2 - \mathbf{C}_1\mathbf{C}_2 + \mathbf{C}_1\mathbf{C}_4 - \mathbf{C}_5 - \mathbf{C}_1\mathbf{C}_3 + \mathbf{C}_3\Big] 1\,\big|_{[\mu]} \ .$$

This example should be sufficient to convince the reader that we have the following general result.

Theorem 2.5

Let $\lambda = (n-k, \gamma_1, \gamma_2, \ldots, \gamma_r)$ with $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_r)$ a partition of k. Then for any $\mu \vdash n \geq k + \gamma_1$ we have

$$\tilde{K}_{(n-k,\gamma_1,\gamma_2,\ldots,\gamma_r),\mu} \;=\; \Big[\sum_{\sigma \in S_{r+1}} sign(\sigma)\; \mathbf{C}_{\gamma_1+\sigma_2-2}\mathbf{C}_{\gamma_2+\sigma_3-3}\cdots\mathbf{C}_{\gamma_r+\sigma_{r+1}-r-1}\Big]\, 1\,\big|_{[\mu]} \ , \qquad 2.51$$

with the convention that $\mathbf{C}_k$ with a negative k is the zero operator and $\mathbf{C}_o$ is the identity.

Proof

From 2.1 we get

$$\tilde{K}_{(n-k,\gamma_1,\gamma_2,\ldots,\gamma_r),\mu} \;=\; \langle\, S^*_{(n-k,\gamma_1,\gamma_2,\ldots,\gamma_r)'} \,,\, \tilde{H}_\mu \,\rangle_* \ .$$

Thus using the expansion

$$S^*_{(n-k,\gamma_1,\gamma_2,\ldots,\gamma_r)'} \;=\; \sum_{\sigma \in S_{r+1}} sign(\sigma)\; e^*_{n-k+\sigma_1-1} e^*_{\gamma_1+\sigma_2-2} e^*_{\gamma_2+\sigma_3-3} \cdots e^*_{\gamma_r+\sigma_{r+1}-r-1}$$

and the definition of the operators $\,^*\partial^{(k)}$ gives

$$\tilde{K}_{(n-k,\gamma_1,\gamma_2,\ldots,\gamma_r),\mu} = \qquad\qquad 2.52$$
$$= \sum_{\sigma \in S_{r+1}} sign(\sigma) \langle\, e^*_{n-k+\sigma_1-1},\ {}^*\partial^{(\gamma_1+\sigma_2-2)}\ {}^*\partial^{(\gamma_2+\sigma_3-3)} \ldots\ {}^*\partial^{(\gamma_r+\sigma_{r+1}-r-1)} \tilde{H}_\mu \,\rangle_* \ ,$$

where we should use the convention that $\,^*\partial^{(k)}$ with k negative is the zero operator and that $\,^*\partial^{(0)}$ is the identity. But now 2.52 gives 2.51 since by the same sequence of steps we used to derive 2.49 from 2.48 we obtain that

$$\langle\, e^*_{n-k+\sigma_1-1},\ {}^*\partial^{(\gamma_2+\sigma_2-2)}\ {}^*\partial^{(\gamma_3+\sigma_3-3)} \ldots\ {}^*\partial^{(\gamma_r+\sigma_{r+1}-r-1)} \tilde{H}_\mu \,\rangle_* = \mathbf{C}_{\gamma_2+\sigma_2-2}\mathbf{C}_{\gamma_3+\sigma_3-3}\cdots\mathbf{C}_{\gamma_r+\sigma_{r+1}-r-1} 1\big|_{[\mu]} \ .$$

Of course the surprising feature of 2.51 is that the parameter n does not occur explicitly on the right hand side. Thus in the computation of $\tilde{K}_{(n-k,\gamma_1,\gamma_2,\ldots,\gamma_r),\mu}$, the information as to the value of n is carried by the partition μ. However, more surprises are in store as we closely examine further examples.

To begin with note that, since for any $p_1 + p_2 + \cdots + p_r = k$ and $\mu \vdash n \geq k$ we have

$$\mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_r}\, 1\, \big|_{[\mu]} \;=\; \langle\, e^*_{n-k} e^*_{p_1} e^*_{p_2} \cdots e^*_{p_r} \,,\, \tilde{H}_\mu \,\rangle_* \;, \tag{2.53}$$

we see that the function

$$\mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_r}\; 1 \tag{2.54}$$

does not change as we permute the factors. So there is no loss in restricting $(p_1, p_2, \ldots, p_n)$ to be a partition. This given, we see that the first six simplest cases are

$$\mathbf{C}_1\, 1\,, \qquad \mathbf{C}_1\mathbf{C}_1 1\,, \qquad \mathbf{C}_2\, 1\,, \qquad \mathbf{C}_1\mathbf{C}_1\mathbf{C}_1\, 1\,, \qquad \mathbf{C}_2\mathbf{C}_1\, 1\,, \qquad \mathbf{C}_3\, 1\,. \tag{2.55}$$

We shall dedicate the remaining part of this section to working out these cases in full detail and induce from our findings the results we need to prove in the rest of this paper.

To begin with, we can deal with $\mathbf{C}_1 1$, $\mathbf{C}_1\mathbf{C}_1\ 1$ and $\mathbf{C}_1\mathbf{C}_1\mathbf{C}_1 1$ all at the same time by using the following general result.

Theorem 2.6

If the m-rational function f is plethystic and generated by $P[x;q,t]$ then the $m+1$-rational function $g = \mathbf{C}_1 f$ is also plethystic and is generated by , P where , is the linear operator on symmetric polynomials defined by setting

$$,\, P \;=\; \tfrac{1}{(1-t)(1-q)} P + \tfrac{1}{(1-1/t)(1-1/q)}\; P[X-z;q,t]\,\Omega\Big[\tfrac{(1-t)(1-q)X-1}{tqz}\Big]\; \Big|_{z^{-1}} \;. \tag{2.56}$$

In particular, if f is Laurent or pure-Laurent, so is $g = \mathbf{C}_1 f$.

Proof

By assumption, we have for every $\mu \vdash n \geq m+1$

$$g_\mu(q,t) \;=\; \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\, f_\nu(q,t) \;.$$

Now if f is generated by $P(x;q,t)$ then we may rewrite this in the form

$$g_\mu(q,t) \;=\; \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\, P[B_\nu;q,t] \;. \tag{2.57}$$

Thus we need to show that for any P we have

$$\sum_{\nu\to\mu} c_{\mu\nu}(q,t)\, P[B_\nu;q,t] \;=\; Q[B_\mu;q,t] \qquad (\ \textit{with} \quad Q = ,\, P\ )\ . \tag{2.58}$$

However, by linearity, we only need to verify this when $P = S_\lambda$ with λ arbitrary. But since we can write

$$S_\lambda[B_\nu] \;=\; S_\lambda[B_\mu - T_{\mu/\nu}] \;=\; \sum_{\lambda/\rho\in V} S_\rho[B_\mu](-T_{\mu/\nu})^{|\lambda/\rho|}\;,$$

setting $P = S_\lambda$ in the left hand side of 2.58 gives

$$LHS(2.58) = \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\, S_\lambda[B_\nu;q,t] = \sum_{\lambda/\rho\in V} S_\rho[B_\mu](-1)^{|\lambda/\rho|} \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\, T_{\mu/\nu}^{|\lambda/\rho|} \ .$$

Grouping terms according to the size of λ/ρ and using 2.7 and 2.13, we get

$$\begin{aligned} LHS(2.58) &= S_\lambda[B_\mu] \sum_{\nu\to\mu} c_{\mu\nu} + \sum_{k\geq 1} (-1)^k \Big(\sum_{\lambda/\rho\in V_k} S_\rho[B_\mu] \Big) \sum_{\nu\to\mu} c_{\mu\nu}\, T_{\mu/\nu}^k \\ &= S_\lambda[B_\mu]\, B_\mu + \frac{1}{M} \sum_{k\geq 1} (-1)^k \Big(\sum_{\lambda/\rho\in V_k} S_\rho[B_\mu] \Big) h_{k+1}[MB_\mu - 1/tq] \ , \end{aligned}$$

where for convenience we have set

$$M = (1-1/t)(1-1/q) \ . \qquad 2.59$$

Thus 2.58 will be established if we show that

$$S_\lambda[X]\, e_1[X] + \frac{1}{M} \sum_{k\geq 1} (-1)^k \Big(\sum_{\lambda/\rho\in V_k} S_\rho[X] \Big) h_{k+1}[MX - 1/tq] \ = \ , S_\lambda \ . \qquad 2.60$$

Note that from the expansion

$$S_\lambda[X - z] = \sum_{\lambda/\rho\in V} S_\rho[X]\, (-z)^{|\lambda|-|\mu|}$$

we get that

$$(-1)^k \sum_{\lambda/\rho\in V_k} S_\rho[X] \ = \ S_\lambda[X-z]\, |_{z^k} \ . \qquad 2.61$$

Substituting this into the left hand side of 2.60 gives

$$\begin{aligned} LHS(2.60) &= S_\lambda[X]\, e_1[X] + \tfrac{1}{M} \sum_{k\geq 1} S_\lambda[X-z]\, |_{z^k} \ h_{k+1}[MX - 1/tq] \\ &= S_\lambda[X]\, e_1[X] + \tfrac{1}{M} \sum_{k\geq 1} S_\lambda[X-z]\, h_{k+1}[\tfrac{MX-1/tq}{z}]\, |_{z^{-1}} \\ &= S_\lambda[X]\, e_1[X] + \tfrac{1}{M} S_\lambda[X-z] \sum_{k\geq 1} h_{k+1}[\tfrac{MX-1/tq}{z}]\, |_{z^{-1}} \ . \end{aligned}$$

However, we may write

$$\sum_{k\geq 1} h_{k+1}[\tfrac{MX-1/tq}{z}] \ = \ \Omega[\tfrac{MX-1/tq}{z}] \ - \ 1 \ - \ \tfrac{Me_1[X]-1/tq}{z} \ ,$$

and we finally derive that

$$\begin{aligned} LHS(2.60) &= S_\lambda[X]\, e_1[X] + \tfrac{1}{M} S_\lambda[X-z] \Big(\Omega[\tfrac{MX-1/tq}{z}] - 1 - \tfrac{Me_1[X]-1/tq}{z} \Big) |_{z^{-1}} \ . \\ &= S_\lambda[X]\, e_1[X] + \tfrac{1}{M} S_\lambda[X-z]\, \Omega[\tfrac{MX-1/tq}{z}]\, |_{z^{-1}} - S_\lambda[X] e_1[X] + \tfrac{1}{Mtq} S_\lambda[X] \\ &= \tfrac{1}{Mtq} S_\lambda[X] + \tfrac{1}{M} S_\lambda[X-z]\, \Omega[\tfrac{MX-1/tq}{z}]\, |_{z^{-1}} \end{aligned}$$

.

Now we can easily see (recalling 2.59) that this is another way of writing the right-hand side of 2.56 with $P = S_\lambda$. This proves 2.60 and completes the proof of 2.56. We should note that these manipulations show that the operator , may also be written in the form

$$, \; P \;=\; P[X]e_1[X] + \sum_{k\geq 1} P[X-z]\,\big|_{z^k \frac{1}{(1-t)(1-q)t^kq^k}} h_{k+1}[(1-t)(1-q)X-1] \;. \qquad 2.62$$

This follows by substituting 2.61 into 2.60 and extracting the factor $1/tq$ out of $h_{k+1}[MX-1/tq]$. This given, we see from the observations made in Remark 2.2 that the only additional denominator factors introduced by an application of , are powers of qt. This establishes our assertions concerning the plethystic nature of $g = \mathbf{C}_1 f$.

Formula 2.62 is perhaps the easiest to program using Stembridge's "SF" MAPLE package. If we do so, three successive applications of , yield that $\mathbf{C}_1 1$, $\mathbf{C}_1\mathbf{C}_1 1$ and $\mathbf{C}_1\mathbf{C}_1\mathbf{C}_1 1$ are plethystic and respectively represented by the following three symmetric polynomials.

$$\begin{aligned}
\mathbf{\Pi}_1 &= \,,\, 1 = e_1\\
\mathbf{\Pi}_{11} &= \,,\,,\, 1 = \frac{1}{tq}\, e_1 - \frac{1}{tq}\, e_1^2 + b\frac{1}{tq}\, e_2\\
\mathbf{\Pi}_{111} &= \,,\,,\,,\, 1 = \frac{1}{t^2q^2}e_1 - \frac{b-1+2tq}{t^3q^3}\, e_1^2 + \frac{b(b-1+tq)}{t^3q^3}\, e_2 \qquad 2.63\\
&\qquad + \;\frac{b-1+tq}{t^3q^3}\, e_1^3 - \frac{b^2-1+(1-tq)tq}{t^3q^3}\, e_1e_2 + \frac{(b-1)[3]_t[3]_q}{t^3q^3}\, e_3 \;,
\end{aligned}$$

where for convenience we have set $b = (1+t)(1+q)$, $[3]_t = 1+t+t^2$ and $[3]_q = 1+q+q^2$.

Of course this process can be continued to obtain pure-Laurent plethystic formulas for any desired $\mathbf{C}_1^k 1$. However, to obtain a similar result for

$$\mathbf{C}_2\, 1\,\big|_{[\mu]} \;=\; \sum_{\nu\subseteq_2\mu} {}^*c^{(2)}_{\mu\nu}(q,t) \qquad\text{and}\qquad \mathbf{C}_3\, 1\,\big|_{[\mu]} \;=\; \sum_{\nu\subseteq_3\mu} {}^*c^{(3)}_{\mu\nu}(q,t) \qquad 2.64$$

we need altogether another ingredient. Of course, the most natural thing to do is to obtain for each operator $\mathbf{C}_k$ a result analogous to the one we obtained for $\mathbf{C}_1$ (as expressed by Theorem 2.6). Now the difficulties encountered by the first author in carrying this out in the early stages of the investigation led to the discovery that every one of the m-rational functions $\mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}1$ can be expressed by alternating applications of $\mathbf{C}_1$ with multiplication by the symmetric polynomial $e_1[X]$. It develops that the additional ingredient that makes this possible is the operator Δ_1 itself. We can understand how this comes about by working on $\mathbf{C}_2 1$ and $\mathbf{C}_3 1$.

Let us begin by noting that formula 1.29 gives

$$\begin{aligned}
\Delta_1\, e_1^*e_{n-1}^* \;&=\; e_1^*e_{n-1}^* \;-\; (e_1^* + \tfrac{1}{z})(e_{n-1}^* + \tfrac{1}{z}e_{n-2}^*)\Omega[-zX]\,\big|_{z^\circ}\\
&=\; e_1^*e_{n-1}^* \;-\; (e_1^*e_{n-1}^* \;+\; e_1^*\tfrac{1}{z}e_{n-2}^* \;+\; \tfrac{1}{z}e_{n-1}^* \;+\; \tfrac{1}{z^2}e_{n-2}^*)\Omega[-zX]\,\big|_{z^\circ}\\
&=\; e_1^*e_1e_{n-2}^* \;+\; e_1e_{n-1}^* \;-\; e_2e_{n-2}^* \;.
\end{aligned}$$

Expanding the last expression in terms of the e^* basis we finally obtain

$$\tfrac{1}{M}\Delta_1\, e_1^*e_{n-1}^* \;=\; e_1^*e_{n-1}^* \;+\; (1+t+q)\, e_1^*e_1^*e_{n-2}^* \;-\; (1+t)(1+q)\, e_2^*e_{n-2}^* \;,$$

where for convenience we have set $M=(1-t)(1-q)$. We next take the $*$-scalar product of both sides $\tilde{H}_\mu[X;q,t]$ for a given $\mu\vdash n$, getting

$$\begin{aligned}\tfrac{1}{M}\langle\, \Delta_1\, e_1^* e_{n-1}^* \,,\, \tilde{H}_\mu \,\rangle_* \;&=\; \langle\, e_1^* e_{n-1}^* \,,\, \tilde{H}_\mu \,\rangle_* \\ &\qquad + (1+t+q)\, \langle\, e_1^* e_1^* e_{n-2}^* \,,\, \tilde{H}_\mu \,\rangle_* \\ &\qquad\qquad - (1+t)(1+q)\, \langle\, e_2^* e_{n-2}^* \,,\, \tilde{H}_\mu \,\rangle_* \;.\end{aligned}$$

Now the "Cauchy" formula in 1.13 together with 1.30 a) imply that the operator Δ_1 is self-adjoint with respect to the $*$-scalar product. Combining this fact with 1.30 a) and 2.53, we can convert this identity into the following equation:

$$\begin{aligned}B_\mu(q,t)\, \mathbf{C}_1 1\,\big|_{[\mu]} \;&=\; \mathbf{C}_1 1\,\big|_{[\mu]} \\ &\qquad + (1+t+q)\, \mathbf{C}_1\mathbf{C}_1 1\,\big|_{[\mu]} \\ &\qquad\qquad - (1+t)(1+q)\, \mathbf{C}_2\, 1\,\big|_{[\mu]} \;.\end{aligned}$$

Solving for $\mathbf{C}_2 1\,|_{[\mu]}$ yields

$$\mathbf{C}_2\, 1\,\big|_{[\mu]} \;=\; \frac{1}{(1+t)(1+q)} \Big(\, (1-B_\mu)\, \mathbf{C}_1 1\,\big|_{[\mu]} \;+\; (1+t+q)\, \mathbf{C}_1\mathbf{C}_1 1\,\big|_{[\mu]} \,\Big) \;.$$

In view of our results concerning $\mathbf{C}_1 1$ and $\mathbf{C}_1\mathbf{C}_1 1$, we see that this identity shows that $\mathbf{C}_2 1$ is plethystic and is represented by the symmetric polynomial

$$\mathbf{\Pi}_2[X;q,t] \;=\; \frac{1}{(1+t)(1+q)} \Big(\, (1-e_1) \;,\; 1 \;+\; (1+t+q) \;,\,,\; 1 \,\Big) \;.$$

To discover the true nature of this polynomial we need only use the formulas given in 2.63. This gives (recalling that $b=(1+t)(1+q)$)

$$\begin{aligned}\mathbf{\Pi}_2 \;&=\; \frac{1}{(1+t)(1+q)} \Big((1-e_1)\, e_1 \;+\; (1+t+q)\, \Big(\frac{1}{tq}\, e_1 \;-\; \frac{1}{tq}\, e_1^2 + b\frac{1}{tq}\, e_2 \Big)\Big) \\ &=\; \frac{1}{(1+t)(1+q)} \Big(\, e_1\, (1+\frac{1+t+q}{tq}) \;-\; e1^2\, (\, 1+\frac{1+t+q}{tq}) \;+\; \frac{(1+t+q)}{tq}\, b\, e_2 \Big) \\ &=\; \frac{e_1}{tq} \;-\; \frac{e_1^2}{tq} \;+\; \frac{1+t+q}{tq}\, e_2 \;,\end{aligned}$$

and we have witnessed the surprising disappearance of the denominator factor $(1+t)(1+q)$, yielding that $\mathbf{C}_2\, 1$ is pure-Laurent. Moreover, we can also apply Theorem 2.6 and deduce that $\mathbf{C}_1\mathbf{C}_2\, 1$ is plethystic pure-Laurent and represented by the symmetric polynomial

$$\mathbf{\Pi}_{12} \;=\; ,\, (\, \frac{e_1}{tq} \;-\; \frac{e_1^2}{tq} \;+\; \frac{1+t+q}{tq}\, e_2 \,) \;.$$

Carrying out this computation using the SF MAPLE package yields

$$\begin{aligned}\mathbf{\Pi}_{12} \;&=\; \frac{1}{t^2q^2}\, e_1 - \frac{(tq+b-1)}{t^3q^3}\, e_1^2 + \frac{b(b-1)}{t^3q^3}\, e_2 \\ &\qquad + \frac{b-1}{t^3q^3}\, e_1^3 - \frac{b^2-tqb-1}{t^3q^3}\, e_1e_2 + \frac{(t+q)[3]_t[3]_q}{t^3q^3}\, e_3\end{aligned} \qquad 2.65$$

To complete our exploration, we are left with the computation of $\mathbf{C}_3\,1$. In this case we begin with

$$\begin{aligned}\Delta_1\, {e_1^*}^2 e_{n-2}^* &= {e_1^*}^2 e_{n-2}^* - (e_1^* + \tfrac{1}{z})^2 (e_{n-2}^* + \tfrac{1}{z} e_{n-3}^*)\,\Omega[-zX]\,|_{z^o} \\ &= {e_1^*}^2 e_{n-2}^* - \big({e_1^*}^2 + 2\tfrac{1}{z}\, e_1^* + \tfrac{1}{z^2}\big)\, e_{n-2}^*\Omega[-zX]\,|_{z^o} - \big({e_1^*}^2 + 2\tfrac{1}{z}\, e_1^* + \tfrac{1}{z^2}\big) e_{n-3}^*\Omega[-zX]\,|_{z^1} \\ &= \ (2e_1^* e_1 - e_2)\, e_{n-2}^* \ + \ ({e_1^*}^2 e_1 - 2e_1^* e_2 + e_3)\, e_{n-3}^* \ .\end{aligned}$$

Expanding the last expression in terms of the e^* basis, we obtain

$$\begin{aligned}\tfrac{1}{M}\Delta_1\, {e_1^*}^2 e_{n-2}^* = (2+t+q)\, {e_1^*}^2 e_{n-2}^* - b\, e_2^* e_{n-2}^* \\ + \ (b+t+q+t^2+q^2)\, {e_1^*}^3 e_{n-3}^* \\ - \ (3b-1+tq^2+2t^2+2q^2+t^2q)\, e_1^* e_2^* e_{n-3}^* \\ + \ [3]_t[3]_q\, e_3^* e_{n-3}^* \ .\end{aligned}$$

Taking the $*$-scalar product of both sides with $\tilde{H}_\mu$ with $\mu \vdash n$ and using the $*$-self-adjointness of Δ_1 together with 1.30 a) we get the equation

$$\begin{aligned}B_\mu \mathbf{C}_1\mathbf{C}_1\, 1\, |[\mu] = \Big(\, (2+t+q)\, \mathbf{C}_1\mathbf{C}_1\, 1 - b\, \mathbf{C}_2\, 1 \\ + \ (b+t+q+t^2+q^2)\, \mathbf{C}_1\mathbf{C}_1\mathbf{C}_1\, 1 \\ - \ (3b-1+tq^2+2t^2+2q^2+t^2q)\, \mathbf{C}_1\mathbf{C}_2\, 1 \\ + \ [3]_t[3]_q\, \mathbf{C}_3\, 1\,\Big)\, |_{[\mu]} \ .\end{aligned} \qquad 2.66$$

This shows that $\mathbf{C}_3\, 1$ is plethystic and that it is represented by the symmetric polynomial

$$\begin{aligned}\mathbf{\Pi}_3 = \frac{1}{[3]_t[3]_q}\Big((e_1 - 2 - t - q)\mathbf{\Pi}_{11} + b\mathbf{\Pi}_2 \\ - (b+t+q+t^2+q^2)\mathbf{\Pi}_{111} \\ + (3b-1+tq^2+2t^2+2q^2+t^2q)\mathbf{\Pi}_{12}\Big)\end{aligned}$$

Substituting into this the expressions that we have already obtained for $\mathbf{\Pi}_{11}, \mathbf{\Pi}_2, \mathbf{\Pi}_{111}$ and $\mathbf{\Pi}_{12}$ gives that

$$\begin{aligned}\mathbf{\Pi}_3 \ = \ \frac{e_1}{t^2q^2} - \frac{b-1}{t^3q^3}\, e_1^2 + \frac{b(t+q)}{t^3q^3}\, e_2 \\ + \frac{t+q}{t^3q^3}\, e_1^3 - \frac{2(b-1)+t^2+q^2-tq}{t^3q^3}\, e_1e_2 \\ + \frac{b^2 - b(1+tq) - tq(1-tq) + t^3 + q^3}{t^3q^3}\, e_3 \ ,\end{aligned}$$

and we see that the denominator $[3]_t[3]_q$ has again disappeared, yielding once again that our desired m-rational function is not only plethystic but pure-Laurent as well. This circumstance has repeated itself on all the data we have been able to obtain. We are thus led to the following:

Pure Laurent Conjecture

For each $\gamma \vdash k$ we have two symmetric polynomials $\mathbf{\Pi}_\gamma(x;q,t)$ and $\mathbf{k}_\gamma(x;q,t)$ of degree $\leq k$ in x with pure-Laurent coefficients such that for any $\mu \vdash n \geq k$ we have

$$\langle\, e_\gamma^* e_{n-k}^* \,,\, \tilde{H}_\mu \,\rangle_* \;=\; \mathbf{\Pi}_\gamma[B_\mu;q,t] \;, \qquad 2.67$$

while for $\mu \vdash n$ and $n \geq k + max(\gamma)$ we have

$$\tilde{K}_{(n-k,\gamma),\mu}(q,t) \;=\; \mathbf{k}_\gamma[B_\mu;q,t] \;. \qquad 2.68$$

We should begin by noting that the polynomials $k_\gamma[x;q,t]$, as we have seen, may be easily obtained from the $\mathbf{\Pi}_\gamma$'s via the relation in 2.51. So we need only be concerned about the $\mathbf{\Pi}$'s. Now it is shown in [10] that the calculations given above can be extended to a general algorithm for constructing all the polynomials $\mathbf{\Pi}_\gamma$. However, this algorithm, as we have already seen in the examples above, produces denominator factors of the form $1-q^r$ and $1-t^s$, and we have not been able to explain through this approach why these factors disappear when we pass to normal forms. In this paper we have been able to get around this problem by proving a variant of Theorem 2.1 that applies to all the operators $\mathbf{C}_k$. The precise nature of the results we have have obtained is given in the last section. Here it is sufficient to say that we have come pretty close to proving the above conjecture in that we have shown it to be true if *pure-Laurent* is simply replaced by *Laurent*.

Although the algorithm given in the last section differs considerably from the one we have sketched above, we can show that it yields exactly the same polynomials. This is a consequence of the following basic fact.

Theorem 2.7

Let $P(x;q,t)$ be a symmetric polynomial of degree d in x with coefficients rational functions of q and t. Then the equalities

$$P[B_\mu(q,t),q,t] \;=\; 0 \qquad (\;\forall\;\; |\mu| \leq d\;) \qquad 2.69$$

force P to vanish identically.

Proof

For any partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_d \geq 0)$ set

$$D_\mu(q,t) \;=\; \sum_{i=1}^{d} t^{d-i}\, q^{\mu_i} \;. \qquad 2.70$$

Note that from I.6 we derive that

$$B_\mu(q,t) \;=\; \tfrac{1-t^d}{(1-t)(1-q)} - \tfrac{1-t^{d-1}}{1-q}\, D_\mu(q,1/t) \;.$$

Thus, from 2.69 we derive that the polynomial

$$Q[Y;q,t] \;=\; P[\tfrac{1-t^d}{(1-t)(1-q)} - \tfrac{1-t^{d-1}}{1-q}\, Y;q,t]$$

satisfies the equalities

$$Q[D_\mu(q,1/t);q,t] \;=\; 0 \qquad (\;\forall\;|\mu|\le d\;)\;.^{(\dagger)} \tag{2.71}$$

Since Q is also symmetric and of degree d in y it has an expansion of the form

$$Q(y;q,t) \;=\; \sum_{|\lambda|\le d} m_\lambda(y)\, c_\lambda(q,t)$$

with coefficients $c_\lambda(q,t)$ rational functions of q and t. This given, the equalities in 2.71 will force Q and P as well to be zero if we can show the nonsingularity of the matrix

$$\|\, m_\lambda[D_\mu(q,1/t)]\,\|_{|\lambda|,|\mu|\le d}\;. \tag{2.72}$$

Now the nonsingularity of the matrix

$$\|\, m_\lambda[D_\mu(q,t)]\,\|_{|\lambda|,|\mu|\le d} \tag{2.73}$$

can be established by means of an argument used by Macdonald in an analogous situation (see [14] p. 334). We sketch it here for sake of completion. The first step is to recognize that the term of highest t-degree in $m_\lambda[D_\mu(q,1/t]$ is

$$t^{\langle\mu,\delta\rangle}\; q^{\langle\mu,\lambda\rangle}\;,$$

where for convenience we have set

$$\delta=(d-1,d-2,\ldots,1,0) \qquad \text{and} \qquad \langle a,b\rangle = a_1b_1+a_2b_2+\cdots+a_db_d\;.$$

Thus it is sufficient to show that

$$det\;\|\,q^{\langle\mu,\lambda\rangle}\|_{|\lambda|,|\mu|\le d}\;\ne\;0\;. \tag{2.74}$$

Now, it develops that given any collection of integral vectors $v^{(1)}, v^{(2)},\ldots,v^{(m)}$, the highest degree term in the polynomial

$$det\;\|\,q^{\langle v^{(i)},v^{(j)}\rangle}\|_{1\le i,j\le m} \;=\; \sum_{\sigma\in S_m} sign(\sigma)\; q^{\langle v^{(\sigma_1)},v^{(1)}\rangle+\langle v^{(\sigma_2)},v^{(2)}\rangle+\cdots+\langle v^{(\sigma_m)},v^{(m)}\rangle}$$

is given by the term corresponding to $\sigma=id$, namely

$$q^{\langle v^{(1)},v^{(1)}\rangle+\langle v^{(2)},v^{(2)}\rangle+\cdots+\langle v^{(m)},v^{(m)}\rangle}\;.$$

This is because the strict inequality (see [14] Ex. 1 p. 341)

$$\langle v^{(1)},v^{(1)}\rangle+\langle v^{(2)},v^{(2)}\rangle+\cdots+\langle v^{(m)},v^{(m)}\rangle \;>\; \langle v^{(\sigma_1)},v^{(1)}\rangle+\langle v^{(\sigma_2)},v^{(2)}\rangle+\cdots+\langle v^{(\sigma_m)},v^{(m)}\rangle$$

$^{(\dagger)}$ We tacitly assume here that the partitions $|\mu|\le d$ are expressed as vectors with d greater or equal to zero components.

valid for $\sigma \neq id$ shows that all the other terms in the expansion of this determinant produce lesser powers of q. In particular this shows that the determinant in 2.74 cannot vanish identically. This establishes the nonsingularity of the matrix in 2.73 as well as that in 2.72, completing the proof.

The following corollary will be needed in the sequel.

Theorem 2.8

If $R(x;q,t)$ is a symmetric polynomial in x with coefficients rational functions of q and t and for all but a finite number of μ we have

$$R[B_\mu(q,t),q,t] \;=\; 0 \;, \qquad 2.75$$

then R vanishes identically.

Proof

Choose m large enough to assure that 2.75 holds true for all $|\mu| \geq m$. Note that we must then have

$$e_m[B_\mu(q,t)]R[B_\mu(q,t),q,t] \;=\; 0 \qquad (\;\forall\mu\;) \qquad 2.76$$

This is because $e_m[B_\mu(q,t)] = 0$ when $|\mu| < m$ and $R[B_\mu(q,t),q,t] = 0$ when $|\mu| \geq m$. Thus the hypotheses of Theorem 2.7 are satisfied for

$$P(x;q,t) = e_m(x)R(x;q,t)$$

and with $d = m + degree\ R$. This gives that P and therefore also R must vanish identically as asserted.

Remark 2.3

We should point out that the assumption that $R(x;q,t)$ be a polynomial is essential for uniqueness. In fact, the function $e_1[X]\ \Omega[-X]$ is not identically 0, and has infinite degree, but it does vanish at all $X = B_\mu$. To see this, note that if $\mu = \emptyset$ then $B_\emptyset = 0$ so $e_1[B_\emptyset] = 0$, while if μ is any other partition, then $\Omega[-B_\mu] = \prod_{s\in\mu}(1 - q^{a'}t^{l'})$ is 0 because the bottom leftmost cell of μ has $a' = l' = 0$.

3. Plethystic formulas for higher order Pieri coefficients. .

This section is dedicated to the proof of Theorem I.3. To this end, our first goal will be the establishment of higher order analogues of Theorems 2.2 and 2.6. We recall that we defined ${}^{+}\partial^{(r)}$ to be the adjoint of multiplication by $h_r\left[\frac{X}{1-t}\right]$ with respect to the scalar product $\langle\ ,\ \rangle_*$, that is, the operator ${}^{+}\partial^{(r)}$ for which

$$\left\langle h_r\left[\tfrac{X}{1-t}\right]P\ ,\ Q\right\rangle_* \;=\; \left\langle P\ ,\ {}^{+}\partial^{(r)}Q\right\rangle_* \;. \qquad 3.1$$

We have also set for a given partition μ

$${}^{+}\partial^{(r)}\tilde{H}_\mu(x;q,t) \;=\; \sum_{\mu/\nu\in V_r} {}^{+}c^{(r)}_{\mu\nu}(q,t)\ \tilde{H}_\nu(x;q,t) \;. \qquad 3.2$$

We have to work here with sums of the form

$$g_\mu(q,t) = \sum_{\nu\,:\,\mu/\nu\in V_r} {}^{+}c^{(r)}_{\mu\nu}(q,t)\; f_\nu(q,t) \ .$$

To be consistent with the notation introduced in the last section, and to take account of the fact that we are working with the coefficients ${}^{+}c^{(r)}_{\mu\nu}$, we should be writing this as

$$g = {}^{+}\mathbf{C}_r\; f \qquad \text{or} \qquad g_\mu = {}^{+}\mathbf{C}_r\; f\,|_{[\mu]} \ .$$

However, we shall systematically only work with plethystic functions

$$f_\mu(q,t) = P[B_\mu(q,t);q,t] \ ,$$

with P a symmetric polynomial with coefficients Laurent in q,t, and it is more convenient to view ${}^{+}\mathbf{C}_r$ as a linear operator acting on P rather than on f. Therefore we shall write instead

$$g_\mu(q,t) \;\; = \;\; {}^{+}\mathbf{C}_r\; P\,|_{[\mu]}$$

as a short hand for

$$g_\mu(q,t) = \sum_{\nu\,:\,\mu/\nu\in V_r} {}^{+}c^{(r)}_{\mu\nu}(q,t)\; P[B_\nu(q,t);q,t] \ . \qquad 3.3$$

Now in complete analogy with our derivation of Theorem I.1 from Theorem 2.2, we can reduce the computations of these sums to sums of the form

$$\tilde g_\mu(q,t) = \sum_{\nu\,:\,\mu/\nu\in V_r} {}^{+}c^{(r)}_{\mu\nu}(q,t)\; P[B_{\mu/\nu}(q,t);q,t] \ , \qquad 3.4$$

where we have set

$$B_{\mu/\nu}(q,t) \;\; = \;\; B_\mu(q,t) - B_\nu(q,t) \ . \qquad 3.5$$

This given, we shall write 3.4 as

$$\tilde g_\mu(q,t) \;\; = \;\; {}^{+}\widetilde{\mathbf{C}}_r\; P\,|_{[\mu]} \ , \qquad 3.6$$

and consider ${}^{+}\widetilde{\mathbf{C}}_r$ as a linear operator acting on symmetric polynomials and yielding m-rational functions.

Comparing 3.4 with the sum appearing in 2.13, we see that $T_{\mu/\nu} = B_{\mu/\nu}$ for $r=1$ and, taking an arbitrary k^{th} power of $T_{\mu/\nu}$, is the one-variable case of plethystic evaluation of an arbitrary symmetric polynomial P at $B_{\mu/\nu}$. Adopting the usual convention of denoting the Kostka-Foulkes polynomials by $K_{\lambda\mu}(t)$, our basic result concerning the sums in 3.3 may be stated as follows:

Theorem 3.1

Let $\widetilde{\Gamma}_r$ be the linear operator on symmetric polynomials defined by setting for the Schur function basis

$$\widetilde{\Gamma}_r S_\lambda \;\; = \;\; t^{\binom{r}{2}} \sum_{\substack{\rho\,:\,\rho/\lambda\in V \\ l(\rho)\le r}} (-qt^{1-r})^{|\rho/\lambda|} \sum_{\kappa\,:\,|\kappa|=|\rho|} S_\kappa\Big[\,(1-q^{-1})X + q^{-1}[r]_t\,\Big] \sum_{\substack{\gamma\,:\,|\gamma|=|\rho| \\ l(\gamma)\le r}} K_{\rho\gamma}(t^{-1}) K^{-1}_{\gamma\kappa}(t^{-1}) \ . \qquad 3.7$$

Then for any symmetric polynomial $P = P(x;q,t)$ *we have*

$$\sum_{\nu\,:\,\mu/\nu\in V_r} {}^{+}c_{\mu\nu}^{(r)}(q,t)\; P[B_{\mu/\nu}(q,t);q,t] \;=\; (\tilde{,}_r\, P)\big[B_\mu\big]\ . \tag{3.8}$$

We should note that if $l(\lambda)\le r$*, then* $\tilde{,}_r S_\lambda$ *is in general a nonhomogeneous symmetric polynomial of degree* $|\lambda|+r$*, and if* $l(\lambda)>r$*, then* $\tilde{,}_r S_\lambda = 0$*.*

The proof of this theorem will be the culmination of a sequence of results, some of which have rather intricate proofs. In order not to get mired into technicalities and lose sight of the line of reasoning we have arranged our presentation so that results of independent interest come first and difficult but purely auxiliary identities come last. But before proceeding with our arguments it will be good to prove two immediate and most important consequences of Theorem 3.1.

Theorem 3.2

Let ${}^{+},_r$ *be the linear operator defined by setting*

$$^{+},_r S_\lambda \;=\; \sum_{\rho\subseteq\lambda}(-1)^{|\rho|}S_{\lambda/\rho}\cdot \tilde{,}_r S_{\rho'}\ . \tag{3.9}$$

Then, on the validity of Theorem 3.1, for any symmetric polynomial $P = P(x;q,t)$ *we have*

$$\sum_{\nu\,:\,\mu/\nu\in V_r} {}^{+}c_{\mu\nu}^{(r)} P[B_\nu] \;=\; ({}^{+},_r\, P)\big[B_\mu\big]\ , \tag{3.10}$$

and the polynomial $Q = {}^{+},_r\,P$ *is the unique finite degree symmetric function satisfying* $Q[B_\mu] = {}^{+}\mathbf{C}_r P\big|_{[\mu]}$*.*

Proof

By linearity, we need only verify 3.10 for $P = S_\lambda$. This given, we have

$$\begin{aligned}
{}^{+}\mathbf{C}_r S_\lambda\big|_{[\mu]} &= \sum_{\nu\,:\,\mu/\nu\in V_r} {}^{+}c_{\mu\nu}^{(r)} S_\lambda[B_\nu] \;=\; \sum_{\nu\,:\,\mu/\nu\in V_r} {}^{+}c_{\mu\nu}^{(r)} S_\lambda\big[B_\mu - B_{\mu/\nu}\big] \\
&= \sum_{\nu\,:\,\mu/\nu\in V_r} {}^{+}c_{\mu\nu}^{(r)} \sum_{\rho\subseteq\lambda} S_{\lambda/\rho}[B_\mu] S_\rho\big[-B_{\mu/\nu}\big] \\
&= \sum_{\rho\subseteq\lambda}(-1)^{|\rho|}S_{\lambda/\rho}[B_\mu] \sum_{\nu\,:\,\mu/\nu\in V_r} {}^{+}c_{\mu\nu}^{(r)} S_{\rho'}\big[B_{\mu/\nu}\big] = \sum_{\rho\subseteq\lambda}(-1)^{|\rho|}S_{\lambda/\rho}[B_\mu]\cdot {}^{+}\tilde{\mathbf{C}}_r S_{\rho'}\big|_{[\mu]}\ .
\end{aligned}$$

Thus from 3.8 we deduce that

$$S_\lambda\big|_{[\mu]} \;=\; \left(\sum_{\rho\subseteq\lambda}(-1)^{|\rho|}S_{\lambda/\rho}[X]\tilde{,}_r S_{\rho'}[X]\right)\Big|_{X=B_\mu}\ ,$$

and this is 3.10 for $P = S_\lambda$. Finally, the uniqueness follows from Theorem 2.7.

Likewise, on the validity of Theorem 3.1 we derive

Theorem 3.3

If $P[X]$ is a symmetric polynomial in X and a Laurent polynomial in q and t, then so are $\widetilde{,}_r P[X]$ and $^+,_r P[X]$. In other words, the operators $^+\widetilde{\mathbf{C}}_r$ and $^+\mathbf{C}_r$ send plethystic Laurent m-rational functions into plethystic Laurent $(m+r)$-rational functions.

Proof

The only denominators introduced by 3.7 are powers of t and q (noting that $K(t)$ is a unitriangular matrix of polynomials, see [14] (III.6, p. 243), and 3.9 does not introduce any further denominators. Linearly extending these to all symmetric functions introduces no further denominators.

We should note that, once we have established Theorem 3.1, by substituting 3.7 into 3.9 we obtain an explicit albeit intricate formula for the coefficients $\phi_{\rho\lambda}(q,t)$ in I.18. In particular we see that Theorem 3.3 together with Theorem 2.7 would then complete the proof of Theorem I.3.

The point of departure in the proof of Theorem 3.1 is a plethystic formula for the coefficients $^+c^{(r)}_{\mu\nu}$. To state it we need to introduce some notation. Let n be a fixed nonnegative integer. Throughout this section, we will consider partitions μ of length at most n; many of the formulas derived will be expressed in terms of the auxiliary parameter n, but they are stabilized so that their values are actually independent of n, provided the partitions involved have length at most n.

Although ultimately we are to determine symmetric functions plethystically at $B_\mu = B_\mu(q,t)$, some expressions will more naturally be expressed in terms of the generator of the weights of the ends of the first n rows of μ, namely

$$Z_{n,\mu} \;=\; \sum_{i=1}^{n} q^{\mu_i-1}t^{i-1} \;.$$

The relation of this to B_μ may be stated as follows.

Lemma 3.1

For all $n \geq l(\mu)$, we have $Z_{n,\mu} = (1-q^{-1})B_\mu + q^{-1}\sum_{i=1}^{n} t^{i-1}$.

Proof

As long as $n \geq l(\mu)$,

$$\begin{aligned} B_\mu \;&=\; \sum_{i=1}^{n} t^{i-1}(1+q+\cdots+q^{\mu_i-1}) \;=\; \sum_{i=1}^{n} t^{i-1}\frac{q^{\mu_i}-1}{q-1} \\ &=\; \frac{q}{q-1}\sum_{i=1}^{n} t^{i-1}q^{\mu_i-1} - \frac{1}{q-1}\sum_{i=1}^{n} t^{i-1} \;=\; \frac{1}{1-q^{-1}}Z_{n,\mu} - \frac{q^{-1}}{1-q^{-1}}\sum_{i=1}^{n} t^{i-1} \;, \end{aligned}$$

and solving for $Z_{n,\mu}$ yields the stated result.

We therefore define

$$Z_n = (1-q^{-1})X + q^{-1}\sum_{i=1}^{n} t^{i-1} \qquad 3.11$$

for arbitrary X. Plethystic expressions in terms of either X or Z_n can always be transformed to expressions in terms of the other by means of this equation, and when $X = B_\mu$ with $l(\mu) \leq n$, we may interpret Z_n as the n letter alphabet $z_1 + \cdots + z_n$ where $z_i = q^{\mu_i - 1} t^{i-1}$.

It will be convenient to specify vertical strips μ/ν by the set of rows of μ from which the final square is removed to obtain ν, that is,

$$I_{\mu/\nu} \;=\; \{\, i \,:\, \mu_i > \nu_i \,\} \;.$$

With this notation, the biexponent generator of a vertical strip is

$$B_{\mu/\nu}(q,t) \;=\; \sum_{i \in I_{\mu/\nu}} z_i \;.$$

We'll consider arbitrary subsets I of $\{1, \ldots, n\}$, not all of which correspond to vertical strips, and their complements $I^c = \{1, \ldots, n\} \setminus I$. Using Macdonald's notation, for any $y = (y_1, \ldots, y_n)$ and any r-subset I , we define

$$A_I(y;t) \;=\; t^{\binom{r}{2}} \prod_{i \in I, j \notin I} \frac{t\, y_i - y_j}{y_i - y_j}$$

and set

$$Y_I \;=\; \sum_{i \in I} y_i \;.$$

We have the following basic formula.

Theorem 3.4

Let μ/ν be a vertical strip and let $I = I_{\mu/\nu}$. Then for all $n \geq l(\mu)$, we have (on specializing Z_n to $Z_{n,\mu}$)

$$^{+}c^{(r)}_{\mu\nu} \;=\; A_I(Z_n;t)\Omega\big[-qt^{1-n} Z_I\big] \;. \qquad 3.12$$

The proof of 3.12 involves manipulating identities given by Macdonald in [14] and since it is quite intricate we postpone it to the end of the section.

Remark 3.1

We should note that the expression in 3.12 is actually independent of n as long as $n \geq l(\mu)$. This follows from the n-free expression for $^{+}c^{(r)}_{\mu\nu}$ given in 3.19 (below), but can also be seen directly. In fact, since $l(\mu) < n+1$, we have $n+1 \notin I$ and $z_{n+1} = q^{-1} t^n$. Thus

$$\frac{A_I(Z_{n+1};t)}{A_I(Z_n;t)} \;=\; \prod_{i \in I} \frac{tz_i - z_{n+1}}{z_i - z_{n+1}} \;=\; \prod_{i \in I} \frac{1 - tz_i/z_{n+1}}{1 - z_i/z_{n+1}} \;=\; \Omega\Big[\tfrac{1-t}{z_{n+1}} Z_I\Big] \;=\; \frac{\Omega\big[-qt^{1-n-1} Z_I\big]}{\Omega\big[-qt^{1-n} Z_I\big]} \;.$$

So the independence follows.

This theorem motivates us to define

$$^{+}c^{(r)}_{I} = A_I(Z_n;t)\Omega\big[-qt^{1-n} Z_I\big]$$

for any subset I of $\{1,\ldots,n\}$. The following fact, already noticed by Macdonald in [14] (Ch. VI.6 p. 332), will considerably simplify the evaluation of some of our sums.

Lemma 3.2

Let n and μ be given, with $n \geq l(\mu)$. If $I \subset \{1,\ldots,n\}$ is not of the form $I_{\mu/\nu}$ for any vertical strip μ/ν, then ${}^{+}c_I^{(r)} = 0$.

Proof

For such a subset I, there is $i \in I$ with $i+1 \notin I$ such that $\mu_i = \mu_{i+1}$. Thus, $t\, z_i = z_{i+1}$. Since the product expansion of $A_I(Z_n;t)$ contains the factor $t\, z_i - z_{i+1} = 0$, the result follows.

In view of 3.12 the next step in proving Theorem 3.1 is to determine how to perform summations involving $A_I(Z_n;t)$. To this end, we have the proposition given below, which should also be of independent interest.

Let us recall that in [14] (Ch. VI.3) Macdonald defines an operator that in plethystic notation may be written

$$D_n^r(q,t)\, P[X] \;=\; \sum_{|I|=r} A_I(x;t) P[X+(q-1)X_I] \;,$$

where X is an n letter alphabet and for any r-subset I of $\{1,\ldots,n\}$, we set $X_I = \sum_{i\in I} x_i$. A generating function for these operators is

$$D_n(u;q,t) \;=\; \sum_{k=0}^{r} u^r\, D_n^r(q,t) \;,$$

where we have used u in place of Macdonald's X because we use X to denote our alphabet. He shows [VI.4, p. 324] that for $l(\lambda) \leq n$, we have

$$D_n(u;q,t) P_\lambda(x;q,t) \;=\; \prod_{i=1}^{n} (1+u\, q^{\lambda_i} t^{n-i})\; P_\lambda(x;q,t) \;.$$

By taking the coefficient of u^r on both sides, we obtain

$$D_n^r(q,t) P_\lambda(x;q,t) \;=\; e_r\left[\textstyle\sum_{i=1}^n q^{\lambda_i} t^{n-i}\right]\, P_\lambda(x;q,t) \;. \qquad 3.13$$

This given, we have

Proposition 3.1 *For all r, n, λ with $r \leq n$,*

$$\sum_{|I|=r} A_I(x;t) P_\lambda\big[X_I;0,t^{-1}\big] \;=\; t^{\binom{r}{2}+(n-r)l(\lambda)} \begin{bmatrix} n-l(\lambda) \\ r-l(\lambda) \end{bmatrix}_t P_\lambda\big[X;0,t^{-1}\big] \;. \qquad 3.14$$

Furthermore, this sum is 0 when $r < l(\lambda)$. The same holds when P_λ is replaced by Q_λ or J_λ on both sides of the equation.

Proof

By I.2 it suffices to prove the result for P_λ, as Q_λ and J_λ are each scalar multiples of P_λ. When $n < l(\lambda)$, also $r < l(\lambda)$, so that P_λ is evaluated at fewer than $l(\lambda)$ variables on both sides of 3.12. Thus both sides of the equation are 0.

So take $n \geq l(\lambda)$. If $r < l(\lambda)$, then $P_\lambda\big[X_I; 0, t^{-1}\big] = 0$ for every term in the summation, and again the summation is 0. The right hand side of the equation is also 0 if we observe the usual convention that the t-binomial coefficient is 0 when its upper number is nonnegative and its lower number is negative.

Finally, take $l(\lambda) \leq r \leq n$. Then, using 3.13, we have

$$e_r\big[\textstyle\sum_{i=1}^n q^{\lambda_i} t^{n-i}\big] P_\lambda[X;q,t] \;=\; D_n^r(q,t) P_\lambda[X;q,t] \;=\; \displaystyle\sum_{|I|=r} A_I(x;t) P_\lambda[X+(q-1)X_I;q,t] \;.$$

Set $q = 0$ to obtain

$$e_r\Big[\textstyle\sum_{i=l(\lambda)+1}^n t^{n-i}\Big] P_\lambda[X;0,t] \;=\; \displaystyle\sum_{|I|=r} A_I(x;t) P_\lambda[X-X_I;0,t] \;=\; \sum_{|I|=r} A_I(x;t) P_\lambda[X_{I^c};0,t] \;.$$

Replacing t by t^{-1}, and noting that

$$A_I(x;t^{-1}) = t^{-\binom{n}{2}} A_{I^c}(x;t) \;,$$

we derive that

$$\begin{aligned} e_r\Big[\textstyle\sum_{i=l(\lambda)+1}^n t^{i-n}\Big] P_\lambda\big[X;0,t^{-1}\big] \;&=\; \displaystyle\sum_{|I|=r} A_I(x;t^{-1}) P_\lambda\big[X_{I^c};0,t^{-1}\big] \\ &=\; t^{-\binom{n}{2}} \sum_{|I|=r} A_{I^c}(x;t) P_\lambda\big[X_{I^c};0,t^{-1}\big] \;. \end{aligned} \qquad 3.15$$

Replace I by I^c and then r by $n-r$ to obtain

$$e_{n-r}\Big[\textstyle\sum_{i=l(\lambda)+1}^n t^{i-n}\Big] P_\lambda\big[X;0,t^{-1}\big] \;=\; t^{-\binom{n}{2}} \displaystyle\sum_{|I|=r} A_I(x;t) P_\lambda\big[X_I;0,t^{-1}\big] \;. \qquad 3.16$$

Now the well known identities

$$e_k\Big[\textstyle\sum_{i=0}^{n-1} t^i\Big] \;=\; t^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_t \qquad \text{and} \qquad e_k\Big[\textstyle\sum_{i=0}^{n-1} t^{-i}\Big] \;=\; t^{\binom{k}{2}-k(n-1)} \begin{bmatrix} n \\ k \end{bmatrix}_t \qquad 3.17$$

give

$$\begin{aligned} e_{n-r}\Big[\textstyle\sum_{i=l(\lambda)+1}^n t^{i-n}\Big] \;&=\; e_{n-r}\Big[\textstyle\sum_{i=0}^{n-l(\lambda)-1} t^{-i}\Big] \\ &=\; t^{\binom{n-r}{2}-(n-r)(n-l(\lambda)-1)} \begin{bmatrix} n-l(\lambda) \\ n-r \end{bmatrix}_t \\ &=\; t^{\binom{n-r}{2}-(n-r)(n-l(\lambda)-1)} \begin{bmatrix} n-l(\lambda) \\ r-l(\lambda) \end{bmatrix}_t \;. \end{aligned}$$

Substituting this into 3.16, we obtain

$$t^{\binom{n-r}{2}-(n-r)(n-l(\lambda)-1)} \begin{bmatrix} n-l(\lambda) \\ r-l(\lambda) \end{bmatrix}_t P_\lambda\big[X;0,t^{-1}\big] \;=\; t^{-\binom{n}{2}} \sum_{|I|=r} A_I(x;t) P_\lambda\big[X_I;0,t^{-1}\big] \;,$$

and collecting and simplifying the powers of t yields

$$\begin{aligned}\sum_{|I|=r} A_I(x;t)P_\lambda[X_I;0,t^{-1}] &= t^{\binom{n}{2}+\binom{n-r}{2}-(n-r)(n-l(\lambda)-1)}\begin{bmatrix}n-l(\lambda)\\ r-l(\lambda)\end{bmatrix}_t P_\lambda[X;0,t^{-1}]\\ &= t^{\binom{r}{2}+(n-r)l(\lambda)}\begin{bmatrix}n-l(\lambda)\\ r-l(\lambda)\end{bmatrix}_t P_\lambda[X;0,t^{-1}]\ ,\end{aligned}$$

as desired.

Remark 3.2

For sake of completeness, we should add that the identity in 3.15 is shown as follows.

$$\begin{aligned}A_I(x;t^{-1}) &= t^{-\binom{r}{2}}\prod_{i\in I,j\notin I}\frac{t^{-1}x_i-x_j}{x_i-x_j}\\ &= t^{-\binom{r}{2}-|I|\cdot|I^c|}\prod_{i\in I,j\notin I}\frac{x_i-tx_j}{x_i-x_j}\\ &= t^{-\binom{r}{2}-r(n-r)-\binom{n-r}{2}}A_{I^c}(x;t)\\ &= t^{-\binom{n}{2}}A_{I^c}(x;t)\end{aligned}$$

As for the identities in 3.17 we note that in $e_k\left[\sum_{i=0}^{n-1}t^i\right]$, the coefficient of t^m is the number of sequences $0\le i_1<\cdots<i_k\le n-1$ with $i_1+\cdots+i_k=m$. Let $\lambda_j=i_j-(j-1)$. Then this is the number of sequences $0\le\lambda_1\le\lambda_2\le\cdots\le\lambda_k\le n-k$ with $\lambda_1+\cdots+\lambda_k=m-\sum_{i=1}^{k}(i-1)=m-\binom{k}{2}$, that is, the number of partitions of $m'=m-\binom{k}{2}$ into at most k parts, each of which is at most $n-k$. It is well-known [15] (pp. 29–30, Prop. 1.3.19) that the coefficient of $t^{m'}$ in $\begin{bmatrix}n\\ k\end{bmatrix}_t$ is the number of partitions of m' into at most k parts with sizes at most $n-k$; multiplying this by $t^{\binom{k}{2}}$ yields the first equation in 3.17.

The second equation in 3.17 follows from the first because the argument of e_k in the second is a fraction $t^{-(n-1)}$ of the argument in the first, and e_k is homogeneous of degree k.

We are now ready to perform the sums needed to prove Theorem 3.1. We do this by computing the action of ${}^{+}\widetilde{\mathbf{C}}_r$ on Schur functions, and then extend it to all symmetric functions by linearity. The following result determines a generating function for this action.

Theorem 3.5

Let $r\le n$. For all partitions μ with $l(\mu)\le n$,

$$\begin{aligned}&\sum_\lambda {}^{+}\widetilde{\mathbf{C}}_r S_\lambda[X]\,\big|_{[\mu]}\, S_\lambda[(1-t^{-1})Y]\\ &\quad= t^{\binom{r}{2}}\sum_\gamma Q_\gamma\left[Y-\tfrac{qt^{1-n}}{1-t^{-1}};0,t^{-1}\right]t^{(n-r)l(\gamma)}\begin{bmatrix}n-l(\gamma)\\ r-l(\gamma)\end{bmatrix}_t P_\gamma[Z_{n,\mu};0,t^{-1}]\ .\end{aligned}\qquad 3.18$$

Proof

We may rewrite the sum on the left hand side of 3.18 in the form

$$\sum_\lambda {}^{+}\widetilde{\mathbf{C}}_r S_\lambda[X]\,\big|_{[\mu]}\, S_\lambda[(1-t^{-1})Y] \;=\; {}^{+}\widetilde{\mathbf{C}}_r \Omega\big[(1-t^{-1})XY\big]\big|_{[\mu]}\ ,$$

with the convention that the operator ${}^{+}\widetilde{\mathbf{C}}_r$ acts on $\Omega\big[(1-t^{-1})XY\big]$ as a symmetric function in the x-variables. This given, by Lemma 3.2, we have (with $Z_n = Z_{n,\mu}$)

$${}^{+}\widetilde{\mathbf{C}}_r P\big|_{[\mu]} \;=\; \sum_{\nu\,:\,\mu/\nu\in V_r} {}^{+}c^{(r)}_{\mu\nu} P[B_{\mu/\nu}] \;=\; \sum_{|I|=r} {}^{+}c^{(r)}_I P[Z_I]$$

whenever $n \geq r$ and $l(\mu) \leq n$. So we evaluate the summation using this alternate form.

$$\begin{aligned}
{}^{+}\widetilde{\mathbf{C}}_r \Omega\big[(1-t^{-1})XY\big]\big|_{[\mu]} &= \sum_{|I|=r} {}^{+}c^{(r)}_I \Omega\big[(1-t^{-1})Z_I Y\big] \\
&= \sum_{|I|=r} A_I(Z_n;t)\Omega\big[-qt^{1-n}Z_I\big]\,\Omega\big[(1-t^{-1})Z_I Y\big] \\
&= \sum_{|I|=r} A_I(Z_n;t)\Omega\Big[(1-t^{-1})Z_I\left(Y - \tfrac{qt^{1-n}}{1-t^{-1}}\right)\Big]
\end{aligned}$$

Now we use the Hall-Littlewood Cauchy formula $\Omega\big[(1-t^{-1})XY\big] = \sum_\gamma Q_\gamma[Y;0,1/t]P_\gamma[X;0,1/t]$ with X replaced by Z_I and Y replaced by $Y - \frac{qt^{1-n}}{1-t^{-1}}$, obtaining

$$\begin{aligned}
{}^{+}\widetilde{\mathbf{C}}_r \Omega\big[(1-t^{-1})XY\big]\big|_{[\mu]} &= \sum_{|I|=r} A_I(Z_n;t)\sum_\gamma Q_\gamma\left[Y - \tfrac{qt^{1-n}}{1-t^{-1}};0,t^{-1}\right]P_\gamma\left[Z_I;0,t^{-1}\right] \\
&= \sum_\gamma Q_\gamma\left[Y - \tfrac{qt^{1-n}}{1-t^{-1}};0,t^{-1}\right]\sum_{|I|=r} A_I(Z_n;t)P_\gamma\left[Z_I;0,t^{-1}\right]\ .
\end{aligned}$$

The identity in 3.14 gives

$${}^{+}\widetilde{\mathbf{C}}_r \Omega\big[(1-t^{-1})XY\big]\big|_{[\mu]} = \sum_\gamma Q_\gamma\left[Y - \tfrac{qt^{1-n}}{1-t^{-1}};0,t^{-1}\right]t^{(n-r)l(\gamma)+\binom{r}{2}}\begin{bmatrix} n-l(\gamma) \\ r-l(\gamma)\end{bmatrix}_t P_\gamma\left[Z_n;0,t^{-1}\right]\ ,$$

which is what we wanted to prove.

We are finally in a position to give our

Proof of Theorem 3.1

We have

$$\begin{aligned}
{}^{+}\widetilde{\mathbf{C}}_r \Omega\big[(1-t^{-1})XY\big]\big|_{[\mu]} &= {}^{+}\widetilde{\mathbf{C}}_r \sum_\lambda S_\lambda[X]S_\lambda[(1-t^{-1})Y]\big|_{[\mu]} \\
&= \sum_\lambda {}^{+}\widetilde{\mathbf{C}}_r S_\lambda[X]\big|_{[\mu]}\cdot S_\lambda[(1-t^{-1})Y]\ .
\end{aligned}$$

Comparing this with the expansion of ${}^{+}\widetilde{\mathbf{C}}_r \Omega\big[(1-t^{-1})XY\big]$ in 3.13, we see that we must expand the Hall-Littlewood functions $Q_\gamma\left[Y - \frac{qt^{1-n}}{1-t^{-1}};0,t^{-1}\right]$ in the basis $\big\{S_\rho[(1-t^{-1})Y]\big\}$. We first use the transition matrix

$$Q_\lambda[X;0,t] = \sum_\mu K_{\mu\lambda}(t)S_\mu[(1-t)X]$$

given in [14] (p. 241), which also may be derived by setting $q = 0$ in I.3, and then we expand the Schur function of a sum:

$$
\begin{aligned}
Q_\gamma\left[Y - \tfrac{qt^{1-n}}{1-t^{-1}}; 0, t^{-1}\right] &= \sum_{\rho\,:\,|\rho|=|\gamma|} K_{\rho\gamma}(t^{-1}) S_\rho\left[(1-t^{-1})\left(Y - \tfrac{qt^{1-n}}{1-t^{-1}}\right)\right] \\
&= \sum_{\rho\,:\,|\rho|=|\gamma|} K_{\rho\gamma}(t^{-1}) S_\rho\left[(1-t^{-1})Y - qt^{1-n}\right] \\
&= \sum_{\rho\,:\,|\rho|=|\gamma|} K_{\rho\gamma}(t^{-1}) \sum_{\lambda\subset\rho} S_\lambda\left[(1-t^{-1})Y\right] S_{\rho/\lambda}[-qt^{1-n}] \\
&= \sum_{\rho\,:\,|\rho|=|\gamma|} K_{\rho\gamma}(t^{-1}) \sum_{\lambda\,:\,\rho/\lambda\in V} S_\lambda\left[(1-t^{-1})Y\right](-qt^{1-n})^{|\rho/\lambda|}
\end{aligned}
$$

because for the single letter qt^{1-n}, the skew Schur function $S_{\rho/\lambda}[-qt^{1-n}] = (-1)^{|\rho/\lambda|} S_{\rho'/\lambda'}[qt^{1-n}]$ is 0 if ρ'/λ' is not a horizontal strip, and is $(-qt^{1-n})^{|\rho/\lambda|}$ if it is a horizontal strip. Plug this into 3.18 and take the coefficient of $S_\lambda[(1-t^{-1})Y]$ to obtain

$$
{}^{+}\widetilde{\mathbf{C}}_r S_\lambda\big|_{[\mu]} = t^{\binom{r}{2}} \sum_\gamma \sum_{\substack{\rho\,:\,\rho/\lambda\in V\\ |\rho|=|\gamma|}} (-qt^{1-n})^{|\rho/\lambda|} K_{\rho\gamma}(t^{-1}) t^{(n-r)l(\gamma)} \begin{bmatrix} n-l(\gamma)\\ r-l(\gamma)\end{bmatrix}_t P_\gamma\left[Z_{n,\mu}; 0, t^{-1}\right] .
$$

We now reduce this to a finite sum. We may restrict the sum over γ to $l(\gamma) \le r$ because $\left[\begin{smallmatrix} n-l(\gamma)\\ r-l(\gamma)\end{smallmatrix}\right]_t = 0$ when $l(\gamma) > r$. We may restrict the sum over ρ to ρ that dominates γ, because $K_{\rho,\gamma}(t^{-1}) = 0$ when ρ does not dominate γ; then $l(\rho) \le l(\gamma) \le r$. Thus ρ/λ is a vertical strip of length at most r, and so there are only finitely many ρ possible, each with only finitely many γ below it in the dominance order.

$$
{}^{+}\widetilde{\mathbf{C}}_r S_\lambda\big|_{[\mu]} = t^{\binom{r}{2}} \sum_{\substack{\rho\,:\,\rho/\lambda\in V\\ l(\rho)\le r}} (-qt^{1-n})^{|\rho/\lambda|} \sum_{\substack{\gamma\,:\,|\gamma|=|\rho|\\ l(\gamma)\le r}} K_{\rho\gamma}(t^{-1}) t^{(n-r)l(\gamma)} \begin{bmatrix} n-l(\gamma)\\ r-l(\gamma)\end{bmatrix}_t P_\gamma\left[Z_{n,\mu}; 0, t^{-1}\right]
$$

We now eliminate n and turn this into a plethystic formula. All occurrences of n can be reduced to occurrences of $u = t^n$, as follows. Rewrite $t^{1-n} = t/u$, $t^{(n-r)l(\gamma)} = (ut^{-r})^{l(\gamma)}$, $Z_{n,\mu} = (1-q^{-1})X + \frac{1-u}{q(1-t)}$ at $X = B_\mu$, and

$$
\begin{bmatrix} n-l\\ r-l\end{bmatrix}_t = \prod_{i=0}^{r-l} \frac{t^{n-l}-t^i}{t^{r-l}-t^i} = \prod_{i=0}^{r-l} \frac{ut^{-l}-t^i}{t^{r-l}-t^i} .
$$

The Z_n substitution is valid when $X = B_\mu$ for all partitions μ and all $n \ge l(\mu)$, while the binomial substitution is valid for all $n \ge r$. These substitutions give us a nonhomogeneous symmetric function in X of degree at most $|\lambda| + r$ that is also a Laurent polynomial in u, so we could write ${}^{+}\widetilde{\mathbf{C}}_r S_\lambda\big|_{[\mu]} = \sum_{|i|<M} a_i[B_\mu]u^i$ for some M and some symmetric functions $a_i[X]$. For any specific choice $X = B_\mu$, we may set $u = t^n$ for any $n \ge l(\mu)$ without affecting the value of ${}^{+}\widetilde{\mathbf{C}}_r S_\lambda\big|_{[\mu]}$. Since u may take on infinitely many different values without affecting the evaluation of this Laurent polynomial, for each

$i \neq 0$ we must have $a_i[B_\mu] = 0$ for all μ; thus, the functions $a_i[X]$ with $i \neq 0$ are identically 0 by Theorem 2.7. So we may choose $u = t^r$ regardless of whether $r \geq l(\mu)$ holds, and thus obtain that

$$ {}^{+}\widetilde{\mathbf{C}}_r S_\lambda\big|_{[\mu]} \;=\; t^{\binom{r}{2}} \sum_{\substack{\rho\,:\,\rho/\lambda\in V\\ l(\rho)\leq r}} (-qt^{1-r})^{|\rho/\lambda|} \sum_{\substack{\gamma\,:\,|\gamma|=|\rho|\\ l(\gamma)\leq r}} K_{\rho\gamma}(t^{-1}) P_\gamma\big[Z_{r,\mu};0,t^{-1}\big] $$

holds for all partitions μ. By using 3.11, the right side is explicitly a plethystic formula in X, which we may denote $(\widetilde{,}_r S_\lambda)[X]$. By Theorem 2.7, it is the unique finite degree symmetric function with the property that $(\widetilde{,}_r S_\lambda)[B_\mu] = {}^{+}\widetilde{\mathbf{C}}_r S_\lambda\big|_{[\mu]}$.

We now manipulate it into a different form. Plug in the transition matrix expressing the Hall-Littlewood functions in terms of the Schur functions [14] (III.6 p. 239)

$$ P_\gamma\big[Z_r;0,t^{-1}\big] \;=\; \sum_{\kappa\,:\,|\kappa|=|\gamma|} K^{-1}_{\gamma\kappa}(t^{-1}) S_\kappa[Z_r] $$

to obtain

$$ \begin{aligned} \widetilde{,}_r S_\lambda &= t^{\binom{r}{2}} \sum_{\substack{\rho\,:\,\rho/\lambda\in V\\ l(\rho)\leq r}} (-qt^{1-r})^{|\rho/\lambda|} \sum_{\substack{\gamma\,:\,l(\gamma)\leq r\\ |\gamma|=|\rho|}} K_{\rho\gamma}(t^{-1}) \sum_{\kappa\,:\,|\kappa|=|\gamma|} K^{-1}_{\gamma\kappa}(t^{-1}) S_\kappa[Z_r] \\ &= t^{\binom{r}{2}} \sum_{\substack{\rho\,:\,\rho/\lambda\in V\\ l(\rho)\leq r}} (-qt^{1-r})^{|\rho/\lambda|} \sum_{\kappa\,:\,|\kappa|=|\rho|} S_\kappa[Z_r] \sum_{\substack{\gamma\,:\,l(\gamma)\leq r\\ |\gamma|=|\rho|}} K_{\rho\gamma}(t^{-1}) K^{-1}_{\gamma\kappa}(t^{-1}) \ , \end{aligned} $$

which establishes formula 3.7. To prove the last assertion in Theorem 3.1, we note that in the sum ${}^{+}\widetilde{\mathbf{C}}_r S_\lambda\big|_{[\mu]}$, the Schur function S_λ is evaluated at the r-letter alphabet $B_{\mu/\nu}$, so if $r < l(\lambda)$, every term of the sum is 0.

We are left with the final task of verifying our basic formula 3.12. In preparing for this, our first step is to relate the ${}^{+}c^{(r)}_{\mu\nu}$'s to the coefficients ${}^{+}d^{(r)}_{\mu\nu}$ given by 2.38, namely

$$ h_r\left[\tfrac{X}{1-t}\right] \tilde{H}_\nu[X;q,t] \;=\; \sum_\mu {}^{+}d^{(r)}_{\mu\nu} \tilde{H}_\mu[X;q,t] \ . $$

By taking the inner product of both sides with some particular $\tilde{H}_\mu$, we obtain

$$ \left\langle h_r\left[\tfrac{X}{1-t}\right] \tilde{H}_\nu \ , \ \tilde{H}_\mu \right\rangle_* \;=\; {}^{+}d^{(r)}_{\mu\nu} \left\langle \tilde{H}_\mu \ , \ \tilde{H}_\mu \right\rangle_* $$

on the one hand, and

$$ \begin{aligned} &= \left\langle \tilde{H}_\nu \ , \ {}^{+}\partial^{(r)} \tilde{H}_\mu \right\rangle_* \;=\; \left\langle \tilde{H}_\nu \ , \ \sum_\gamma {}^{+}c^{(r)}_{\mu\gamma} \tilde{H}_\gamma \right\rangle_* \\ &= {}^{+}c^{(r)}_{\mu\nu} \left\langle \tilde{H}_\nu \ , \ \tilde{H}_\nu \right\rangle_* \end{aligned} $$

on the other; using $\left\langle \tilde{H}_\mu \ , \ \tilde{H}_\mu \right\rangle_* = \tilde{h}_\mu \tilde{h}'_\mu$, we conclude

$$ {}^{+}c^{(r)}_{\mu\nu} \;=\; {}^{+}d^{(r)}_{\mu\nu} \cdot \frac{\tilde{h}_\mu \tilde{h}'_\mu}{\tilde{h}_\nu \tilde{h}'_\nu} \ . \qquad 3.19 $$

Next plug in the value of ${}^{+}d^{(r)}_{\mu\nu}$ given by 2.39,

$$ {}^{+}c^{(r)}_{\mu\nu} \;=\; \psi'_{\mu/\nu}(q,t^{-1})\frac{\tilde h_\nu(q,t)}{\tilde h_\mu(q,t)}\cdot\frac{\tilde h_\mu(q,t)\tilde h'_\mu(q,t)}{\tilde h_\nu(q,t)\tilde h'_\nu(q,t)} \;=\; \psi'_{\mu/\nu}(q,t^{-1})\cdot\frac{\tilde h'_\mu(q,t)}{\tilde h'_\nu(q,t)} \;, $$

so that, by 1.21,

$$ {}^{+}c^{(r)}_{\mu\nu} \;=\; t^{n(\mu)-n(\nu)}\psi'_{\mu/\nu}(q,t^{-1})\cdot\frac{h'_\mu(q,t^{-1})}{h'_\nu(q,t^{-1})} \;. \qquad 3.20 $$

We will also need the following auxiliary identity:

Lemma 3.3

(See [14], pp. 337–338.)

$$ h'_\mu(q,t) \;=\; \Omega\Big[-(1-q^{-1})^{-1}\Big(\textstyle\sum_{i=1}^{n}\big(q^{\mu_i}t^{n-i}-1\big)+\big(t^{-1}-1\big)\sum_{1\le i<j\le n}q^{\mu_i-\mu_j}t^{j-i}\Big)\Big] \;. \qquad 3.21 $$

Proof

We have

$$ h'_\mu(q,t) \;=\; \prod_{s\in\mu}\Big(1-q^{a_\mu(s)+1}t^{l_\mu(s)}\Big) \;=\; \Omega\Big[-\textstyle\sum_{s\in\mu}q^{a_\mu(s)+1}t^{l_\mu(s)}\Big] \;. \qquad 3.22 $$

We will evaluate

$$ S \;=\; (1-q^{-1})\sum_{s\in\mu}q^{a_\mu(s)+1}t^{l_\mu(s)} \;. $$

The columns of length j are those numbered $\mu_{j+1}+1,\mu_{j+1}+2,\ldots,\mu_j$. The contribution to S of the cells in row i that are in columns of length j, for each $j\ge i$, is

$$ (1-q^{-1})(q^{\mu_i-\mu_{j+1}}+\cdots+q^{\mu_i-\mu_j+1})t^{j-i} \;=\; (q^{\mu_i-\mu_{j+1}}-q^{\mu_i-\mu_j})t^{j-i} \;, $$

and there are no cells in row i in columns of length smaller than i. The total contribution of row i to S is obtained by summing over j:

$$ \begin{aligned} \sum_{j=i}^{n}\big(q^{\mu_i-\mu_{j+1}}-q^{\mu_i-\mu_j}\big)\,t^{j-i} \;&=\; \sum_{j=i}^{n}q^{\mu_i-\mu_{j+1}}t^{j-i}-\sum_{j=i}^{n}q^{\mu_i-\mu_j}t^{j-i} \\ &=\; \sum_{j=i+1}^{n+1}q^{\mu_i-\mu_j}t^{j-1-i}-\sum_{j=i}^{n}q^{\mu_i-\mu_j}t^{j-i} \;. \end{aligned} $$

Separate out term $j=n+1$ in the first sum and $j=i$ in the second:

$$ \begin{aligned} &=\; q^{\mu_i-\mu_{n+1}}t^{n+1-1-i}-q^{\mu_i-\mu_i}t^{i-i}+\sum_{j=i+1}^{n}\big(q^{\mu_i-\mu_j}t^{j-1-i}-q^{\mu_i-\mu_j}t^{j-i}\big) \\ &=\; q^{\mu_i}t^{n-i}-1+\big(t^{-1}-1\big)\sum_{j=i+1}^{n}q^{\mu_i-\mu_j}t^{j-i} \;. \end{aligned} $$

Finally, sum this over all i to obtain

$$S \;=\; (1-q^{-1})\sum_{s\in\mu} q^{a_\mu(s)+1}t^{l_\mu(s)} \;=\; \sum_{i=1}^{n}\left(q^{\mu_i}t^{n-i}-1\right)+\left(t^{-1}-1\right)\sum_{1\le i<j\le n} q^{\mu_i-\mu_j}t^{j-i}\;.$$

Plug this into 3.22 to obtain 3.21.

We are now in position to give our

Proof of Theorem 3.4

We start with rewriting $h'_\mu(q,t^{-1})/h'_\nu(q,t^{-1})$ using 3.21:

$$\frac{h'_\mu(q,t^{-1})}{h'_\nu(q,t^{-1})} = \Omega\Big[-(1-q^{-1})^{-1}\left(\textstyle\sum_{i=1}^n \left(q^{\mu_i}-q^{\nu_i}\right)t^{i-n}+(t-1)\sum_{1\le i<j\le n}\left(q^{\mu_i-\mu_j}-q^{\nu_i-\nu_j}\right)t^{i-j}\right)\Big]\;.$$

Let $I=I_{\mu/\nu}$ and break this up according to which i,j are in I, using $\nu_i=\mu_i$ if $i\notin I$ or μ_i-1 if $i\in I$. In the first sum, terms with $i\notin I$ contribute nothing, and in the second sum, terms with i and j both in I or both not in I contribute nothing.

$$\begin{aligned}
&= \Omega\left[-\sum_{i\in I}\frac{q^{\mu_i}-q^{\mu_i-1}}{1-q^{-1}}t^{i-n}-(t-1)\left(\sum_{\substack{i<j\\ i\in I\\ j\notin I}}\frac{q^{\mu_i-\mu_j}-q^{\mu_i-1-\mu_j}}{1-q^{-1}}t^{i-j}+\sum_{\substack{i<j\\ i\notin I\\ j\in I}}\frac{q^{\mu_i-\mu_j}-q^{\mu_i-\mu_j+1}}{1-q^{-1}}t^{i-j}\right)\right]\\
&= \Omega\left[-\sum_{i\in I}q^{\mu_i}t^{i-n}-(t-1)\left(\sum_{\substack{i<j\\ i\in I\\ j\notin I}}q^{\mu_i-\mu_j}t^{i-j}-\sum_{\substack{i<j\\ i\notin I\\ j\in I}}q^{\mu_i-\mu_j+1}t^{i-j}\right)\right]\\
&= \Omega\big[-\textstyle\sum_{i\in I}q^{\mu_i}t^{i-n}\big]\cdot\displaystyle\prod_{\substack{i\in I\\ j\notin I\\ i<j}}\frac{1-q^{\mu_i-\mu_j}t^{i-j+1}}{1-q^{\mu_i-\mu_j}t^{i-j}}\prod_{\substack{i\notin I\\ j\in I\\ i<j}}\frac{1-q^{\nu_i-\nu_j}t^{i-j}}{1-q^{\nu_i-\nu_j}t^{i-j+1}}
\end{aligned} \qquad 3.23$$

Now we rewrite 2.29 in terms of I and t^{-1} as

$$\psi'_{\mu/\nu} \;=\; \prod_{\substack{i\notin I\\ j\in I\\ i<j}}\frac{(1-q^{\nu_i-\nu_j}t^{i-j+1})(1-q^{\mu_i-\mu_j}t^{i-j-1})}{(1-q^{\nu_i-\nu_j}t^{i-j})(1-q^{\mu_i-\mu_j}t^{i-j})}\;. \qquad 3.24$$

Plug 3.24 and 3.23 into 3.20, noting that the products with ν cancel, to obtain

$${}^{+}c^{(r)}_{\mu\nu} \;=\; \Omega\big[-\textstyle\sum_{i\in I}q^{\mu_i}t^{i-n}\big]\cdot t^{n(\mu)-n(\nu)}\displaystyle\prod_{\substack{i\notin I\\ j\in I\\ i<j}}\frac{1-q^{\mu_i-\mu_j}t^{i-j-1}}{1-q^{\mu_i-\mu_j}t^{i-j}}\cdot\prod_{\substack{i\in I\\ j\notin I\\ i<j}}\frac{1-q^{\mu_i-\mu_j}t^{i-j+1}}{1-q^{\mu_i-\mu_j}t^{i-j}}\;.$$

Express this in terms of $z_k=q^{\mu_k-1}t^{k-1}$:

$$= \Omega[-\textstyle\sum_{i\in I} qt^{1-n} z_i] \cdot t^{n(\mu)-n(\nu)} \prod_{\substack{i\notin I\\ j\in I\\ i<j}} \frac{1-z_i/(t\,z_j)}{1-z_i/z_j} \cdot \prod_{\substack{i\in I\\ j\notin I\\ i<j}} \frac{1-t\,z_i/z_j}{1-z_i/z_j}$$

$$= \Omega[-qt^{1-n} \textstyle\sum_{i\in I} z_i] \cdot t^{n(\mu)-n(\nu)} \prod_{\substack{i\notin I\\ j\in I\\ i<j}} t^{-1} \prod_{\substack{i\notin I\\ j\in I\\ i<j}} \frac{t\,z_j - z_i}{z_j - z_i} \cdot \prod_{\substack{i\in I\\ j\notin I\\ i<j}} \frac{t\,z_i - z_j}{z_i - z_j} \ .$$

Swap i and j in the second product and then combine it with the third:

$$= \Omega[-qt^{1-n} Z_I] \cdot t^{n(\mu)-n(\nu)} \prod_{i\notin I, j\in I, i<j} t^{-1} \prod_{i\in I, j\notin I} \frac{t\,z_i - z_j}{z_i - z_j}$$

$$= \Omega[-qt^{1-n} Z_I] \cdot t^{n(\mu)-n(\nu)} \prod_{i\notin I, j\in I, i<j} t^{-1} \cdot t^{-\binom{r}{2}} A_I(Z_n;t)$$

Finally, we obtain

$$^{+}c^{(r)}_{\mu\nu} = \Omega[-qt^{1-n} Z_I]\, A_I(Z_n;t) t^{n(\mu)-n(\nu)-\binom{r}{2}-K} \ ,$$

where K is the number of pairs (i,j) with $i \notin I$ and $j \in I$ and $1 \le i < j \le n$. However,

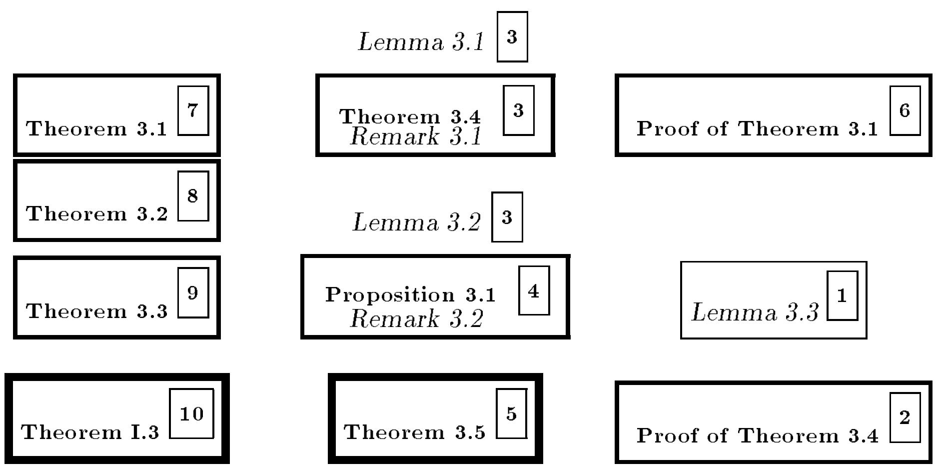
$$K = \sum_{i\notin I, j\in I, i<j} 1 = \sum_{j\in I, i<j} 1 - \sum_{j\in I, i\in I, i<j} 1 = \sum_{j\in I} (j-1) - \binom{r}{2}$$

while

$$n(\mu) - n(\nu) = \sum_{j\ge 1} (j-1)(\mu_j - \nu_j) = \sum_{j\in I} (j-1)$$

so that $n(\mu) - n(\nu) - \binom{r}{2} - K = 0$. This completes the proof of Theorem 3.4.

A visual display of what we have shown in this section should be helpful at this point.



In the diagram above we have arranged the results obtained into three columns. The order of presentation may be obtained by reading from left to right and down each column. The logical sequence of implications which culminates with the proof of Theorem 3.1 and ultimately leads to the proof of Theorem I.3 may be visualized by following the numbers in the small boxes.

4. Laurent nature of $\mathbf{k}_\gamma(x;q,t)$ and polynomiality of $\tilde{K}_{\lambda\mu}(q,t)$.

In this section we shall use the tools developed in sections 2 and 3 to establish the results announced in the introduction. To begin with we should point out that our proof of Theorem 2.6, in particular formula 2.60, shows that the operators , in I.14 and 2.56 are one and the same. Thus Theorem I.2 is just another way of stating Theorem 2.6. Since we have seen in section 3 that Theorem I.3 is a corollary of Theorem 3.1, the next result that remains to be shown is Theorem I.1. Now we have seen in section 2 (formula 2.51) that the coefficients $\tilde{K}_{\lambda\mu}(q,t)$ are relatively simple linear combinations of the m-rational functions

$$\mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_r}\,1\,|_{[\mu]} \;=\; \langle\, e^*_{n-k}e^*_{p_1}e^*_{p_2}\cdots e^*_{p_r}\;,\;\tilde{H}_\mu\,\rangle_* \;. \qquad 4.1$$

Thus we can obtain Theorem I.1 by deriving from the results of section 3 that all of these functions are plethystic and that they are represented by symmetric polynomials with Laurent coefficients.

Before we can proceed, we need to introduce some notation and make some preliminary observations. For convenience let us set, for any symmetric polynomial P,

$$P^+[X] \;=\; P[\tfrac{X}{1-t}] \;.$$

Note that from the definition in 3.1 we get, for any $\gamma\vdash k$ and any $\mu\vdash n\geq k$

$$\langle\, e^*_{n-k}h^+_{\gamma_1}h^+_{\gamma_2}\cdots h^+_{\gamma_r}\;,\;\tilde{H}_\mu\,\rangle_* \;=\; \langle\, e^*_{n-k}\;,\;{}^+\partial^{(\gamma_1)}\,{}^+\partial^{(\gamma_2)}\cdots{}^+\partial^{(\gamma_r)}\,\tilde{H}_\mu\,\rangle_* \;.$$

Thus the same steps that gave us 2.50 and Theorem 3.2 yield us the identity

$$\langle\, e^*_{n-k}h^+_{\gamma_1}h^+_{\gamma_2}\cdots h^+_{\gamma_r}\;,\;\tilde{H}_\mu\,\rangle_* \;=\; {}^+\mathbf{C}_{\gamma_1}\;{}^+\mathbf{C}_{\gamma_2}\cdots{}^+\mathbf{C}_{\gamma_r}\;1\,|_{[\mu]} \;. \qquad 4.2$$

More precisely, we have the following auxiliary result.

Proposition 4.1

For any $\rho\vdash k$ and any $n\geq k$, the m-rational function

$$\langle\, e^*_{n-k}h^+_{\rho_1}h^+_{\rho_2}\cdots h^+_{\rho_r}\;,\;\tilde{H}_\mu\,\rangle_* \qquad 4.3$$

is always plethystic and is uniquely represented by the polynomial

$${}^+\mathbf{\Pi}_\rho(x;q,t) \;=\; {}^+,\,_{\rho_1}\;{}^+,\,_{\rho_2}\cdots{}^+,\,_{\rho_r}\;1 \;. \qquad 4.4$$

Moreover, ${}^+\mathbf{\Pi}_\rho(x;q,t)$ has a Schur function expansion of the form

$${}^+\mathbf{\Pi}_\rho(x;q,t) \;=\; \sum_{|\lambda|\leq k} S_\lambda(x)\;{}^+a_{\lambda\rho}(q,t) \;, \qquad 4.5$$

with coefficients $^{+}a_{\rho\gamma}(q,t)$ *Laurent polynomials in* q *and* t.

Proof

The plethystic nature of these functions is an immediate consequence of 4.2. In fact, successive applications of Theorem 3.2, yield that

$$\langle\, e^*_{n-k} h^+_{\rho_1} h^+_{\rho_2}\cdots h^+_{\rho_r} \,,\, \tilde H_\mu \,\rangle_* \;=\; {}^{+}\mathbf{\Pi}_\rho\big[B_\mu(q,t)\,;\,q,t\,\big] \qquad (\ \forall\quad \mu\vdash n\geq k\ )\ . \qquad 4.6$$

This, together with Theorem 2.7, also implies that $^{+}\mathbf{\Pi}_\gamma(x;q,t)$ is unique. The Laurent nature of the $^{+}a_{\lambda\rho}(q,t)$ follows from Theorem 3.3.

Lemma 4.1

For any given vector $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_r)$ *of positive integers adding up to* k *we have the expansion*

$$e^*_{\gamma_1}(x)e^*_{\gamma_2}(x)\cdots e^*_{\gamma_r}(x) \;=\; \sum_{\rho\vdash k} {}^{+}h_\rho(x)\, f_{\rho,\gamma}(q,t)\ , \qquad 4.7$$

where the coefficients $f_{\rho,\gamma}(q,t)$ *are rational functions of* q *and* t *whose denominator factors are all of the form*

$$(1-q^i) \qquad\qquad (\ \textit{with}\quad i\geq 1\ ) \qquad 4.8$$

Proof

Expanding each of the factors in the left hand side of 4.7 by means of the the "dual" Cauchy formula in 2.41, namely the identity

$$e_m\big[\tfrac{X}{(1-t)(1-q)}\big] \;=\; \sum_{\lambda\vdash m} h_\lambda\big[\tfrac{X}{1-t}\big]\; f_\lambda\big[\tfrac{1}{1-q}\big]$$

we obtain 4.7 with the coefficients $f_{\rho,\gamma}(q,t)$ products of of forgotten basis elements plethystically evaluated at $1/(1-q)$. The statement concerning their denominators is easily derived by expanding each $f_\lambda(x)$ in terms the Schur and using one of the standard formulas giving $S_\lambda\big[\frac{1}{1-q}\big]$.

This places us in a position to derive the basic result of this paper.

Theorem 4.1

For a partition $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_r)\vdash k$ *set*

$$\mathbf{\Pi}_\gamma(x;q,t) \;=\; \sum_{\rho\vdash k} {}^{+}\mathbf{\Pi}_\rho(x;q,t)\, f_{\rho,\gamma}(q,t)\ , \qquad 4.9$$

where the coefficients $f_{\rho,\gamma}(q,t)$ *are the same as those in 4.7. Then we necessarily have*

$$\langle\, e^*_{n-k} e^*_{\gamma_1} e^*_{\gamma_2}\cdots e^*_{\gamma_r} \,,\, \tilde H_\mu \,\rangle_* \;=\; \mathbf{\Pi}_\gamma\big[B_\mu(q,t)\,;q,t\,\big] \qquad (\ \forall\ \mu\vdash n\geq k\ )\ , \qquad 4.10$$

and (surprisingly) $\mathbf{\Pi}_\gamma(x;q,t)$ *also has a Schur function expansion of the form*

$$\mathbf{\Pi}_\gamma(x;q,t) \;=\; \sum_{|\lambda|\leq k} S_\lambda(x)\, a_{\lambda\gamma}(q,t) \qquad 4.11$$

with coefficients $a_{\lambda\gamma}(q,t)$ *Laurent polynomials in* q *and* t*. Again, we should add that* $\mathbf{\Pi}_\gamma(x;q,t)$ *is the unique polynomial satisfying 4.10.*

Proof

From 4.7 we derive that

$$\langle\, e^*_{n-k}e^*_{\gamma_1}e^*_{\gamma_2}\cdots e^*_{\gamma_r}\ ,\ \tilde H_\mu\,\rangle_* \;=\; \sum_{\rho\vdash k}\langle\, e^*_{n-k}\,{}^{+}h_\rho(x)\ ,\ \tilde H_\mu\,\rangle_*\, f_{\rho,\gamma}(q,t)\ ,$$

and 4.6 yields that

$$\langle\, e^*_{n-k}e^*_{\gamma_1}e^*_{\gamma_2}\cdots e^*_{\gamma_r}\ ,\ \tilde H_\mu\,\rangle_* \;=\; \sum_{\rho\vdash k}{}^{+}\mathbf{\Pi}_\rho\big[B_\mu(q,t)\,;q,t\,\big]\, f_{\rho,\gamma}(q,t)\ . \tag{4.12}$$

This gives 4.10. Our proof would be complete if it wasn't for the fact that the coefficients $f_{\rho,\gamma}(q,t)$ have denominators. However, we are in a better position here than we where in section 2 where we could not explain the disappearance of the denominators in the final expressions we obtained for $\mathbf{\Pi}_2$ and $\mathbf{\Pi}_3$. This is due to the fact that the factors in 4.8 involve only the variable q. To see how this comes about, note that from the definition in I.15 we can see that the $*$-scalar product is symmetric in q and t. The same is also true for all the polynomials $e^*_m(x)$. Now, 4.10 with μ replaced by μ' gives

$$\mathbf{\Pi}_\gamma\big[B_{\mu'}(q,t)\,;q,t\,\big] \;=\; \langle\, e^*_{n-k}e^*_{\gamma_1}e^*_{\gamma_2}\cdots e^*_{\gamma_r}\ ,\ \tilde H_{\mu'}\,\rangle_*\ .$$

However, the stated symmetries together with 1.55 yield that interchanging q and t results in the identity

$$\mathbf{\Pi}_\gamma\big[B_{\mu'}(t,q)\,;t,q\,\big] \;=\; \langle\, e^*_{n-k}e^*_{\gamma_1}e^*_{\gamma_2}\cdots e^*_{\gamma_r}\ ,\ \tilde H_{\mu}\,\rangle_*$$

and thus, using 4.10 again, we derive that

$$\mathbf{\Pi}_\gamma\big[B_{\mu'}(t,q)\,;t,q\,\big] \;=\; \mathbf{\Pi}_\gamma\big[B_{\mu}(q,t)\,;q,t\,\big]\ . \tag{4.13}$$

Now, it is easily verified that for any partition μ we have $B_{\mu'}(t,q)=B_\mu(q,t)$. Thus 4.13 may also be written in the form

$$\mathbf{\Pi}_\gamma\big[B_{\mu}(q,t)\,;t,q\,\big] \;=\; \mathbf{\Pi}_\gamma\big[B_{\mu}(q,t)\,;q,t\,\big]\ .$$

Since this is to hold true for every $\mu\vdash n\geq k$, we are in a position to use Theorem 2.8 and reach the final conclusion that

$$\mathbf{\Pi}_\gamma\big[X;t,q\,\big] \;\equiv\; \mathbf{\Pi}_\gamma\big[X;q,t\,\big]\ .$$

This in particular implies that we must have

$$a_{\lambda\gamma}(q,t) \;=\; a_{\lambda\gamma}(t,q) \tag{4.14}$$

for all the coefficients $a_{\lambda\gamma}(q,t)$ occurring in 4.11.

Let us take assessment of what we have put together. On the one hand, computing $\mathbf{\Pi}_\mu$ from 4.9 and 4.4 we derive from Proposition 4.1 and Lemma 4.1 that each $a_{\lambda\gamma}(q,t)$ may be expressed as a rational function in q,t whose denominator has only factors of the form

$$1-q^i\quad(\text{with }\ i\geq 1\,)\quad\text{ and }\quad t^r\ ,\ q^s\quad(\text{with }\ r,s\geq 1\,)\ .$$

On the other hand, interchanging q and t in this first expression and using 4.14 we obtain a second expression which exhibits $a_{\lambda\gamma}(q,t)$ as a a rational function in q,t whose denominator has only factors of the form

$$1-t^i \quad (\text{with } i\geq 1) \quad \text{and} \quad q^r\ ,\ t^s \quad (\text{with } r,s\geq 1)\ .$$

These two facts force us to the conclusion that the factors $1-q^i$ in the first expression and the factors $1-t^i$ in the second expression must disappear upon reducing these two expressions to their normal form. This leaves powers of t and q as the only possible denominator factors and establishes the Laurent nature of each coefficient $a_{\lambda\gamma}(q,t)$. Since the uniqueness part can again be obtained from Theorem 2.8, our proof is now complete.

Let us now extend the definition of the polynomials $\mathbf{\Pi}_\gamma(x;q,t)$ to polynomials $\mathbf{\Pi}_p(x;q,t)$ indexed by an arbitrary sequence of integers $p=(p_1,p_2,\ldots,p_r)$ by setting $\mathbf{\Pi}_p(x;q,t)=\mathbf{\Pi}_\gamma(x;q,t)$ if the non-vanishing components of p can be rearranged to a partition γ and setting $\mathbf{\Pi}_p(x;q,t)\equiv 0$ otherwise. This given, we can derive the following final conclusions regarding the coefficients $\tilde{K}_{\lambda\mu}(q,t)$.

Theorem 4.2

For a partition $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_r)\vdash k$ set

$$\mathbf{k}_\gamma(x;q,t)=\sum_{\sigma\in S_{r+1}} sign(\sigma)\ \mathbf{\Pi}_{(\gamma_1+\sigma_2-2\ ,\ \gamma_2+\sigma_3-3\ ,\ \ldots\ ,\ \gamma_r+\sigma_{r+1}-r-1)}(x;q,t)\ . \qquad 4.15$$

Then for $\lambda=(n-k,\gamma_1,\gamma_2,\ldots,\gamma_r)$ and $n-k\geq\gamma_1$ we have

$$\tilde{K}_{\lambda\mu}(q,t) \ = \ \mathbf{k}_\gamma\left[\,B_\mu(q,t)\,;q,t\,\right] \qquad (\ \forall \quad \mu\vdash n\geq k+\gamma_1\ )\ . \qquad 4.16$$

Moreover, the polynomials $\mathbf{k}_\gamma(x;q,t)$ have Schur function expansions of the form

$$\mathbf{k}_\gamma(x;q,t) \ = \ \sum_{|\lambda|\leq k} S_\lambda(x)\,\mathbf{k}_{\lambda\gamma}(q,t) \qquad 4.17$$

with $\mathbf{k}_{\lambda\gamma}(q,t)$ Laurent polynomials and symmetric in q and t. Finally, when we carry out the plethystic evaluation in 4.16, we obtain an expression for $\tilde{K}_{\lambda\mu(q,t)}$ as a polynomial with integer coefficients with t-degree bounded by $n(\mu)$ and q-degree bounded by $n(\mu')$

Proof

The identity in 4.15 is an immediate consequence of 2.51 and our convention about the polynomials $\mathbf{\Pi}_p$. The Laurent nature and symmetry of the $\mathbf{k}_{\lambda\gamma}(q,t)$ follows from 4.15 and the corresponding properties of the coefficients $a_{\lambda\gamma}(q,t)$ in 4.11. As for the remaining assertions we can reason as follows. We have now two distinct ways to compute the $\tilde{K}_{\lambda\mu}(q,t)$. We can obtain them by applying the crude algorithm sketched in the proof of Proposition 2.6 and derive that they can be given a first rational expression with denominator factors all of the form

$$(q^r-t^s) \qquad (\ \text{with}\ \ r+s\geq 1\ )\ .$$

On the other hand we can use 4.16 and derive that they can be given a second rational expression with denominator factors all of the form

$$q^r t^s \qquad\qquad (\text{ with } r, s \geq 0) \ .$$

The equality of these two expressions forces the conclusion that the factors $q^r - t^s$ in the first expression and the factors $q^r t^s$ in the second must disappear when we bring those expressions to their normal form. This proves the polynomiality of the $\tilde{K}_{\lambda\mu}(q,t)$. The degree bounds are forced by the identity in 1.51. Indeed, by equating coefficients of S_λ there we obtain the equality

$$\tilde{K}_{\lambda\mu}(q,t) \ = \ t^{n(\mu)} q^{n(\mu')} \, \tilde{K}_{\lambda'\mu}(1/q, 1/t)$$

which shows that whatever polynomial $\tilde{K}_{\lambda'\mu}(q,t)$ might be, $\tilde{K}_{\lambda\mu}(q,t)$ itself being a polynomial must be of degree at most $n(\mu)$ in t and $n(\mu')$ in q. This completes our proof.

Corollary 4.1

The original Kostka-Macdonald coefficients $K_{\lambda\mu}(q,t)$ are polynomials in q and t with integer coefficients, of degree at most $n(\mu)$ in t and $n(\mu')$ in q.

Proof

For our present purposes, we may write the defining equality I.5 in the form

$$K_{\lambda\mu}(q,t) \ = \ \tilde{K}_{\lambda\mu}(q, 1/t) \, t^{n(\mu)} \qquad 4.18$$

and our assertions are easily seen to follow from what we have just finished proving for the $\tilde{K}_{\lambda\mu}(q,t)$.

Remark 4.1

It might be good to have in one single place the succession of formulas which, together with formula 4.15, constitute our algorithm for computing the polynomials $\mathbf{k}_{\lambda\gamma}(q,t)$. The list is as follows.

(1) The definition of $\tilde{\Gamma}_r$ in 3.7:

$$\tilde{\Gamma}_r S_\lambda \ = \ t^{\binom{r}{2}} \sum_{\substack{\rho \,:\, \rho/\lambda \in V \\ l(\rho) \leq r}} (-qt^{1-r})^{|\rho/\lambda|} \sum_{\kappa \,:\, |\kappa| = |\rho|} S_\kappa \Big[\, (1-q^{-1})X + q^{-1}[r]_t \, \Big] \sum_{\substack{\gamma \,:\, |\gamma| = |\rho| \\ l(\gamma) \leq r}} K_{\rho\gamma}(t^{-1}) K^{-1}_{\gamma\kappa}(t^{-1}) \ .$$

(2) The definition of ${}^{+}\Gamma_r$ in 3.9:

$${}^{+}\Gamma_r S_\lambda \ = \ \sum_{\rho \subseteq \lambda} (-1)^{|\rho|} S_{\lambda/\rho} \cdot \tilde{\Gamma}_r S_{\rho'} \ .$$

(3) Formula 4.4, giving ${}^{+}\mathbf{\Pi}_\rho$:

$${}^{+}\mathbf{\Pi}_\rho(x;q,t) \ = \ {}^{+}\Gamma_{\rho_1} \, {}^{+}\Gamma_{\rho_2} \cdots {}^{+}\Gamma_{\rho_r} \, 1 \ .$$

(4) Finally formula 4.9, which gives us $\mathbf{\Pi}_\gamma(x;q,t)$:

$$\mathbf{\Pi}_\gamma(x;q,t) = \sum_{\rho \vdash k} \, {}^{+}\mathbf{\Pi}_\rho(x;q,t) \, f_{\rho,\gamma}(q,t) \ ,$$

and formula 4.7, which gives us the coefficients $f_{\rho,\gamma}(q,t)$:

$$e^*_{\gamma_1}(x)e^*_{\gamma_2}(x)\cdots e^*_{\gamma_r}(x) = \sum_{\rho\vdash k} {}^+h_\rho(x)\, f_{\rho,\gamma}(q,t) \ .$$

This algorithm has been implemented in MAPLE using Stembridge's "SF" package. The code itself and tables of the 40 polynomials $\mathbf{k}_\gamma$ needed to compute the $\tilde{K}_{\lambda\mu}$ up to partitions of $n \le 12$ may be obtained via anonymous FTP from macaulay.ucsd.edu. To give an idea what these polynomials look like, we have attached at the end of the paper, a table of 12 that allow the computation of certain finite families of $\tilde{K}_{\lambda\mu}$. It may also be used to compute the polynomials $K_{\lambda\mu}(q,t)$ by means of 4.18. This table, used in conjunction with the formula

$$\tilde{K}_{\lambda\mu}(q,t) \ = \ k_{(\lambda_2,\lambda_3,\ldots)}[B_\mu(q,t);q,t] \ = \ t^{n(\mu)}q^{n(\mu')}k_{(\lambda'_2,\lambda'_3,\ldots)'}[B_\mu(1/q,1/t);1/q,1/t] \ ,$$

is all that is needed to compute all the $\tilde{K}_{\lambda\mu}(q,t)$ up to $n=8$. Recalling that we have set $\mathbf{k}_\gamma[X;q,t] = \sum_\nu \mathbf{k}_{\gamma\nu}(q,t)S_\nu[X]$, the Laurent polynomial $\mathbf{k}_{\gamma\nu}(q,t)$ is given as a matrix, with rows $i=0,-1,-2,\ldots$ downward from the top and columns $j=0,-1,-2,\ldots$ leftward from the right. The entry in row i, column j, is the coefficient of q^it^j. A negative coefficient $-m$ is denoted $\overline{m}$.

We should mention that Theorem 2.2, and in particular the identity

$$\sum_{\nu\to\mu} c_{\mu\nu}\, T^k_{\mu/\nu} \ = \ \frac{1}{(1-t)(1-q)t^kq^k}\, h_{k+1}[(1-t)(1-q)B_\mu - 1] \ , \qquad 4.19$$

were discovered within the probabilistic setting of the q,t-hook walk introduced in [10]. Although we have essentially given here two different proofs of 4.19, we believe that some aspects of the original proof should be of interest. In fact, there are a number of interesting questions that arise in this connection which should lead to further work. To proceed we need to state some of the identities proved in [10].

First of all we recall that the q,t-hook walk defined in [10] is simply a weighted version of the random walk discovered by Greene-Nijenhuis-Wilf [11]. Given $\mu \vdash n$, a sequence of random variables $Z_1, Z_2, \ldots, Z_s, \ldots$ is constructed where the value of each Z_i is a cell of the diagram of μ and given that $Z_i = s$ then Z_{i+1} is randomly chosen to be one the other cells of the "hook" of s in μ. More precisely, if $Z_{i+1} = s'$ then s' is directly EAST or directly NORTH of s. Moreover, if there are $i \ge 0$ cells of μ between s and s' then the transition probability is given by

$$P[Z_{i+1}=s'|Z_i=s] \ = \ \begin{cases} q^i\frac{t^{l_\mu(s)}(1-q)}{t^{l_\mu(s)}-q^{a_\mu(s)}} & \text{when } s' \text{ is EAST of } s \ , \\ t^i\frac{q^{a_\mu(s)}(t-1)}{t^{l_\mu(s)}-q^{a_\mu(s)}} & \text{when } s' \text{ is NORTH of } s \ . \end{cases}$$

The walk is started by choosing $Z_1 = s$ with probability

$$P[Z_1 = s] \ = \ \frac{t^{l'_\mu(s)}q^{a'_\mu(s)}}{B_\mu(q,t)} \ .$$

The walk ends when it reaches one of the corners of μ. We shall then write $Z_{end}=\mu/\nu$ (for a $\nu\to\mu$) when the walk ends at the cell we must remove from μ to get ν.

It is clear from the definition that a walk which starts at a cell s must end at a corner μ/ν that is weakly NORTH-EAST of s. Recalling the definition of $R_{\mu/\nu}$ and $C_{\mu/\nu}$ made in the introduction, we shall denote here by $R_{\mu/\nu}(s)$ and $C_{\mu/\nu}(s)$ respectively the collection of cells of $R_{\mu/\nu}$ and $C_{\mu/\nu}$ that are strictly NORTH-EAST of s. We also let $r[s]$ and $c[s]$ denote the two cells of $R_{\mu/\nu}\cup\{\mu/\nu\}$ and $C_{\mu/\nu}\cup\{\mu/\nu\}$ that are respectively directly NORTH and directly EAST of s. Of course, unless the starting cell s is already one of the corners, at least one of $R_{\mu/\nu}(s)$ and $C_{\mu/\nu}(s)$ will not be empty and one of $r[s]$ and $c[s]$ will not be μ/ν.

Now, it is shown in [10] (Theorem 2.2) that the conditional probability of the walk ending at μ/ν when it starts at s may be computed from the following identity:

$$P\big[\,Z_{end}=\mu/\nu\mid Z_1=s\,\big]\;=\;A(r[s])\,B(c[s])\prod_{r\in R_{\mu/\nu}(s)}\frac{\tilde h'_\mu(r)}{\tilde h'_\nu(r)}\prod_{c\in C_{\mu/\nu}(s)}\frac{\tilde h_\mu(c)}{\tilde h_\nu(c)}\;, \tag{4.20}$$

where the functions $A(r)$, $B(c)$ are defined by setting

$$A(r)=\begin{cases}1 & \text{if } r \text{ is a corner,}\\ \frac{t^{l_\mu(r)}(1-q)}{t^{l_\mu(r)}-q^{a_\mu(r)}} & \text{otherwise,}\end{cases}\qquad\text{and}\qquad B(c)=\begin{cases}1 & \text{if } c \text{ is a corner,}\\ \frac{q^{a_\mu(c)}(t-1)}{t^{l_\mu(c)}-q^{a_\mu(c)}} & \text{otherwise}\,.\end{cases}$$

It is also shown in [10] (Theorem 2.3) that the coefficients $c_{\mu\nu}(q,t)$ may be expressed in the form

$$c_{\mu\nu}(q,t)\;=\;\sum_{s\ll=\mu/\nu}q^{a'(s)}t^{l'(s)}\,P\big[Z_{end}=\mu/\nu\mid Z_1=s\big]\;, \tag{4.21}$$

where the symbol $s\ll=\mu/\nu$ means that μ/ν is weakly NORTH-EAST of s.

In this vein, we can give a probabilistic interpretation to any of our sums

$$\sum_{\nu\to\mu}c_{\mu\nu}(q,t)\;f_\nu(q,t)\;.$$

To be precise, (by a slight abuse of notation) we can set $f_{Z_{end}}=f_\nu(q,t)$ when $Z_{end}=\mu/\nu$ and consider it a random variable. This given, we can write

$$\sum_{\nu\to\mu}f_\nu(q,t)P\big[Z_{end}=\mu/\nu\mid Z_1=s\big]\;=\;E\big[\,f_{Z_{end}}\mid Z_1=s\,\big]\;,^{(\dagger)}$$

and use 4.21 to obtain that

$$\sum_{\nu\to\mu}c_{\mu\nu}(q,t)\;f_\nu(q,t)\;=\;\sum_{s\in\mu}q^{a'(s)}t^{l'(s)}\,E\big[\,f_{Z_{end}}\mid Z_1=s\,\big]\;. \tag{4.22}$$

(†) We read this as the "conditional expectation of $f_{Z_{end}}$ given that $Z_1=s$."

Two particular cases of this formula are worth noting here. The first is obtained by setting $f \equiv 1$. This gives

$$\sum_{\nu \to \mu} c_{\mu\nu}(q,t) \;=\; \sum_{s \in \mu} q^{a'(s)} t^{l'(s)} \;=\; B_\mu(q,t) \;,$$

which yields a probabilistic proof of 2.7. The other case is when f_μ is taken to be the "Hilbert series" $F_\mu(q,t)$. This permits us to rewrite the recursion I.13 in the form

$$F_\mu(q,t) \;=\; \sum_{s \in \mu} q^{a'(s)} t^{l'(s)} \, E[\, F_{Z_{end}}(q,t) \mid Z_1 = s \,] \;.$$

Now data obtained at the time of the writing of [10] revealed that the expression

$$E[\, F_{Z_{end}}(q,t) \mid Z_1 = s \,] \;, \qquad 4.23$$

computed using 4.20 and available tables of the $F_\nu(q,t)$, always yields a polynomial in q,t with integer coefficients. It was then conjectured there that this should be true in full generality. The efforts to prove this conjecture led to a close study of the expression in 4.20 and its rewriting in terms of weights of corner cells of the partition μ. More precisely, using the same notation as in section 2, we can state the following result.

Theorem 4.3

If $A_i, A_{i+1}, \ldots, A_j$ are the corners of μ that are weakly NORTH-EAST *of s, then for any $i \le r \le j$ we have*

$$P[Z_{end} = A_r | Z_1 = s] \;=\; t^{\alpha_i} q^{\beta_j} \, \frac{1}{x_r} \prod_{i \le r' \le j}^{(r' \ne r)} \frac{1}{x_r - x_{r'}} \prod_{i \le r' \le j-1} (x_r - u_{r'}) \;. \qquad 4.24$$

Proof

Starting with 4.20 and carrying out cancellations and manipulations entirely analogous to those carried out in the proof of 2.16, we obtain 4.24 without additional difficulties. We can omit the details.

Recalling the definition of $T_{\mu/\nu}$ given at the start of section 2, and setting $T_{Z_{end}} = T_{\mu/\nu}$ when $Z_{end} = \mu/\nu$, we derive from 4.24 the following analogue of 2.13.

Theorem 4.5

Under the same hypotheses as above, for any $k \ge 1$ we have

$$E[\, T^k_{Z_{end}} | Z_1 = s \,] \;=\; t^{\alpha_i} q^{\beta_j} h_{k-1}[x_i + x_{i+1} + \cdots + x_j - u_i - u_{i+1} - \cdots - u_{j-1}] \qquad 4.25$$

Proof

From 4.24 we get that

$$E[\, T^k_{Z_{end}} | Z_1 = s \,] \;=\; t^{\alpha_i} q^{\beta_j} \sum_{r=i}^{j} x_r^{k-1} \prod_{i \le r' \le j}^{(r' \ne r)} \frac{1}{x_r - x_{r'}} \prod_{i \le r' \le j-1} (x_r - u_{r'})$$

and 4.25 follows by equating coefficients of t^{k-1} in the partial fraction expansion

$$\frac{\prod_{i\le r\le j-1}(1-tu_r)}{\prod_{i\le r\le j}(1-tx_r)} \;=\; \sum_{r=i}^{j} \prod_{i\le r'\le j}^{(r'\neq r)} \frac{1}{x_r-x_{r'}} \prod_{i\le r'\le j-1}(x_r-u_{r'})\;\frac{1}{1-tx_r}\;.$$

Formula 4.25 has the following remarkable corollary:

Theorem 4.6

Suppose that f is plethystic Laurent and that $f_\mu(q,t)$ always evaluates to a polynomial with integer coefficients; then likewise for any $s\in\mu$, the conditional expectation $E[f_{Z_{end}}\mid Z_1=s\,]$ always evaluates to a polynomial with integer coefficients.

Proof

Let f be represented by the polynomial $P[x;q,t]$ with Schur function expansion

$$P(x;q,t) \;=\; \sum_{|\lambda|\le d} c_\lambda(q,t)\,S_\lambda(x)\;.$$

Then for any $\nu\rightarrow\mu$ we can write

$$f_\nu(q,t)=P[B_\nu(q,t);q,t]=P[B_\mu-T_{\mu/\nu};q,t]=\sum_{|\lambda|\le d} c_\lambda(q,t)\,S_\lambda[B_\mu-T_{\mu/\nu}]$$

$$=\sum_{|\lambda|\le d} c_\lambda(q,t)\sum_{k=0}^{d}(-1)^k\,T_{\mu/\nu}^k\Big(\sum_{\lambda/\rho\in V_k} S_\rho[B_\mu]\Big)\;.$$

We thus can write

$$E[f_{Z_{end}}\mid Z_1=s\,] \;=\; \sum_{|\lambda|\le d} c_\lambda(q,t)\sum_{k=0}^{d}(-1)^k\,E[\,T_{Z_{end}}^k\mid Z_1=s\,]\Big(\sum_{\lambda/\rho\in V_k} S_\rho[B_\mu]\Big)\;. \qquad 4.26$$

Since by hypothesis the coefficients $c_\lambda(q,t)$ are Laurent, we deduce from the polynomiality of the right hand side of 4.25 that the expression in 4.26 has only denominator factors of the form t^iq^j. On the other hand we can compute $E[f_{Z_{end}}\mid Z_1=s\,]$ directly from the formula

$$E[\,f_{Z_{end}}\mid Z_1=s\,] \;=\; \sum_{\nu\rightarrow\mu} f_\nu(q,t)P\big[Z_{end}=\mu/\nu\mid Z_1=s\,\big]\;, \qquad 4.27$$

with the conditional probability $P\big[Z_{end}=\mu/\nu\mid Z_1=s\,\big]$ as given by 4.20. But now we can see, from the form of 4.20 and the assumed polynomiality of $f_\nu(q,t)$, that the computation of $E[f_{Z_{end}}\mid Z_1=s\,]$ via this route yields a rational expression with denominator factors all of the form t^i-q^j. We can thus conclude, as we have done many times before, that when the two rational expressions given by 4.26 and 4.27 are reduced to normal form, all the denominators must necessarily cancel. This completes our proof.

Remark 4.2

Since, by definition, we have

$$F_\mu(q,t) \;=\; \sum_\lambda f_\lambda \, \tilde{K}_{\lambda\mu}(q,t) \;, \tag{4.28}$$

we see that one of the consequences of Theorem 4.2 is that $F_\mu(q,t)$ is a polynomial with integer coefficients. Moreover, 4.16 gives us that for each integer n, we can construct a polynomial $\Phi_n(x;q,t)$ whose Schur function expansion has Laurent coefficients and is such that

$$F_\mu(q,t) \;=\; \Phi_n[B_\mu(q,t);q,t] \qquad (\;\forall \quad \mu \vdash n\;)\;. \tag{4.29}$$

Thus we can apply Theorem 4.6 and conclude that the conditional expectation

$$E[\,F_{Z_{end}}(q,t)\mid Z_1 = s\,]$$

is always a polynomial with integer coefficients. Thus what was observed in [10] from numerical data does hold true in full generality.

We should mention that our data also shows that the polynomials $E[F_{Z_{end}}(q,t)|Z_1 = s]$ may have some negative coefficients. Yet, the weighted sum

$$F_\mu(q,t) \;=\; \sum_{s\in\mu} q^{a'(s)}t^{l'(s)}\, E[\,F_{Z_{end}}(q,t)\mid Z_1 = s\,] \tag{4.30}$$

always turns out to yield a polynomial with positive coefficients, consistent with the Macdonald conjecture that the $K_{\lambda\mu}(q,t)$ themselves have positive coefficients. Another puzzling problem is to understand how the recursion in 4.27 is related to the bigraded S_n-modules studied in [7]. Such an understanding might suggest an approach to establishing the positive integrality of the polynomials $F_\mu(q,t)$. In this connection we must point out that Theorem 2.6 and the recursion

$$F_\mu(q,t) \;=\; \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\, F_\nu(q,t)$$

yield that the polynomial $\Phi_n(x;q,t)$ giving 4.29 may be computed from the formula

$$\Phi_n(x;q,t) \;=\; ,^n\, 1 \;.$$

As a last item, we wish to show that Theorem 4.3 does yield a q,t-hook walk proof of the identity in 2.13. To carry this out we need some notation. Let $A_i = (\alpha_i,\beta_i)$ for $i = 1,\ldots,m$ denote the corners of μ as before and let $B_\mu(i,j)$ denote the portion of the diagram of μ defined by

$$B_\mu(i,j) \;=\; \{\, s\in\mu \;:\; \alpha_{i+1} < l'_\mu(s) \le \alpha_i \;;\; \beta_{j-1} < a'_\mu(s) \le \beta_j \,\}\;.$$

It is easy to see that when s varies in $B_\mu(i,j)$ the corners of μ that are weakly NORTH-EAST of s remain constantly

$$A_i\;,\;A_{i+1}\;,\;\ldots\;,\;A_j$$

Let us then denote by s_{ij} the NORTH-EAST corner of $B_\mu(i,j)$; this is the cell with coleg α_i and coarm β_j. Moreover set

$$B_\mu(i,j;q,t) \;=\; \sum_{s\in B_\mu(i,j)} t^{l'_\mu(s)} q^{a'_\mu(s)} \;.$$

This given, we may rewrite 4.22, with f_ν replaced by $T^k_{\mu\nu}$, in the form

$$\sum_{\nu\to\mu} c_{\mu\nu}(q,t)\;T^k_{\mu\nu} \;=\; \sum_{i=1}^{m}\sum_{j=i}^{m} B_\mu(i,j;q,t)\;E\big[\,T^k_{Z_{end}}\;|\;Z_1=s_{ij}\,\big]\;. \qquad 4.31$$

Now, we can easily deduce from the geometry of a diagram μ with corners $A_1, A_2, \ldots, A_m$ that

$$B_\mu(i,j;q,t) = t^{\alpha_{j+1}+1} q^{\beta_{i-1}+1} \frac{(1-t^{\alpha_j-\alpha_{j+1}})(1-q^{\beta_i-\beta_{i-1}})}{(1-t)(1-q)} = \frac{(t^{\alpha_{j+1}}-t^{\alpha_j})(q^{\beta_{i-1}}-q^{\beta_i})}{(1-1/t)(1-1/q)}\;.$$

Thus we can write

$$t^{\alpha_i} q^{\beta_j} B_\mu(i,j;q,t) \;=\; \frac{(u_j-x_j)(u_{i-1}-x_i)}{(1-1/t)(1-1/q)}\;.$$

Substituting this and 4.25 into 4.31 gives

$$\begin{aligned}(1-1/t)(1-1/q)&\sum_{\nu\to\mu} c_{\mu\nu}(q,t)\;T^k_{\mu\nu}\\ &= \sum_{i=1}^{m}\sum_{j=i}^{m}(u_j-x_j)(u_{i-1}-x_i)h_{k-1}[x_i+\cdots+x_j-u_i-\cdots-u_{j-1}]\\ &= h_{k+1}[x_1+\cdots x_m-u_o-\cdots u_m]\;,\end{aligned}$$

where the last equality follows from standard symmetric function identities. Replacing $x_1+\cdots x_m-u_o-\cdots u_m$ by the right hand side of 2.15, we finally obtain

$$\sum_{\nu\to\mu} c_{\mu\nu}(q,t)\;T^k_{\mu\nu} \;=\; \frac{1}{(1-\frac{1}{t})(1-\frac{1}{q})} h_{k+1}\Big[(1-\tfrac{1}{t})(1-\tfrac{1}{q})B_\mu(q,t)-\tfrac{1}{tq}\Big]\;,$$

which is another way of writing formula 2.13.

This completes the q,t-hook walk proof of Theorem 2.2.

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	$S_\emptyset$	S_1	$q^{-1}t^{-1}S_2$	$S_{1,1}$	$q^{-2}t^{-2}S_3$	$q^{-1}t^{-1}S_{2,1}$	$S_{1,1,1}$
$k_\emptyset$	1	0	0	0	0	0	0
k_1	$\bar{1}$	1	0	0	0	0	0
k_2	0	$\begin{matrix} 0 & \bar{1} \\ 1 & 0 \end{matrix}$	$\bar{1}$	$\begin{matrix} 1 & 0 \\ & 1 \end{matrix}$	0	0	0
$k_{1,1}$	1	$\bar{1}$	0	1	0	0	0
k_3	0	$\begin{matrix} 0 & 0 & 0 \\ 0 & \bar{1} & 0 \\ 1 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 1 \\ \bar{1} & \bar{1} & 0 \\ & \bar{1} & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & \bar{1} & 0 \\ 1 & 1 & 0 & \bar{1} \\ & 1 & 1 & 0 \\ & & 1 & 0 \end{matrix}$	$\begin{matrix} 1 & 0 \\ & 1 \end{matrix}$	$\begin{matrix} \bar{1} & 0 & 0 \\ & \bar{1} & 0 \\ & & \bar{1} \end{matrix}$	$\begin{matrix} 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 \\ & & 1 & 0 \\ & & & 1 \end{matrix}$
$k_{2,1}$	0	$\begin{matrix} 0 & 1 \\ \bar{1} & 0 \end{matrix}$	$\begin{matrix} 0 & 1 \\ \bar{1} & 0 \end{matrix}$	$\begin{matrix} 0 & \bar{1} & \bar{1} \\ 1 & 1 & \bar{1} \\ & 1 & 0 \end{matrix}$	1	$\begin{matrix} \bar{1} & 0 \\ & \bar{1} \end{matrix}$	$\begin{matrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{matrix}$
$k_{1,1,1}$	$\bar{1}$	1	0	$\bar{1}$	0	0	1

	$S_\emptyset$	S_1	$q^{-1}t^{-1}S_2$	$S_{1,1}$	$q^{-2}t^{-2}S_3$	$q^{-1}t^{-1}S_{2,1}$
k_4	0	$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \bar{1} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ \bar{1} & \bar{1} & \bar{1} & 1 & 0 \\ & \bar{1} & \bar{1} & 0 & 0 \\ & & \bar{1} & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{1} & \bar{1} & 0 & 0 \\ 1 & 1 & 1 & \bar{1} & \bar{1} & 0 \\ & 1 & 1 & 1 & \bar{1} & 0 \\ & & 1 & 1 & 0 & 0 \\ & & & 1 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 & \bar{1} & 0 \\ 1 & 1 & 1 & 0 & \bar{1} \\ & 1 & 2 & 1 & 0 \\ & & 1 & 1 & 0 \\ & & & 1 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 & 1 & 0 & 0 \\ \bar{1} & \bar{1} & \bar{1} & 0 & 1 & 0 \\ & \bar{1} & \bar{2} & \bar{1} & 0 & 1 \\ & & \bar{1} & \bar{2} & \bar{1} & 0 \\ & & & \bar{1} & \bar{1} & 0 \\ & & & & \bar{1} & 0 \end{matrix}$
$k_{3,1}$	0	$\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ \bar{1} & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 & \bar{1} \\ 0 & 1 & 2 & 0 \\ \bar{1} & \bar{1} & 1 & 0 \\ & \bar{1} & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 & 1 & 0 \\ 0 & \bar{1} & \bar{2} & \bar{1} & 1 \\ 1 & 1 & 0 & \bar{2} & 0 \\ & 1 & 1 & \bar{1} & 0 \\ & & 1 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & \bar{1} & \bar{1} \\ 1 & 2 & 1 & \bar{1} \\ & 1 & 2 & 0 \\ & & 1 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 1 & 1 & 0 \\ \bar{1} & \bar{2} & \bar{1} & 1 & 1 \\ & \bar{1} & \bar{3} & \bar{1} & 1 \\ & & \bar{1} & \bar{2} & 0 \\ & & & \bar{1} & 0 \end{matrix}$
$k_{2,2}$	0	0	$\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ \bar{1} & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & \bar{1} & \bar{1} & 0 \\ 1 & 0 & \bar{1} & 0 \\ & 1 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & \bar{1} \\ 1 & 1 & 0 \\ & 1 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 1 & 0 \\ \bar{1} & \bar{1} & 0 & 1 \\ & \bar{1} & \bar{1} & 0 \\ & & \bar{1} & 0 \end{matrix}$
$k_{2,1,1}$	0	$\begin{matrix} 0 & \bar{1} \\ 1 & 0 \end{matrix}$	$\begin{matrix} 0 & \bar{1} \\ 1 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 & 1 \\ \bar{1} & \bar{1} & 1 \\ & \bar{1} & 0 \end{matrix}$	$\begin{matrix} 0 & \bar{1} \\ 1 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 & 0 \\ \bar{1} & 0 & 1 \\ & \bar{1} & 0 \end{matrix}$

	$S_{1,1,1}$	$q^{-3}t^{-3}S_4$	$q^{-2}t^{-2}S_{3,1}$	$q^{-2}t^{-2}S_{2,2}$	$q^{-1}t^{-1}S_{2,1,1}$	$S_{1,1,1,1}$
k_4	$\begin{matrix} 0&0&0&\bar{1}&0&0&0 \\ 1&1&1&0&\bar{1}&\bar{1}&0 \\ &1&2&2&1&\bar{1}&0 \\ &&1&2&2&0&\bar{1} \\ &&&1&2&1&0 \\ &&&&1&1&0 \\ &&&&&1&0 \end{matrix}$	$\begin{matrix} \bar{1}&0&0&0 \\ &\bar{1}&\bar{1}&0 \\ &&\bar{1}&0 \\ &&&\bar{1} \end{matrix}$	$\begin{matrix} 1&0&0&0&0 \\ &1&1&0&0 \\ &&1&1&0 \\ &&&1&0 \\ &&&&1 \end{matrix}$	$\begin{matrix} 0&0 \\ \bar{1}&0 \end{matrix}$	$\begin{matrix} \bar{1}&0&0&0&0&0 \\ &\bar{1}&\bar{1}&0&0&0 \\ &&\bar{1}&\bar{1}&0&0 \\ &&&\bar{1}&\bar{1}&0 \\ &&&&\bar{1}&0 \\ &&&&&\bar{1} \end{matrix}$	$\begin{matrix} 1&0&0&0&0&0&0 \\ &1&1&1&0&0&0 \\ &&1&1&1&0&0 \\ &&&1&1&1&0 \\ &&&&1&1&0 \\ &&&&&1&0 \\ &&&&&&1 \end{matrix}$
$k_{3,1}$	$\begin{matrix} &0&0&\bar{1}&\bar{1}&\bar{1}&0 \\ &1&2&2&0&\bar{2}&\bar{1} \\ &&1&3&3&0&\bar{1} \\ &&&1&3&2&\bar{1} \\ &&&&1&2&0 \\ &&&&&1&0 \end{matrix}$	$\begin{matrix} &\bar{1}&\bar{1}&0 \\ &&\bar{1}&\bar{1} \\ &&&\bar{1} \end{matrix}$	$\begin{matrix} &1&1&0&0 \\ &&1&2&0 \\ &&&1&1 \\ &&&&1 \end{matrix}$	$\begin{matrix} \bar{1}&0 \\ &\bar{1} \end{matrix}$	$\begin{matrix} &\bar{1}&\bar{1}&0&0&0 \\ &&\bar{1}&\bar{2}&0&0 \\ &&&\bar{1}&\bar{2}&0 \\ &&&&\bar{1}&\bar{1} \\ &&&&&\bar{1} \end{matrix}$	$\begin{matrix} &1&1&1&0&0&0 \\ &&1&2&2&1&0 \\ &&&1&2&2&0 \\ &&&&1&2&1 \\ &&&&&1&1 \\ &&&&&&1 \end{matrix}$
$k_{2,2}$	$\begin{matrix} &&0&0&\bar{1}&\bar{1}&0 \\ &&1&1&1&\bar{1}&\bar{1} \\ &&&1&2&1&\bar{1} \\ &&&&1&1&0 \\ &&&&&1&0 \end{matrix}$	$\begin{matrix} &&\bar{1}&0 \\ &&&\bar{1} \end{matrix}$	$\begin{matrix} &&1&0&0 \\ &&&1&0 \\ &&&&1 \end{matrix}$	$\begin{matrix} &0 \end{matrix}$	$\begin{matrix} &&\bar{1}&0&0&0 \\ &&&\bar{1}&\bar{1}&0 \\ &&&&\bar{1}&0 \\ &&&&&\bar{1} \end{matrix}$	$\begin{matrix} &&1&0&1&0&0 \\ &&&1&1&1&0 \\ &&&&1&1&1 \\ &&&&&1&0 \\ &&&&&&1 \end{matrix}$
$k_{2,1,1}$	$\begin{matrix} &&&0&\bar{1}&\bar{1}&\bar{1} \\ &&&1&1&0&\bar{1} \\ &&&&1&1&\bar{1} \\ &&&&&1&0 \end{matrix}$	$\begin{matrix} &&&\bar{1} \end{matrix}$	$\begin{matrix} &&&1&0 \\ &&&&1 \end{matrix}$	$\begin{matrix} &\bar{1} \end{matrix}$	$\begin{matrix} &&&\bar{1}&0&0 \\ &&&&\bar{1}&0 \\ &&&&&\bar{1} \end{matrix}$	$\begin{matrix} &&&1&1&1&0 \\ &&&&1&1&1 \\ &&&&&1&1 \\ &&&&&&1 \end{matrix}$

$k_{3,2}$

$S_\emptyset$

0

S_1

0

$q^{-2}t^{-2}S_2$

0	0	0	$\bar{1}$
0	1	2	0
$\bar{1}$	$\bar{1}$	1	0
	$\bar{1}$	0	0

$q^{-1}t^{-1}S_{1,1}$

0	0	0	1	1
0	$\bar{1}$	$\bar{2}$	$\bar{1}$	1
1	1	$\bar{1}$	$\bar{2}$	0
	1	1	$\bar{1}$	0
		1	0	0

$q^{-2}t^{-2}S_3$

0	0	0	0	0	1
0	0	$\bar{1}$	$\bar{3}$	$\bar{2}$	0
1	2	3	0	$\bar{3}$	0
	1	3	3	$\bar{1}$	0
		1	2	0	0
			1	0	0

$q^{-1}t^{-1}S_{2,1}$

0	0	0	0	0	$\bar{1}$	0
0	0	1	3	2	0	$\bar{1}$
$\bar{1}$	$\bar{2}$	$\bar{3}$	0	4	2	0
	$\bar{1}$	$\bar{3}$	$\bar{4}$	0	3	0
		$\bar{1}$	$\bar{3}$	$\bar{3}$	1	0
			$\bar{1}$	$\bar{2}$	0	0
				$\bar{1}$	0	0

$k_{3,2}$

$S_{1,1,1}$

0	0	0	0	0	1	1	0
0	0	$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{2}$	1	1
1	2	3	1	$\bar{3}$	$\bar{5}$	$\bar{2}$	1
	1	3	5	3	$\bar{3}$	$\bar{3}$	0
		1	3	5	1	$\bar{3}$	0
			1	3	3	$\bar{1}$	0
				1	2	0	0
					1	0	0

$q^{-3}t^{-3}S_4$

0	0	0	1	2	0
$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	1	2
	$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{2}$	1
		$\bar{1}$	$\bar{3}$	$\bar{3}$	0
			$\bar{1}$	$\bar{2}$	0
				$\bar{1}$	0

$q^{-2}t^{-2}S_{3,1}$

0	0	0	$\bar{1}$	$\bar{2}$	0	0
1	2	3	2	$\bar{1}$	$\bar{3}$	0
	1	3	6	3	$\bar{1}$	$\bar{2}$
		1	3	6	2	$\bar{1}$
			1	3	3	0
				1	2	0
					1	0

$q^{-2}t^{-2}S_{2,2}$

0	0	1	0
$\bar{1}$	$\bar{1}$	0	1
	$\bar{2}$	$\bar{1}$	0
		$\bar{1}$	0

$k_{3,2}$

$q^{-1}t^{-1}S_{2,1,1}$

0	0	0	1	2	0	0	0
$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	1	3	1	0
	$\bar{1}$	$\bar{3}$	$\bar{6}$	$\bar{4}$	0	3	0
		$\bar{1}$	$\bar{3}$	$\bar{7}$	$\bar{4}$	1	2
			$\bar{1}$	$\bar{3}$	$\bar{6}$	$\bar{2}$	1
				$\bar{1}$	$\bar{3}$	$\bar{3}$	0
					$\bar{1}$	$\bar{2}$	0
						$\bar{1}$	0

$S_{1,1,1,1}$

0	0	0	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{1}$	0	0
1	2	3	3	1	$\bar{2}$	$\bar{3}$	$\bar{2}$	0
	1	3	6	7	4	$\bar{1}$	$\bar{3}$	$\bar{1}$
		1	3	7	8	4	$\bar{2}$	$\bar{1}$
			1	3	7	7	1	$\bar{2}$
				1	3	6	3	$\bar{1}$
					1	3	3	0
						1	2	0
							1	0

$q^{-4}t^{-4}S_5$

1	1	1	0	0
	1	2	2	0
		1	2	1
			1	1
				1

$q^{-3}t^{-3}S_{4,1}$

$\bar{1}$	$\bar{1}$	$\bar{1}$	0	0	0
	$\bar{1}$	$\bar{2}$	$\bar{3}$	0	0
		$\bar{1}$	$\bar{2}$	$\bar{3}$	0
			$\bar{1}$	$\bar{2}$	$\bar{1}$
				$\bar{1}$	$\bar{1}$
					$\bar{1}$

$k_{3,2}$

$q^{-3}t^{-3}S_{3,2}$

1	0	0
	2	0
		1

$q^{-2}t^{-2}S_{3,1,1}$

1	1	1	0	0	0	0
	1	2	3	0	0	0
		1	2	4	0	0
			1	2	3	0
				1	2	1
					1	1
						1

$q^{-2}t^{-2}S_{2,2,1}$

$\bar{1}$	0	0	0
	$\bar{2}$	0	0
		$\bar{2}$	0
			$\bar{1}$

$q^{-1}t^{-1}S_{2,1,1,1}$

$\bar{1}$	$\bar{1}$	$\bar{1}$	0	0	0	0	0
	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$	0	0	0
		$\bar{1}$	$\bar{2}$	$\bar{4}$	$\bar{1}$	0	0
			$\bar{1}$	$\bar{2}$	$\bar{4}$	$\bar{1}$	0
				$\bar{1}$	$\bar{2}$	$\bar{3}$	0
					$\bar{1}$	$\bar{2}$	$\bar{1}$
						$\bar{1}$	$\bar{1}$
							$\bar{1}$

$S_{1,1,1,1,1}$

1	1	1	1	1	0	0	0	0
	1	2	3	3	2	1	0	0
		1	2	4	4	3	1	0
			1	2	4	4	2	0
				1	2	4	3	1
					1	2	3	1
						1	2	1
							1	1
								1