

# THE *DNR* SYSTEM AS A CONCEPTUAL FRAMEWORK FOR CURRICULUM DEVELOPMENT AND INSTRUCTION<sup>1</sup>

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One of the main questions raised in the Foundations for the Future Conference was: What conceptual systems provide powerful foundations for success in mathematics or science classes? The chief goal of this chapter is to outline a conceptual framework—called *DNR-based instruction*<sup>2</sup> (or *DNR*, for short)—claiming to be such a system. *DNR-based instruction* stipulates conditions for achieving the critical goals of provoking students’ *intellectual need* to learn mathematics, helping them acquire<sup>3</sup> mathematical *ways of understanding* and *ways of thinking*, and assuring that they internalize and retain the mathematics they learn.

A critical element of *DNR-based instruction* is that mathematics teaching must not appeal to gimmicks, entertainment, or contingencies of reward and punishment, but focus solely on the learner’s *intellectual need* by fully utilizing humans’ remarkable capacity to be puzzled. Nor should mathematics curricula compromise the mathematical integrity of their contents. A subject matter is mathematical only if it adheres to and maintains the essential nature of the mathematics discipline. Thus, for example, a “geometry curriculum” is not geometry if deductive reasoning is not among its eventual objectives. Teaching correct mathematics, however, is not necessarily correct teaching. A teacher may maintain the mathematical integrity of the content he or she is presenting but neglect the *intellectual need* of the students or be mistaken as to what constitutes such a need for them. As a proof-free “geometry curriculum” does not teach geometry, an intellectually-purposeless “algebra curriculum,” one in which students’ actions are socially rather than intellectually driven, does not teach students. In *DNR-based instruction* the integrity of the content taught and the *intellectual need* of the student are equally central. The mathematical integrity of a curricular content is determined by the *ways of understanding* and *ways of thinking* which have evolved in many centuries of mathematical practice and continue to be the ground for scientific advances.

The positions expressed in the previous paragraph reflect some of the elements constituting the worldview underlying *DNR-based instruction*. *DNR* can be thought of as

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<sup>1</sup> To Appear in: FOUNDATIONS FOR THE FUTURE IN MATHEMATICS EDUCATION (R. Lesh, E. Hamilton, & J. Kaput Eds.), Erlbaum

<sup>2</sup> Italics are used in this paper to indicate that the term will be subsequently defined, whereas underlining is used to add emphasis. While it is unavoidable that the reader assigns her or his meanings to a term that is not defined at the time of its first appearance, it is necessary that later, after the term has been defined, the reader adjusts the meaning of the term accordingly.

<sup>3</sup> In current mathematics education literature, verbs such as “to acquire” and “to attain” are often avoided because, some argue, they connote passive learning, and, instead, the verb “to construct” is commonly used. In this paper, such verbs will be used freely and synonymously; their intended meaning is drawn from Piaget’s theory of equilibration.

a system consisting of: (a) premises (explicit assumptions underlying the *DNR* concepts and claims), (b) concepts (referred to as *DNR determinants*), and (c) *instructional principles* (claims about the potential effect of teaching actions on *student learning*). Not every *DNR instructional principle* is explicitly labeled as such. The system states only three foundational principles, the *duality principle*, the *necessity principle*, and the *repeated-reasoning principle*; hence, the acronym *DNR*. The other principles in the system are derivable from and organized around these three principles. Collectively, the three components that comprise the system—premises, determinants, and *instructional principles*—constitute a unified theoretical perspective about the learning and teaching of mathematics—a perspective that provides a language and tools to formulate and address critical curricular and instructional concerns. Some of these concerns were raised in the Foundations for the Future Conference; for example: What are some of the essential characters of problem-solving situations that lead to meaningful learning? What types of experiences facilitate or retard development?

The *DNR* system has been discerned from and, in turn, implemented in a long series of studies into the learning and teaching of mathematics in the elementary, secondary, and undergraduate levels, as well as studies with teachers of each of these levels. Earlier publications introduced a number of aspects of the *DNR* system—not always with this acronym and not necessarily in the version presented here (see, for example, Harel, 1990, 1997, 1998, 2001; Harel & Sowder, 2005). It goes beyond the scope of this paper to do more than outlining the system and briefly pointing to its potential in guiding curriculum development and instruction. A more extensive publication is underway which will describe the system in its entirety, lay out its complete theoretical foundations, and demonstrate its capacity to constitute a conceptual framework for designing, developing, and implementing mathematics curricula.

The chapter is organized in four sections: The first section lays out several of the underlying premises and determinants of *DNR*. The second section outlines the three chief *DNR instructional principles*—*duality*, *necessity*, and *repeated reasoning*—along with examples of learning-teaching situations that fulfill or violate them, corresponding to experiences that facilitate or retard development. The third section discusses several instructional activities from an ongoing intervention with algebra teachers and points to their rationale in terms of the *DNR-based instruction* perspective. The fourth, and last, section recapitulates the essential elements discussed in the paper.

## **Underlying Assumptions and Concepts**

### *Premises*

*DNR instructional principles* are not primary, in that they are based on certain premises and incorporate foundational concepts, called *DNR determinants*. The premises were not determined a priori, before *DNR* was formulated. Rather, they emerged in a process of reflection on and exploration of justifications for the *DNR* principles, and in

defining the *DNR* determinants, as we will see. There are eight *DNR* premises;<sup>4</sup> for the purpose of this paper, only four are needed:

1. ***Subjectivity Premise:*** Any observations humans claim to have made are due to the attribution of their mental structure to their environment.
2. ***Knowledge Development Premise:*** The process of knowing is developmental in the sense that it proceeds through a continual tension between accommodation and assimilation.
3. ***Teaching Premise:*** Construction of scientific knowledge is not spontaneous. There will always be a difference between what one can do under expert guidance or in collaboration with more capable peers and what one can do without guidance.
4. ***Mathematics Epistemology Premise:*** Knowledge of mathematics consists of all the *ways of understanding* and *ways of thinking* that have evolved throughout the history of mathematics.

As the reader might have recognized, the first and second premises are inherited from known theories—the subjectivity premise from Piaget’s constructivism and the knowledge development premise from the Piagetian theory of equilibration. Likewise, the teaching premise is Vygotsky’s known concept of *ZPK* (Zone of Proximal Knowledge).

#### *Determinants*

As to the *DNR* determinants, it is beyond the scope of this paper to discuss each of them or provide the theoretical foundations and motivations for the ones presented here. I will focus only on the most essential for the presentation of the three *DNR instructional principles*.

#### **A Triad of Determinants: Mental Act, Way of Understanding, and Way of Thinking**

Humans’ construction of knowledge involves numerous *mental acts* such as representing, interpreting, defining, computing, conjecturing, inferring, proving, structuring, symbolizing, transforming, generalizing, applying, modeling, connecting, predicting, reifying, classifying, formulating, searching, anticipating, and problem solving. *DNR-based instruction’s* focus is not just on what mental acts students perform and how often they perform them but also on the *products* and *characters* of these mental acts. The distinction between *product* and *character* of a mental act is central, for it delineates the content of the cognitive objectives at which *DNR-based instruction* aims as well as the knowledge held by the learner:

*Product* is a particular outcome of a mental act carried out by an individual, whereas *character* is a particular feature of that mental act. Respectively, they are referred to as a *way of understanding* and a *way of thinking* associated with the mental act.

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<sup>4</sup> They are accepted as premises in the *DNR* theoretical perspective, but they might have been substantiated empirically or theoretically elsewhere.

To illustrate the two categories of knowledge, ways of understanding and ways of thinking, consider the following example: Two first graders, Aaron and Betty, solve the problem “ $3 + 4 = ?$ ”. From conversations with the two, we may observe that Aaron interprets the “ $=$ ” sign merely as a command—add 3 and 4 and write the result in the place of the question mark—whereas Betty interprets the sign as equality between two quantities—the quantity that results from combining the two quantities 3 and 4 and an unknown quantity to be found. These different interpretations are products of Aaron’s and Betty’s interpreting act—they are their ways of understanding the “ $=$ ” sign in the string of symbols “ $3 + 4 = ?$ ”. We may infer on the basis of multitude of observations the characters of, or the ways of thinking associated with, Aaron and Betty’s interpreting acts. We may find, for example, that while Aaron’s interpreting act is characteristically devoid of quantitative considerations, Betty’s is quantitatively based.

Ways of thinking persist throughout grade levels from elementary school to college. Consider, for example, the string of symbols  $y = \sqrt{6x - 5}$ . Some of the *desirable* ways of understanding this string are:

1.  $y = \sqrt{6x - 5}$  is an “equation”—a condition on the quantities  $x$  and  $y$ .
2.  $y = \sqrt{6x - 5}$  is a “real-valued function”—for a real number  $x$ , there corresponds the value  $\sqrt{6x - 5}$
3.  $y = \sqrt{6x - 5}$  is a “proposition-valued function”—for an ordered pair of real numbers  $(x, y)$ , there corresponds the proposition values “true” or “false.”

These *desirable* ways of understanding are markedly different from many of high school and college students’ ways of understanding equality (a string of symbols of the form  $A = B$ , where  $A$  and  $B$  are algebraic expressions, or functions). For many students, equality represents no quantitative reality except that symbols must be transformed according to some rules to get an answer judged correct or wrong by the teacher or textbook—consistent with Aaron’s behavior. These students’ behaviors manifest a way of thinking in which the mental act of symbolizing is characteristically free from meaningful quantitative referents. This behavior is referred to as the *non-referential symbolic* way of thinking.

Ways of thinking seem to be classifiable into three categories: *problem-solving approaches*, *proof schemes*, and *beliefs about mathematics*. Although these three categories may not constitute the entire universe of ways of thinking, they definitely represent an important portion of it.

### *Problem-Solving Approaches*

A problem-solving act is not of the same status as the other mental acts listed above. Any of these acts is, in essence, a problem-solving act. The acts of interpreting and generalizing, as well as those of inferring, structuring, symbolizing, proving and so on. are essentially acts of problem solving. Despite this, the distinction among the different mental acts is cognitively and pedagogically important, for it enables us to better understand the nature of mathematical practice by individuals and communities in the

classroom and throughout history, and, accordingly, set explicit cognitive objectives for instruction.

The actual “solution” one provides—viewed as such by oneself—is a way of understanding because it is a particular product of one’s problem-solving act. A problem-solving approach, on the other hand, is a way of thinking. For example, each of the approaches “look for a simpler problem,” “consider alternative possibilities while attempting to solve a problem,” and “look for a key word in the problem statement” characterizes, at least partially, one’s problem-solving act; hence, they are instances of ways of thinking. Since *heuristics* are problem-solving approaches, they are also ways of thinking. In the literature, the term *heuristic* is often used for successful problem-solving approaches: “Heuristic strategies are rules of thumb for successful problem solving, general suggestions that help an individual to understand a problem better or to make progress toward its solution” (Schoenfeld, 1985, p. 23). I use the term “heuristic” in this sense, and further stress that, consistent with the subjectivity premise, the judgment as to whether a problem-solving approach is a heuristic is made from the viewpoint of the observer, not the observed. The observer, usually a mathematically experienced individual, is unlikely to judge the approach “look for a key word in the problem statement” as a heuristic. A student, on the other hand, may deem this approach successful if in her or his experience its application has resulted in a correct answer in a large percentage of cases—a not uncommon scenario in current school mathematics due to the type of problems students are assigned.

### *Proof Schemes*

Like the problem-solving act, the mental act of *proving*, too, has a special status in that it is an act that is involved in one way or another in any mathematical activity. Indeed many ways of thinking are characters of the problem-solving and *proving* acts. While problem-solving approaches are instances of ways of thinking associated with the problem-solving act, *proof schemes* are the ways of thinking associated with the *proving* act. The mental act of *proving* has a special meaning in *DNR-based instruction*. As defined in Harel and Sowder (1998):

*Proving* is the act employed by a person to remove or instill doubts about the truth of an assertion.

Any observation one makes can be conceived either as a *conjecture* or as a *fact*:

A *conjecture* is an assertion made by a person who has doubts about its truth. A person’s assertion ceases to be a conjecture and becomes a *fact* in her or his view once the person becomes certain of its truth.

There, in Harel and Sowder (1998), a distinction was made between two instances of the proving act: *ascertaining* and *persuading*:

*Ascertaining* is the act an individual employs to remove her or his own doubts about the truth of an assertion (or about the truth of its negation), whereas *persuading* is the act an individual employs to remove others’ doubts about the truth of an assertion (or of its negation).

Conceptually and methodologically, it is difficult to separate these two acts, for in ascertaining for oneself, one considers how to persuade others, and vice versa.

Pedagogically, however, the distinction is critically important, for students need to learn that it is necessary that they ascertain for themselves before attempting to convince others, and that what constitutes ascertainment for themselves may not convince others. The notions which have just been defined are the basis for the concept of *proof scheme*:

A *proof scheme* is a character of one's collective acts of ascertaining and persuading; hence, it is a way of thinking.

Note that while a proof scheme is a way of thinking, a proof—a particular chain of arguments one offers to ascertain for oneself or to convince others—is, by definition, a way of understanding. To illustrate, consider two individuals, say Aimee and Bob, who, through some experiences, made the conjecture that for any real numbers,  $a$ ,  $b$ ,  $c$ , and  $d$ ,  $(a+b)/(a-b) = (c+d)/(c-d)$  whenever  $a/b = c/d$ . Aimee evaluated this conjecture in several specific cases and concluded that it is true. Bob, on other hand, applied a different approach, such as:

$$a/b = c/d \Rightarrow a/b + 1 = c/d + 1 \Rightarrow (a+b)/b = (c+d)/d \Rightarrow (a+b)/(c+d) = b/d$$

and

$$a/b = c/d \Rightarrow a/b - 1 = c/d - 1 \Rightarrow (a-b)/b = (c-d)/d \Rightarrow (a-b)/(c-d) = b/d$$

So:

$$(a+b)/(a-b) = (c+d)/(c-d).$$

Both Aimee and Bob have carried out the mental act of proving, but each produces a different outcome—a different proof. Respectively, these two proofs are Aimee's and Bob's ways of understanding the reason for why the conjecture is true. Clearly, the character of Aimee's proving act—her proof scheme—is different from the character of Bob's proving act: while the former is empirical the latter is deductive. In Harel and Sowder (1998), a broad taxonomy of students' proof schemes was identified by examining their proofs. Later, in Harel (in press), this taxonomy was refined and extended to capture more students' proof schemes as well as those of the mathematics community throughout history (see also, Harel and Sowder, in press).

#### *Beliefs about Mathematics*

Problem-solving approaches and proof schemes are ways of thinking internal to mathematics; they characterize the mental acts one carries out in doing mathematics. It is necessary to distinguish them from *beliefs*: one's views of mathematics. More specifically, *beliefs* here are restricted to the character of one's interpretation of (a) what mathematics is, (b) how it is created, and (c) its intellectual or practical benefits. Examples of these beliefs include, respectively, (a) "Formal mathematics has little or nothing to do with real thinking or problem solving" (Schoenfeld, 1985), (b) "The solution of a problem should not take more than five minutes" (Schoenfeld, 1985), and (c) "It is advantageous to have multiple interpretations of a mathematical concept" (Harel, 1998).

#### **Ways of Understanding and Ways of Thinking: Desirable versus Institutionalized**

The terms way of understanding and way of thinking do not imply correct knowledge, as can be seen from the examples discussed this far. In referring to what

students know, the terms only indicate the knowledge—correct or erroneous, useful or impractical—currently held by the students. It must be highlighted, however, that in DNR-based instruction the ultimate goal is for students to develop ways of understanding and ways of thinking compatible with those that have been institutionalized in the discipline of mathematics, those the mathematics community at large accepts as correct and useful in solving mathematical and scientific problems. For example, the goal is to gradually refine current students’ proof schemes toward the proof scheme shared and practiced by the mathematicians of today. Such a scheme exists, by the epistemology premise, and is believed to be part of the ground for scientific advances in mathematics.

This goal is meaningless without considering the knowledge development premise, which implies that students’ learning necessarily involves the construction of imperfect and even erroneous ways of understanding and deficient, or even faulty, ways of thinking. Teachers must be aware of this natural phenomenon when working toward a cognitive objective, and their teaching actions must be consonant with it. In particular, they must attempt to identify students’ current ways of understanding and ways of thinking, regardless of their quality, and help students gradually refine them.

During this process of refinement, teachers may set their cognitive objectives in terms of *desirable*, rather than institutionalized, ways of understanding or ways of thinking. A way of understanding or way of thinking is *desirable* if a teacher sets it as an intermediate cognitive objective toward one held and practiced by the mathematics community at large. Clearly, any institutionalized way of understanding or way of thinking is also desirable, but the converse is not necessarily true. To illustrate this distinction between “desirable” and “institutionalized,” consider the following example:

Several studies (e.g., Dubinsky, 1986, Ernest, 1984, Harel, 2001, Brown, 2003) have shown that students often have major difficulties with the concept of mathematical induction. There are many reasons for the difficulties students experience with this concept; the one that is relevant to the discussion here is that the standard instructional treatments introduce the formal principle of mathematical induction too abruptly and without ensuring that the students have a need—an *intellectual need*—for it. In a *DNR-based* teaching experiment, reported in Harel (2001), the concept was introduced in four instructional phases, each aiming at necessitating for the students a new way of understanding. Each phase is structured so that the repeated application of that way of understanding can lead to the construction of a particular way of thinking associated with the act of proving. The cognitive objectives in the first three phases were intermediate in the sense they consisted of desirable, not necessarily institutionalized, ways of understanding and ways of thinking. In Phase 1, for example, the students were engaged for a relatively long period of time in working on problems typified<sup>5</sup> by:

- Find an upper bound to the sequence  $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \dots$
- Let  $n$  be a positive integer. Show that any  $2^n \times 2^n$  chessboard with one square removed can be tiled using L-shaped pieces, where each of the pieces covers three squares.

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<sup>5</sup> For the precise characterization of these problems, see Harel (2001).

The general solution approach that emerged from these problems was referred to in Harel (2001) as *quasi-induction*, which is exemplified by the following solution given by one of the students in the teaching experiment to first problem:

Since  $\sqrt{2}$  is less than 2,  $2 + \sqrt{2}$  is less than 4, and so  $\sqrt{2 + \sqrt{2}}$  is less than 2.

Since  $\sqrt{2 + \sqrt{2}}$  is less than 2,  $2 + \sqrt{2 + \sqrt{2}}$  is less than 4. Hence,  $\sqrt{2 + \sqrt{2 + \sqrt{2}}}$  is less than 2. And so on.

Thus, the intermediate, desirable cognitive objective of this phase was for students to construct quasi-induction as a way of thinking—as a method of proof by which they ascertain for themselves or persuade others about the truth of certain kind of mathematical propositions. In this stage, the goal was not for students to prove such propositions by the institutionalized method, the formal principle of mathematical induction. That goal was left to fourth stage, the final stage of the instructional treatment.

The focus of DNR-based instruction on the two categories of knowledge, ways of understanding and ways of thinking, is entailed from the system of premises presented in the opening of Section 1. First, as was mentioned earlier, there is a need to identify students' ways of understanding and ways of thinking as they currently are—independent of their quality—for, by the knowledge development premise, students' construction of new knowledge is based on what they already know. Second, by the teaching premise, teachers' expert guidance is necessary to enable students develop new knowledge. Hence, it is essential that teachers build models for students' knowledge to inform their teaching actions; without such models teachers' planning and implementation of instruction is likely to be unguided and haphazard. Third, while necessary, such models are not sufficient: teachers' actions must be directed, in addition, by cognitive objectives formed in terms of desirable or institutionalized ways of understanding and ways of thinking, which, by the epistemology premise, comprise mathematical knowledge.

### **The Concept of Instructional Principle and Its Related Determinants: Teaching Action and Student Learning**

What is an *instructional principle*? Consider the common conception “In sequencing mathematics instruction, start with what is easy.” This conception might be interpreted as implying a cause-effect link between a *teaching action*—that of sequencing mathematics instruction—and *student learning*. The *teaching action* might be viewed as a likelihood condition: starting with what is easy for the students may help students learn. It might be viewed as a necessary condition: for students to learn teachers must start with what is easy for them. Or it might be viewed as a sufficient condition: starting with what is easy for the students will help students learn. Textbook authors and teachers seem to use this conception mostly in its necessary-condition form. This example can be abstracted to define the notion of *instructional principle*:



An *instructional principle* is a conception implying an effect of a teaching action on *student learning*. The teaching action may be conceived as a likelihood condition, necessary condition, or sufficient condition for the effect to take place.<sup>6</sup>

By *teaching action* is meant:

A curricular or instructional measure or decision a teacher carries out for the purpose of achieving a cognitive objective, establishing a new didactical contract, or implementing an existing one.

Of critical importance is the definition of *student learning*:

*Student learning* is a continuum of disequilibrium-equilibrium phases together with (a) the *ways of understanding* and *ways of thinking* that the learner utilizes or newly constructs during the various phases and (b) the cognitive, social, and affective stimuli that result from or instigate these phases.

Research in mathematics education has offered useful models for learning trajectories of various mathematical concepts and ideas. However, relative to the broad scope of this definition these models are incomplete. This is expected due to the enormous empirical and theoretical difficulties in building models that incorporate phases of disequilibrium-equilibrium, their utilized or resultant *ways of understanding* and *ways of thinking*, and the cognitive, social, and affective stimuli that result from and instigate the various phases. Such comprehensive models, however, are desirable and the hope is that they will be constructible in the future. Ideally, the concept of “instructional principle”—a general conception about an effect of a teaching action on student learning—should be understood in terms of this encompassing definition of “student learning.” However, given the obvious difficulties in fully incorporating all the elements of student learning, such an interpretation may not be practical. For this reason, *DNR* instructional principles will be limited in their scope, in that they are general conceptions about the effect of teaching actions on students’ ways of understanding and ways of thinking and the intellectual stimuli that are needed for disequilibrium-equilibrium phases. While affective stimuli are important, they are not central to the theoretical perspective presented in this chapter.

## **DNR’s Three Foundational Instructional Principles**

### *The Duality Principle*

The *duality principle* concerns the developmental interdependency between what the students produce and the character of their mental acts—between their ways of understanding and their ways of thinking. Before formulating the principle, let us consider one of the findings in our research on proof schemes. In all of the teaching experiments I conducted on this question, the most noticeable, prevalent, and persistent way of thinking among the participants—students as well as teachers—was the *empirical proof scheme*. That is, a scheme by which one proves (attempts to remove or instill

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<sup>6</sup> What is an instructional principle for one person might be a myth for another. See, for example, Harel and Sowder (2005) for a discussion on the pedagogical ineffectiveness of the conception “In sequencing mathematics instruction, start with what is easy.”

doubts about the truth of a conjecture) either inductively—by relying on evidence from examples of direct measurements of quantities, substitutions of several numbers in algebraic expressions, and so on—or perceptually—by relying on evidence from visual or tactile perceptions. This way of thinking is especially robust among both students and teachers when they observe a pattern suggesting that one can generate as many examples as wants in support of the conjecture. This finding has been observed by other researchers as well (see Goetting, 1995; Chazan, 1993).

Based on the knowledge development premise, students do not come to school as blank slates, ready to acquire desirable knowledge independently of what they already know. Rather, what students know now constitutes a basis for what they will know in the future. This is true for all ways of understanding and ways of thinking associated with any mental act; the mental act of proving is of no exception. In daily life, the methods of justification available to humans are largely empirical. From childhood, when humans seek to justify or account for a particular phenomenon, they are likely to do so based on similar or related phenomena in their past. Given that the number of such phenomena in one's past is unavoidably finite, people's judgments are typically empirical. Through such repeated experience, humans' hypothesis evaluation evolves to be dominantly empirical, whereby the empirical proof scheme is established as a way of thinking characterizing their mental act of proving. A person's repeated experiences of justifying particular assertions and of accounting for particular phenomena in daily life thus shape the character of her or his mental act of proving. Summarizing:

1. The ways of understanding associated with one's mental act of proving impact the way of thinking associated with that mental act.

Of equal importance is the converse of this statement. Continuing with our discussion of the empirical proof scheme, this way of thinking does not disappear upon entering school, nor does it fade away effortlessly when students take mathematics classes. It continues to impact the particular justifications students produce for mathematical assertions—their way of understanding why particular assertions are true or false. It takes enormous instructional effort for students to recognize the limits of empirical evidence in mathematics and to construct alternative, deductively-based proof schemes. Even mathematically able students are not immune from the impact of the empirical proof scheme (see Fischbein and Kedem, 1982). Summarizing:

2. The ways of thinking associated with one's mental act of proving impact the way of understanding associated with that mental act.

This brief account points to a reciprocal cause-effect relation between ways of understanding and ways of thinking with respect to the mental act of proving: through everyday experience in justifying assertions and accounting for phenomena occurring in their physical and social environment, humans' mental act of hypothesis evaluation evolves to be empirically based. In turn, the way of thinking that most students bring to mathematics classes is dominantly the empirical proof scheme, which they readily apply in solving mathematical problems involving justification. Thus, together, Statements 1 and 2 above express a dual effect associated with the mental act of proving. The *duality principle* asserts that this dual effect is developmental and applies to all mental acts. Specifically,

**The Duality Principle:** Students develop ways of thinking only through the construction of ways of understanding, and the ways of understanding they produce are determined by the ways of thinking they possess.

There is reciprocity between ways of understanding and ways of thinking claimed in the duality principle: a change in ways of thinking brings about a change in ways of understanding, and vice versa. The claim intended is, in fact, stronger: not only do these two categories of knowledge affect each other but a change in one cannot occur without an appropriate change in the other. This assertion constitutes an instructional principle because it implies an effect of a teaching action on student learning. Specifically:

- a. Students would be able to construct a way of thinking associated with a certain mental act or refine or modify an existing one only if they are helped to construct suitable ways of understanding associated with that mental act.

Conversely:

- b. Students would be able to construct a way of understanding associated with a certain mental act or refine or modify an existing one only they are helped to construct suitable ways of thinking associated with that mental act in the form of problem-solving approaches, proof schemes, or beliefs about mathematics.

Thus, for example, the teaching action of preaching ways of thinking to students would have no effect on the quality of the ways of understanding they would produce.

Similarly, talking to them about the nature of proof in mathematics, advising them to use particular heuristics, or telling them what beliefs about mathematics to adopt would have minimal or no effect on the quality of the proofs and solutions they will produce and the particular beliefs about mathematics they will hold. Only by producing desirable ways of understanding—by ways of carrying out mental acts of, for example, interpreting mathematical ideas through reading, writing, and oral communication, solving mathematical problems, and proving mathematical assertions—can students construct desirable ways of thinking.

Current textbooks and teaching practices do not pay enough attention to ways of thinking. For example, seldom do they consider questions such as: What specific ways of thinking do common instructional activities in algebra—such as simplifying, factoring, and so on.—promote? Or, what exactly are the ways of thinking that can or should be promoted by the use of computer technology in algebra? An immediate implication of the duality principle is that it is essential that teachers form instructional goals in terms of ways of thinking and devise and use appropriate instructional activities through which students can build ways of understanding that can potentially help them construct desirable ways of thinking. This implication leads to two critical questions: What constitute such appropriate instructional activities? What is the nature of instructional treatments that can help students construct desirable ways of understanding and ways of thinking? These questions are addressed by the other two *DNR* principles.

#### *The Necessity Principle*

Fundamental to *DNR-based instruction* is the knowledge development premise, which entails that problem solving should not be just a goal but also the means for

learning mathematics (see also Brownell 1946; Davis, 1992; Hiebert, 1997). “Problem solving” is usually defined as “[engagement] in a task for which the solution method is not known in advance” (NCTM, 2000). Many of the situations students encounter in school satisfy this definition and yet they do not constitute problem solving because the situations are not *intrinsic*, but *alien*, to the students.

When we talk about problems in the context of mathematics curricula, we refer to learning-teaching events that involve two sets of interpretations (i.e., ways of understanding): the set that belongs to the poser of the problem and the set that belongs to the one to whom the problem is posed. An important consequence of this simple observation is that the two sets are not always identical and, in many cases, do not even intersect. Teachers are often unaware of this. They might present a problem to their class and incorrectly assume that their students share their interpretation(s) of the problem. Conversely, students might pose a question to their teacher, who either encounters difficulty making sense of what they are asking or interprets their question differently from what the students intended. Of particular interest are the scenarios where what is conceived as a problem by the teacher is unproblematic to the students, and vice versa. In general, a situation that is problematic to one person but is unproblematic to another is referred to as *intrinsic* (*I*) to the first and *alien* (*A*) to the second. Accordingly, mathematics problems in a learning-teaching setting are of four categories: (*A, I*), (*I, A*), (*I, I*), and (*A, A*), where the first component of these ordered pairs refers to the student, and the second to the teacher. (*I, I*)—the situation where a problem as stated is intrinsic to both—is the only desired category among the four. As long as the problem is intrinsic to both, significant learning is likely to occur, even when—or perhaps especially when—the teacher’s interpretation is different from those of the students.

Unfortunately, none of the other three cases, (*A, I*), (*I, A*), and (*A, A*), is uncommon. The case (*A, A*), for example, occurs especially when the teacher lacks basic understanding of the mathematical content he or she is teaching—a not uncommon phenomenon that has been documented in the literature (Ball, 1990; Cohen, 1991; Ma, 1999; Post, T., Harel, G., Behr, M. & Lesh, R., 1991; Simon, 1993). To illustrate, consider the following episode: A ninth-grade teacher requires his class to accompany each assertion written on the left-hand side of two-column proofs by a “justification” on the right-hand side, both in algebra and in geometry. For example, a student in this class justified the assertions, “ $AB \cong AB$ ” by the phrase “reflexive property;” the assertion “If  $\angle ABC = 30^\circ$  and  $\angle CBD = 45^\circ$ , then  $\angle ABD = 75^\circ$ ,” by “additive property;” and the assertion “ $a + b = b + a$ ” by “commutative property.” It turned out that neither the student nor the teacher understood the meaning of these phrases, and both viewed the three assertions as obvious—ones that require no justification—but each felt compelled to follow rules: the student had to follow those imposed by her teacher, and the teacher those imposed by the textbook. Thus, the task of justification was alien to both the teacher and his student—a clear (*A, A*) case. On the other hand, if this teacher possessed the *axiomatic proof scheme*—the way of thinking that mathematical structures are determined by a set of axioms—he would have understood the meaning and role of the

reflexive and additive properties in Euclidean geometry, and of the commutative property in algebra, and in turn the situation would have been intrinsic to him. Cases of the  $(A, I)$  category are perhaps the most common in mathematics education in all levels, but especially at the college level, where the instructors have deep understanding of the problems they present to their students but usually are unaware that their appreciation of the problems is not always shared by their students. This supports the claim that while mathematical knowledge is indispensable for quality teaching, it is not sufficient. Teachers must also know how to address students as learners. An important application of the *necessity principle* is to assure that the mathematics students' learning grows out of problems intrinsic to them:

**The Necessity Principle:** For students to learn what we intend to teach them, they must have a need for it, where by 'need' is meant *intellectual need*, not social or economic need.

Most students, even those who are eager to succeed in school, feel intellectually aimless in mathematics classes because we—teachers—fail to help them realize an *intellectual need* for what we intend to teach them. The term *intellectual need* refers to a behavior that manifests itself internally with learners when they encounter an intrinsic problem—a problem they understand and appreciate. For example, students might encounter a situation that is incompatible with, or presents a problem that is unsolvable by, their existing knowledge. Such an encounter is intrinsic to the learners for it stimulates a desire within them to search for a resolution or a solution, whereby they might construct new knowledge. There is no guarantee that the learners construct the knowledge sought or any knowledge at all, but whatever knowledge they construct is meaningful to them since it is integrated within their existing cognitive schemes as a product of effort that stems from and is driven by their personal, intellectual need. Whereas one should not underestimate the importance of students' social need (e.g., mathematical knowledge can endow me with a respectable social status in my society) and economic need (e.g., mathematical knowledge can help me obtain comfortable means of living) as learning factors, teachers should not and cannot be expected to stimulate—let alone fulfill—these needs. Intellectual need, on the other hand, is prime responsibility of teachers and curriculum developers.

### *The Repeated Reasoning Principle*

Even if ways of understanding and ways of thinking are necessitated through students' *intellectual need*, there remains the task of ensuring that students internalize, organize, and retain this knowledge. Namely, the question of concern is: What are the conditions on instructional treatments that are necessary for students to internalize, organize, and retain desirable ways of understanding and ways of thinking? Research has shown that repeated experience, or practice, is a critical factor in these cognitive processes (see, for example, Cooper, 1991). The emphasis of *DNR-based instruction* is on repeated reasoning that reinforces desirable ways of understanding and ways of thinking. Repeated reasoning, not mere drill and practice of routine problems, is essential to the process of internalization, which is a conceptual state where one is able apply

knowledge autonomously and spontaneously. The sequence of problems must continually call for reasoning through the situations and solutions, and must respond to the students' changing intellectual needs. This is the basis for the *repeated reasoning principle*.

**The Repeated Reasoning Principle:** Students must practice reasoning in order to internalize, organize, and retain ways of understanding and ways of thinking.

Consider the following scenario (reported in Harel & Sowder, 2005): Two elementary school children, Sam and Tami, were taught division of fractions. Sam was taught in a typical method, where he was presented with the rule  $(a/b) \div (c/d) = (a/b) \cdot (d/c)$ , and the rule was introduced to him in a meaningful context and with an adequate justification that he understood. Tami, on the other hand, was presented with no rule. Each time she encountered a division of fraction problem, she explained its meaning and the rationale of her solution. Sam and Tami were assigned homework problems on division of fractions. Sam solved all the problems correctly, and gained, as a result, a good mastery of the division rule. It took Tami a much longer time to do her homework. Here is what Tami—a real third-grader—said when she worked on  $(4/5) \div (2/3)$ :

How many  $2/3$ s in  $4/5$ ? I need to find what goes into both [meaning: a unit-fraction that divides  $4/5$  and  $2/3$  with no remainders].  $1/15$  goes into both. It goes 3 times into  $1/5$  and 5 times into  $1/3$ , so it would go 12 times into  $4/5$  and 10 times into  $2/3$  (She writes:  $4/5 = 12/15$ ;  $2/3 = 10/15$ ;  $(4/5) \div (2/3) = (12/15) \div (10/15)$ ). How many times does  $10/15$  go into  $12/15$ ? ... How many times do 10 things go into 12 things? ... One time and  $2/10$  of a time ... which is 1 and  $1/5$ .

Tami had opportunities for reasoning of which Sam was deprived. Tami practiced reasoning and computation; Sam practiced only computation. Further, Tami eventually abstracted the division rule and learned an important lesson about mathematical efficiency—a way of thinking Sam had little chance to acquire. Further, though repeated solutions of this kind, Tami would likely develop the way of thinking of interpreting situations in terms of linear change—what is known in the literature as proportional reasoning.

The repeated reasoning principle is complementary to the other two principles in that its aim is for students to internalize the ways of understanding and ways of thinking acquired through the application of the other two principles. Through repeated reasoning in solving intrinsic problems the application of ways of understanding and ways of thinking become autonomous and spontaneous. This principle, as the other two principles in the *DNR* system, has been a keystone in our teaching experiments with students and teachers, and we attribute the internalization of important ways of thinking by our participants to the persistent application of this principle in our teaching. For example, in a linear algebra teaching experiment for mathematics and engineering majors, the aim was to eradicate students' faulty proof schemes by helping them develop alternative, deductive proof schemes. The students were engaged in numerous analyses

of linear algebra questions in terms of systems of linear equations and, in turn, in terms of the meaning of row operations on the system's equations. The intellectual benefit of this activity is that students built images of the structure of the row echelon form of a matrix and the implication of row reduction to span and linear independence. Row reduction became for these students a conceptual tool—a way of thinking—to approach problems on the existence and uniqueness of solutions to linear systems and problems on span and linear independence in  $C^n$ .

### **DNR-Based Instruction Intervention: Examples from a Professional Development Class for Teachers**

A three-year NSF-funded project is underway to systematically examine the effect of DNR-based instruction on the teaching practice of algebra teachers and on the achievement of their students. I will conclude this paper with a discussion of instructional activities from this study and point to their rationale in terms of the underlying DNR-based instruction theoretical perspective.

It is clear from the formulation of the duality principle that the subjectivity premise is its most critical foundation. In our work with teachers a major effort is made to help them realize that the approach a student chooses to solve a problem depends on her or his way of understanding the problem; hence, not only are multiple solutions to a given problem possible but in many cases are inevitable. Adhering to the duality principle, the teachers come to this realization through their own multiple solutions to mathematical problems. In turn, they gradually acquire the ways of thinking, “a problem can have multiple solutions” and “it can be advantageous to solve a problem in different ways.” Similarly, through their own interpretations of concepts, they learn that “a concept can have multiple interpretations” and “it can be advantageous to possess multiple interpretations of a concept.” These ways of thinking, although essential in the learning and creation of mathematics, are often absent from teachers' and students' repertoires of reasoning. We engage teachers in activities aimed at promoting multiple ways of understanding concepts. For example, through their solutions to different problems the teachers learn that the concept of fraction, say  $3/4$ , can be understood in different ways and it is advantageous to understand it in different ways. Such ways of understanding include: *unit fraction* ( $3/4$  is the sum of  $1/4 + 1/4 + 1/4$ ); *partition* ( $3/4$  is the quantity that results from dividing 3 units into 4 equal parts); *measurement* ( $3/4$  is the measure of a 3 cm long segment with a 4-cm unit ruler); *solution to an equation* ( $3/4$  is the solution to  $4x = 3$ ); *part-whole* ( $3/4$  is 3 out of 4 units). Similarly, the teachers solved problems through which they learned multiple ways of understanding algebraic expressions of the form  $y = f(x)$  (e.g.,  $y = 3x^2 + x - 2$ ) and the significance of each interpretation. For example, they see how one can understand  $y = f(x)$  in terms of a condition on the variables  $x$  and  $y$ , or in terms of a real-valued function, or in terms of proposition-valued function, as we have discussed earlier. These newly acquired desirable ways of understanding by the teachers are contrasted with the interpretations they or their students commonly possess. For example, through the actual work of

students, the teachers see that for many students a string of symbols such as  $y = 3x^2 + x - 2$  represents no quantitative reality (the non-referential proof scheme), except possibly that the “=” sign is understood as a “do something signal,” where one side of the equation is reserved for the operation to be carried out and the other side for its outcome, as was documented by Behr, Erlwanger, & Nichols (1976).

In accordance with the necessity principle, the different interpretations of concepts are necessitated for the teachers through their own solutions to problems. For example, on one occasion the teachers, working in small groups, investigated the condition under which one fraction, say  $\alpha$ , is smaller than another fraction, say  $\beta$ , where the difference between the numerators of  $\alpha$  and  $\beta$ , which is given to be an integer, is equal to the ratio of their denominators. After 35 minutes, one of the teachers, Colleen, presented her group’s solution. The group’s solution process involved many quadratic inequalities. During Colleen’s presentation there were numerous questions and suggestions from the class regarding the solution process. After about 25 minutes of classroom discussion, the final answer that the class derived from a complex system of inequalities was summarized by Colleen as follows:

We are to find when is  $n/d < (n+x)/(d/x)$ , where  $n$  and  $d$  are integers and  $x$  is a non-zero number. The solution is:

- For  $-4 < n$  or  $n > 0$ ,  $x > (-n + \sqrt{n^2 + 4n})/2$  or  $x < (-n - \sqrt{n^2 + 4n})/2$ ;
- For  $-4 < n < 0$ , all  $x$  values work
- For  $n = 0$  or  $n = -4$ ,  $x \neq -n/2$ .

Following this presentation, one of the teachers, Ken, suggested checking a few cases to see if they agree with the final answer. (“Evaluate your solution” is a way thinking strongly stressed throughout the intervention.) After about five minutes of group work, Ken indicated that he noticed a strange phenomenon: when different forms of the same fraction are used different results are obtained. He demonstrated his observation with the equivalent fractions  $-2/5$  and  $-6/15$ . Indeed, for  $-2/5$  any value of  $x$  works but for  $-6/15$  only  $x < 3 - \sqrt{3}$  or  $x > 3 + \sqrt{3}$  work. This provided a great opportunity—intended and anticipated—for the instructor to point to the necessity to differentiate between “fraction” and “rational number.” Although the ratio is the same, the fraction is not, and the solution with a different fraction, even if the ratio is the same, is also different. The solution for the fraction  $-2/5$  is different from the solution for the fraction  $-6/15$ . This result was particularly amazing for the teachers. Their own solution to a problem that was intrinsic to them necessitated for them the distinction between “fraction” and “rational number.”

On another occasion, the teachers worked on justifying the quadratic formula. Prior to this problem, the teachers had repeatedly worked with many quadratic functions, finding their roots by essentially completing the square. In doing so they repeatedly transformed a given equation  $ax^2 + bx + c = 0$  into an equivalent equation of the form  $(x+T)^2 = L$  for some terms  $T$  and  $L$ , in order for them to solve for  $x$  (as  $-T + \sqrt{L}$  and



$-T - \sqrt{L}$ ). They then abstracted this process to develop the quadratic formula. To get to the desired equivalent form, they understood the reason and need for dividing through by  $a$ , bringing  $c/a$  to the other side of the equation, and completing the square. For these teachers, the symbolic manipulation process was goal oriented and conditioned by quantitative considerations; namely, transformations are applied with the intention to achieve a predetermined intrinsic goal. In this case, the teachers practiced the way of thinking of transforming an algebraic expression into a desired form without altering its original quantitative value. This way of thinking—which is, in our view, one of the essential characteristics of algebraic reasoning—was known among the teachers as the changing-the-form-without-changing-the-value habit of mind. We see here the simultaneous implementation of the duality principle, the necessity principle, and the repeated reasoning principle. In particular, the repeated application of this habit of mind helped the participant teachers internalize it, whereby they become autonomous and spontaneous in applying it.

### **Recapitulation**

Ways of understanding refer to products of mental acts, such as particular interpretations, generalization, solutions, justifications, and so on. which students produce as they are engaged in mathematical activities. Ways of thinking, on the other hand, are characters of one's mental acts; problem-solving approaches, proof schemes, and beliefs about mathematics are categories of ways of thinking. The non-referential symbolic proof scheme—a scheme by which one ascertains oneself or persuades others of the truth of a conjecture on the basis of the appearance of symbols alone, without attending to their quantitative or functional referents—is an example of undesirable way of thinking. Heuristics are desirable problem-solving approaches, judged so by an observer. Beliefs about mathematics are restricted to one's interpretation of what mathematics is, how it is created, and its intellectual or practical benefits. They might be considered as didactical contracts, in Brousseau's (1997) terms, or norms, in Cobb and Yackel's (1996) terms, which are linked to one's perceptions of mathematics.

Since reasoning deductively is the single most central way of thinking in mathematics, mathematical proof must be a central focus of mathematics instruction. There is little attempt in current mathematics curricula to alter the character of students' mental act of proving by helping them acquire deductively-based proof schemes. On the contrary, the empirical proof scheme that students acquire in daily life is reinforced throughout the school years, where a strong emphasis is put on justification by examples, physical manipulatives, and, in particular, reliance on generalizing from finite patterns. As a consequence, students remain in this vicious cycle where their empirical proof scheme affects the kind of justifications they produce, and the justifications they produce reinforce the empirical proof scheme they possess.

In sum, the theoretical perspective presented here entails four essential elements for DNR-based instruction. First, in designing, developing, and implementing mathematics curricula, ways of thinking and ways of understanding must be the ultimate cognitive objectives; they must be addressed simultaneously, for each affects the other. Second, meaningful concepts can only be elicited through solutions to problems that

provoke intellectual needs for students. The concepts elicited and the ways they are elicited constitute and, at the same time, impact students' ways of thinking and ways of understanding. Third, ways of thinking and ways of understanding are organized, internalized, and retained through repeated solutions to intrinsic problems—problems the students understand and appreciate. Fourth, and lastly, proof and justification must be institutionalized as central means to create mathematics and as norms for mathematical discourse. As critical means of fostering desirable proof schemes, instruction must focus on carefully chosen problems—conceived as such by the students—through whose solution students gradually modify their ways of thinking as to what constitutes ascertainment and persuasion in mathematics. Such problems might be called "proof-eliciting problems" (Harel and Lesh, 2003; see also Lesh and Doerr, 2003).

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