Case Studies of Mathematics Majors' Proof Understanding, Production, and Appreciation[1]

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Abstract

Proof understanding, production, and appreciation (PUPA) are important parts of a mathematician's repertoire. Many U.S.A. university students, however, have difficulty with proof. One intent of this study was to examine the development of such students' PUPAs and possibly to identify significant influences on that development, through interviews of the students throughout their studies in mathematics. Three case studies show a great variance in the development of the students' proof skills. Some students come to university with excellent PUPAs and continue to thrive in a proof environment. Others enter university with poor PUPAs and unfortunately graduate without a significant change in their proof skills and attitudes. Still others come with poor proof skills but do show some growth during their undergraduate mathematics programs. Results of teaching experiments suggest that making proofs "tangible" is a means of helping those with poor PUPAs to grow in their proof understandings and abilities.

Key words: Prospective teachers, mathematical proof, teaching proof

Abbreviations:

PUPA – proof understanding, production, and appreciation

Executive Summary

Three case studies of students specializing in mathematics in a U.S.A university show striking differences in an important aspect of mathematics: One's abilities in proof understanding, proof production, and proof appreciation (PUPA). Drawn from interviews of 36 university students throughout roughly the latter half of their university coursework in mathematics, the case studies illustrate the dramatic difference in proof attainment of students who finish an undergraduate degree in mathematics. "Ann" has hardly grown at all, recognizes her proof weaknesses, but despairs at being able to produce an acceptable mathematical proof even at the end of her degree program. "Ben," on the other hand, enters university with an admirable PUPA and naturally leaves with a presumably strengthened one. "Carla" is perhaps more typical, showing initial reliance on unsophisticated methods of justification, but showing growth during her coursework and finishing not with an entirely satifactory PUPA but a perhaps acceptable one. Possible implications from the interview study include these:

> Students must not come to expect proof production to happen easily but must learn that coming up with a proof often involves considerable mental work.
>
> Deliberate planning for PUPA nurturing in the undergraduate curriculum will be necessary to further most students' PUPAs.
>
> Helping entering students with weak PUPAs should be an instructional concern.

Teaching experiments at another university suggest that there are at least a few principles that might guide our teaching for PUPA. One of these, "Make proofs tangible," flows from Harel's "necessity principle" for mathematics instruction: For students to learn, they must see an intellectual need for what we intend to teach them (1998). "Make proofs tangible" means that the proofs we offer must, in the students' eyes, include mathematical objects familiar to the students (concreteness), have the underlying idea made clear (convincingness), and clearly entail a need for the justifications of the steps of the proof (essentiality).

Background

Alibert and Thomas (1991) point out that for an individual there can be two goals in establishing certainty about a statement: convincing oneself, and convincing others. Using "proof" in the sense of "justification," Harel (1994; Harel & Sowder, 1998) has coined the phrase *proof scheme* to describe these two aspects. From several teaching experiments and a cross-check at another university, the categories of proof schemes in Fig. 1 seem serviceable in describing university students (cf. Harel, in press; Harel & Sowder, 1998).

Figure 1. Categories of proof schemes.

To most readers the labels in Figure 1 probably suggest that the deductive proof schemes are at some stage the ones valued in mathematics. A given person may, however, exercise different proof schemes, perhaps at times accepting the statements of others considered to be more expert (an authoritarian proof scheme) or settling for studying a specific example or two of a statement (an empirical proof scheme). A mathematician, of course, would be aware of the tentativeness of such approvals and the necessity of going beyond such efforts before being willing to endorse a statement as "mathematically proved," knowing that authorities are sometimes incorrect and that a generalization supported by even a large number of examples cannot be trusted unequivocally. (It must be pointed out, as Bell [1976] notes, that there are important roles of proof besides verifying results, like gaining insight or organizing a body of knowledge. These certainly play a role in proof appreciation.)

Because they show up frequently in the case studies, the ritual proof scheme and the symbolic-non-quantitative proof scheme are elaborated on here. (Some of the other proof schemes are also exemplified in the discussion of the case studies.) Students who accept an argument as a proof solely because of its form are showing a ritual proof scheme. To them form, because it *suggests* substance, makes a proof, and thus blocks their ability to detect a false argument. For example, any two-column "proof," or any extensive argument with symbols and equations would be accepted as proofs, just because such has the appearances of a proof. The same ritualistic sort of reaction was also noted by Healy and Hoyles (2000), in that their British secondary school students felt that algebraically presented arguments were most valued by their teachers, even though the students themselves felt that algebraic forms did not communicate or explain well. Furthermore, most students accepted an incorrect algebraic argument, "being swayed by the presence of the algebraic form" (p. 412). In another external conviction proof scheme, students who reason uncritically with symbols are showing the symbolic-non-quantitative proof scheme. Such a student might, for example, write the derivative of the product of two functions as the product of their derivatives—

$\frac{d}{dx}$ f(x)g(x) = f'(x)g'(x)--just because that equality preserves the existing structure suggested by the cases of the derivatives of the sum and the difference of two functions. Letting the symbols carry an argument is not entirely bad, of course; when the referents for the symbols and meanings for the operations on the symbols are in mind, the power of symbols is well known.

The study reported here had two components. One component followed prospective mathematics teachers during much of their undergraduate programs, via regular interviews; the case studies part of this report comes from this component. The second component featured teaching experiments, with the aim of identifying principles that might be useful in promoting students' PUPAs; one such principle is described following the case studies section.

The Case Studies: Overview

It is difficult to generalize about post-secondary education in the U.S. University students in the U.S., however, most often take about two-thirds of their coursework in areas outside of their specialty (their "major"). Students with very strong secondary school background may start their university mathematics work with multivariable calculus, but most start with beginning calculus. Over four years a mathematics major will take roughly

14 half-year-long courses, each course with its own examinations and marks. When students have completed all this work, they are awarded a "bachelor's" degree in mathematics.

The students in these case studies were enrolled in such a program. During the first two years, they typically would have studied calculus for a year and a half and taken half-year courses in discrete mathematics, beginning linear algebra, and computer programming. Proofs are usually presented at least occasionally in the courses during these two years, but very few of the students will have been expected to write any proofs except during the standard discrete mathematics chapter-long treatment of "methods of proof." Hence their exposure to proof to this point has been minimal, and most of their personal proof-writing goes back to their secondary school year-long course in Euclidean geometry, which usually involved at least some student-written proofs.

The case-study students were mathematics majors at an urban university in the southwestern U.S. This university has a large mathematics department, delivering courses in mathematics, statistics, and computer science for the whole student body. Mathematics majors may sub-specialize in pure mathematics, applied mathematics, statistics, or mathematics for teaching secondary school.

The first cohort of interviewed students was selected randomly from class lists for the discrete mathematics and linear algebra courses. These students were then contacted by telephone or in person and invited to participate in a study being run by the department "to see how mathematics majors' ideas about topics in mathematics evolve over the course of their undergraduate studies in mathematics." The actual intent, of course, was much narrower: To study how undergraduate mathematics majors' ideas about proof evolved over the course of their programs. No one refused to be interviewed, and each was interviewed individually for about an hour each semester (half-school-year) until they were graduated, transferred to another school, or withdrew from school. In a few cases, a student attended a summer session mathematics course; such students were also interviewed after those courses. Students were paid a nominal fee for each interview. The interviews extended over three years to allow us to follow the first cohort through to the completion of their programs (many students take more than four years to complete their undergraduate program at this university). Thirty-six students were involved, with varying numbers of interviews depending on when they entered (and exited) the study.

Each interview was semi-structured, with some questions planned to make the student as comfortable as possible in the interview environment and with other questions related to the content of the courses they were currently taking or had taken, with the intent of examining their proof understanding, production, and appreciation (PUPA). Occasionally the interviewer pursued a path that was viewed as a potential source of further information about a student's PUPA. The interviewer was most often one of the authors, but a doctoral student also carried out a small number of interviews. The interviews were audiotaped and transcribed.

For this report, the case studies of three students–Ann, Ben, and Carla (all names are pseudonyms)–were selected to illustrate the diverse nature of graduates of the program. Ann is the disappointment; although she received a bachelor's degree with acceptable marks in her mathematics courses, her proof skills on graduation were virtually the same as when she was first interviewed: weak. In stark contrast to Ann, Ben showed a strong PUPA throughout the interviews. Carla is the in-between case; she entered with weak PUPA but finished better although not at Ben's level.

The English and grammar in some of the quotes have been altered for easier reading. A few implications for instruction or curriculum, labeled "possible messages," are interspersed with our discussion of the students. Table 1 gives an overview of the coursework the students had completed and were enrolled in at the time of the interviews, which took place toward the end of the semester. The “transition” course attempts to bridge the gap between the first university mathematics courses and the later ones with more abstract material and with a proof emphasis; such courses are increasingly common in U. S. universities.

Table 1. Courses beyond Calculus 1-3 and computer programming enrolled in at time of interview.

<u>Case Study 1: Ann</u>

As Table 1 shows, at the time of the first interview, Ann was enrolled in discrete mathematics. She had completed three semesters of calculus, beginning linear algebra, a programming course, and a non-proof-oriented probability-statistics course. Hence, her personal experience with proof in college had been primarily as an observer, but she had formed an image of proofs.

Ann: ...In linear, he proved them, we didn't. But he proved all the formulas and...the different equalities and reasons for the formula being used. So the teacher is proving them.

Int(erviewer): …And (are proofs) on the tests?

Ann: No!

Int: How do you feel about proving?

Ann: Of course I don't like them, but I guess it's because you have to gather so much information in order to be able to prove it, and if you don't know part of the information, or if it doesn't pop out of your head right away, then you don't know how to get it all together.

Int: So, you're thinking of high school geometry, or...?

Ann: Basically, I'm looking at calculus and linear algebra. There're just so many things that are out there that you don't think of them right away. And if you don't have all of them, you can't put them together in a proof. So they're very hard, you know.

Here is a possible message: Instructors should convey the genuine mental labor involved in doing proofs. The information needed for a proof does not, of course, “pop out” of the instructor’s head but is searched for and reconstructed. Instructors might also explain whether what they are doing in a particular proof is something natural, like referring to a definition or earlier theorem, or is one of the standard "tricks of the trade" in proof making. (In a later interview, Ann did say that she believed that some of her teachers had made this latter sort of comment, but that there was not time to write them in her notes.)

Ann does acknowledge the importance of proof:

> Ann: Well, if you can't prove them, you're just kind of saying, 'Well, okay here are these big, long formulas.' Now are they really true?...Or is there really a reason why I'm doing this? The reasons for formulas and theorems are really important.

She does like the kinds of proof she has encountered in discrete mathematics:

> Ann: I don't like proving, but I guess the proving that we do in terms of logic, we're only dealing with the true and false. We're not dealing with twelve other different things, and you see they're true, or it's not false...And you can prove those with just a basic couple of steps. You don't have to, like in calculus, prove a whole page.
>
> Int: Oh, I see, it's long proofs that...
>
> Ann: Yeah, I guess so.

During the first interview, Ann showed evidence, as many students do, of relying on the inductive proof scheme (based on examples), although she is ostensibly but perhaps ritualistically aware of the use of variables in arguments. For example, she was shown a specific example of a 2x2 matrix A, with $(A^T)^{-1}$ and $(A^{-1})^T$ calculated; she was asked whether the equality would always be the case.

> Ann: ...I think it is. I think it is true.
>
> Int: How would you convince me, or convince somebody else that it is true?

Ann: I guess you would just have to go through these, or go through a couple of other ones, or maybe deal with letters so they can see that...if the letters work, then that means you can plug in anything for those letters.

Int: Which is most convincing, doing some more examples or doing it with letters?

Ann: To me personally, I think doing more examples of it. But if you can show that it would work for anything, well, that's even more convincing. But for me, I like to see the numbers, and be able to see that, Yes!, it works for this case, and this case, and this case.

Ann's preference for examples is expressed in another problem, in which she was to judge the arguments of different hypothetical students for the uniqueness of the inverse of a matrix.

Ann: ...[In reacting to a correct, general proof] You can prove this for anything, even if it's a 100 by 100 or whatever. This [a 3x3 numerical example, worked out with a system of equations to get a unique solution] on the other hand is only good for small matrices that are easier to calculate. So I guess it will depend on the size you're looking at. But I think they still will both be right--it's just a matter of which way the student wanted to do it. I know I like the numbers. That's my personal preference. I like numbers.

Ann's first interview also showed evidence of a generic proof scheme: The argument with a specific matrix could serve as a surrogate for an argument with any matrix. "...Even though you're still only using a 2 by 2, this matrix can be anything." Hence Ann's first interview showed a dominance by the inductive proof scheme, with possible hints of the ritualistic and generic schemes.

Ann's second interview occurred after a summer course in abstract algebra. One interview problem gave several numerical instances of $a^2 - b^2$, with $a = b + 1$, so that $a^2 - b^2 = (a - b)(a + b) = a + b$ in the instances given, with "$a^2 - b^2 = a + b$ for all $a > b$" a hypothetical student's conclusion. Ann's task was to tell whether the student was correct. She had decided that "if it was one number away," it would be true.

Int: Suppose someone says, 'Well, I can see that it checks in these examples, but if I did some other example, would it come out ok?' What would you say to that person?

Ann: I guess we could go ahead and check them out and see if their two numbers would work. So just check them and see....

[Ann later describes the pattern by writing $a^2 – (a – 1)^2 = a + (a – 1)$.]

Int: ...What would you do to settle this [for all cases]?

Ann: If it keeps following this pattern. You'd have to, I mean, you'd have to go from infinity to 0. As long as each case worked, then this [$a^2 – b^2 = a + b$] would be the formula.

Int: Did we look at any numbers over three digits?

Ann: No. Is that ok?

Int: Do you think we should?

Ann: Yeah (chuckles). [Ann checks that $100^2 - 99^2$ does equal 100 + 99.] Uh, so in order to prove this, am I going to have to go through every digit, I mean, four digit, five digit number, six digit numbers?

Int: That's really my question to you, how do you go about proving something like this?

Ann: I don't know....I just don't even know where to begin....I don't know how to prove that they are equal without doing from zero to infinity, that these are going to work.

One might have thought that after an exposure to the general arguments of an abstract algebra course, Ann would almost instantly, after writing $a^2 – (a – 1)^2 = a + (a – 1)$, proceed to a proof. She seemed not to realize that establishing that equation would cover all the "zero to infinity" of numerical cases on which she was focusing; her dominant proof scheme was still the inductive one. In particular, she had not internalized the proof scheme that we call “arithmetical algebra,” to indicate that it is deductive and involves (meaningful) symbols. Although no doubt Ann had seen such algebraic proofs, she had not internalized them in the sense that they were not a ready part of her personal repertoire. She plaintively noted, "I just don't even know where to begin."

The default nature of Ann's inductive proof scheme and a ritualistic use of variables continued to be clear in the third interview (during a semester in which she was taking an axiomatic geometry course and a "transition" course intended to serve as a bridge from calculus-type courses to proof-oriented ones). She seems to be increasingly aware that variables give generality and that proofs require generality. Ann is reacting to "Is it true that if x and y are factors of n, then xy will be a factor of n?"

Int. (after Ann says she thinks the statement is true): How would you convince somebody of this?...

Ann: I don't know. It's something that your intuition tells you....You'd probably have to set up some sort of proof. But you've already got letters in there. I don't know how you would show that, other than examples.

During her last two interviews, Ann showed no progress, and an increasing discouragement about proofs. She was memorizing proofs in one course and recognized that without understanding the proofs, she was forgetting them quickly. She spoke favorably about proofs by mathematical induction, but could not carry one out, and relied on a specific case to buttress her belief in the truth of a statement. She recognized that one counterexample showed that a statement was false, but when she failed to find a counterexample, she did not know how to proceed. During the final interview, we noted that proof had been a continuing concern of Ann's throughout her program and asked her what we could do differently to make proof easier.

Ann: I don't know. I've always said, a proving class. But then after looking at this stuff, you know, you can't just take a class and learn how to do it. You just kind of got

> to feel it, and just keep doing it over and over. But I think we should have started earlier. I mean, I don't remember proving anything in Calculus 1. And if they are proving things, the teacher does it up on the board, you write down and memorize it. So I think if we had started, even in junior high, basic things, I think that would have created a much better atmosphere. But that doesn't have to do with the college program.

Here is another possible message: Clearly we missed the boat with Ann and her PUPA. Was it her fault or ours? "When good teaching leads to bad results," part of a clever title by Schoenfeld (1988) for an article in which he describes high school teaching that was directed toward test preparation, might apply here. Certainly her instructors were competent mathematically, and the department prides itself on its concern for teaching. Might a greater, conscious, and explicit concern for PUPA have made a difference for Ann?

Case Study 2: Ben

Ben was clearly different from Ann from the very beginning of the interviews. He remarked that he had found his secondary school geometry course to be "a much fun course" and that geometry was "the first class I think I really excelled at." His approach to a combinatorics problem was to try to remember the proof of a relevant formula that he had forgotten, because "the formula would come out of that," evidence that Ben already had formed an appreciation of proofs as sources of insight or at least as a more reliable long-term memory support than just memorizing a formula. Indeed, Ben said, "As I went through my early calculus, they [the book] would give a theorem and I would try and prove it...and then I'd read their proof."

In contrast to most students, Ben rarely showed evidence of an empirical proof scheme. Even in the first interview, he preferred to deal with an abstract matrix rather than a specific matrix, or even a matrix with variables as entries; "Do matrix algebra, rather than work out a matrix's elements...which is a lot simpler, if you can pull it off...And it shows the concept." We interpreted such statements as evidence of a transformational proof scheme, and perhaps even an axiomatic one: Ben chose to express a matrix by a single letter rather than even a general $[a_{ij}]$ form, because he could then easily reason about matrices using properties of matrices rather than through direct computation. Ben was not completely fault-free, however. During the first interview he endorsed one proof of a result, when the proof given was actually a proof of the converse of the result, a mistake he did not repeat, however, in later similar problems in that interview and later ones.

In Ben's case, the two semesters of advanced calculus (= beginning real analysis) was a revelation: "...it's the first course of mathematics." An accident of having two different instructors for the two semesters exposed Ben to two alternative axiomatic bases.

> Ben: ...it offered some confusion at first because you've been developing things to look this way. But suddenly you're coming back this way and developing things that were foundations in one as consequences in the second one.

This exposure might have given Ben an appreciation at the level of the axiomatizing proof scheme, which involves an ability to handle alternative axioms for a given domain of study. (Crediting Ben with such an appreciation is an inference, however, since we did not plan any items to test directly for the axiomatizing proof scheme.) Nonetheless, we offer this possible message: Where is an undergraduate mathematics major exposed to alternative sets of axioms? In Ben's case, it first happened serendipitously and not as a part of a plan for the course of study. (Non-Euclidean geometry, a prime course for examination of alternative axiom sets, is an elective in all programs for mathematics majors at this university.) As Table 1 suggests, Ben was also unusual in that he took more courses than the typical major (he was undecided about which of two directions to go—applied mathematics or teaching high school).

Why was Ben different? One element of his success might be that he entered the university having had a very proof-favorable experience in secondary school geometry. This preparation predisposed him to react positively to proof demands and opportunities. Here is a possible message: What can we do for our mathematics majors who have <u>not</u> had a positive experience with proof in their secondary school work? This concern is doubly important for our prospective secondary school mathematics teachers, because they will be providing the proof background to the next generations of entering university students.

Case Study 3: Carla

Carla's first interview occurred toward the end of a semester in which she was enrolled in beginning linear algebra and the "transition" course, and after she had finished three semesters of calculus. She was not comfortable with proof and, even when specific cases were given, started her arguments by examining still other cases. Yet she recognized that verifying specific cases was not enough.

> Carla: ...you can't really say if he's correct or not...based on six problems. [And for another problem] He's only given two examples, so you can't really say the statement is true.

She had learned that she needed a "general argument." When asked where she learned to look for these general arguments, she cited the “transition” course.

> Int.: ...Not in high school geometry?
>
> Carla: I didn't really pay much attention to that back then. It's like I was just trying to get the answer; I was not really concerned about how you got it, and why you got it.

Commentators have pointed to the U.S. K-12 curriculum as being derelict in establishing a sense-making attitude toward mathematics. Harel (in press) feels that the K-12 emphasis is too much on getting results and too little on the reasons for those results. This opinion is supported by Manaster's (1998) analysis of eighth-grade data from the Third International Mathematics and Science Study: U.S. teachers, in contrast to those in Japan and Germany, never asked for any reasoning in their mathematics lessons. Even though Carla had had high-school geometry, the usual locus for a serious introduction to proof in U.S. curricula, she apparently learned from her experience that getting answers was all that was of concern. Even though the 1989 National Council of Teachers of

Mathematics statement of standards made a strong call for "mathematics as reasoning," that was not a part of mathematics to Carla until her fourth semester of college mathematics. It should be no surprise then that she incorrectly accepted a proof of a converse as an acceptable proof, perhaps because it read like a proof, a behavior of one with a ritualistic proof scheme. Nor should it be a surprise that she proceeded with only numerical work in investigating a puzzle that invites the introduction of variables; she used an inductive proof scheme rather than a transformational proof scheme even though she had verbalized an awareness of the need for a general argument.

Carla's next semester included a course in discrete mathematics that includes attention to logic and to proof methods (and is usually taken before the transition course that had been Carla's first extensive exposure to proof in college). In the last interview she would credit the discrete mathematics course with her first "gain of learning how to do proofs and so I think to me that was an important course." In her second interview she continued to know that any number of specific cases is insufficient in mathematics.

Carla: I'd just like a proof or something.

Int.: Why would anybody want to see a proof?

Carla: Well, to make sure that it's always going to work. Because if you use letters, no matter what number you put for that letter you can always get that...no matter what number you put, you can put a big old number.

Carla had absorbed an important reason for using variables, but her use of variables seemed almost ritualistic rather than transformational, since a transformational proof scheme involves some sort of anticipation about where the transformation will lead. This appeared to be absent with Carla. She may have been en route to an internalized proof scheme involving variables, but she was not there yet. Using n as a variable suggested to her a proof by mathematical induction, a topic in the discrete mathematics course she was taking.

Carla: ...Try more examples, and then if I see it works every time I try another example, I would use letters.

Int.: So what do you mean by "using letters" here?

Carla: Well, for this one, 1 + 3 = 4, so let's set *n* equal to one and find the formula for something that represents that number....Two times *n* plus one is odd, if *n* is an odd number. So if *n* is equal to 1, then this would be three. Three plus one would equal four--something like that...I would do it for the next one...General, but that takes a long time.

Even though the discrete mathematics course included attention to logic, Carla still could not deal with a proof of the contrapositive and did not recognize that a given proof was actually for the converse of the given statement. During the third interview, she felt more confident about "proof by contradiction and things that in the past I was clueless about." Yet she confused "disprove" with "prove with a proof by contradiction" and ritualistically labelled a proof as a proof by contradiction because it started with the word "suppose." The following excerpt even makes one wonder whether, when she asserted that examples cannot give a proof, Carla is just trying to say what she thought the interviewer would expect.

Carla: ...I tried one [example] and it worked, and I'm sure if I try another one, it's going to work again.

...

Int.: So if I'm in doubt, I should look at an example? And if the example works...?

Carla: Well, I mean, I just can't say it's true because of that.

Carla’s Interview 4 showed that she had made progress. She was enrolled in advanced calculus, abstract algebra, and foundations of geometry, all proof-intensive courses. (Although her mark for abstract algebra was acceptable, she later decided to re-take the course since she was planning graduate school and did not feel secure in her knowledge of abstract algebra.) For the advanced calculus course, she said, "...it's just fascinating how you can actually prove things from the first calculus courses. You just didn't have any clues as to what was going on," showing an appreciation of the insight role that proof can play. She also proceeded confidently on a proof by mathematical induction, even though she was not quite able to finish it. She was surprised, and pleased, by her performance on "Is it true that if x is a factor of m and of n, then x is a factor of m – n?" She started by thinking of values for m and n but before verifying the result with those values switched to variables. After she had finished her proof, the interviewer asked her to retrace her thinking.

Int.: I'm curious here because you started off looking at a numerical example. You didn't even finish that....So what happened here?

Carla: I don't know, it just...I just said, well if I just think abstract. I guess I just thought more about thinking without numbers, which is something funny because last time I couldn't do something like this....On problems like this one I was always looking at numbers before I started looking at letters.

Int.: Did you try to think more in terms of letters today, or did it just happen?

Carla: It just happened. I'm surprised myself!

At this stage it would seem that Carla had indeed grown in her PUPA, with a possible and gratifying message that the more intensive attention to proof from her three mathematics course was definitely having an influence. Her fifth interview occurred during a semester in which she was studying non-Euclidean geometry, an elective course but one of the few opportunities for most undergraduates to develop an axiomatizing proof scheme, marked as it is by the ability to deal with alternative sets of axioms. She summarized the course with "where changing one postulate in Euclidean geometry makes a difference, and it makes you go into all these other types of geometry." Rather than the axiomatizing proof scheme, however, Carla appears to have been using a Greek-axiomatic proof scheme, in which reasoning with axioms is accepted but the axioms have a perceptual or experiential basis. She and the interviewer have been discussing ideal points in hyperbolic geometry.

Int.: What is your comfort level with this right now?

Carla: Honestly, with ideal points? I, I mean I, I take it as it is. I don't really like, I wouldn't bet money on it, you know what I mean? But I understand it. Like I go-with-the-flow kind of thing....It's like based on logic more than on [reality].

Unfortunately, Carla also showed some regression during her last two interviews (5 and 6). Her first reaction showed an empirical proof scheme for the question, "For positive integer n, is $f(x) = x^n$ continuous at $x = 1$?"

Carla: Yeah, it's going to be continuous.

Int.: How come?

Carla: Um, well, 'cause I'm just thinking of x^2, x^3, x^5. They're all continuous.

Carla also gave inductive responses that focused on specific groups or specific sets, for questions about whether there was any relationship between cyclic and abelian groups, and about the cardinality of the power set of a set.

The Case Studies: Concluding Remarks

For most students the PUPA acronym can analogously be interpreted in its biological sense, as the transition form from a larva to the adult insect. At the beginning the PUPA for a student like Carla may have had an extremely immature form, and although the student's PUPA develops positively during the undergraduate career, it may not reach a completely mature status. Ann, with her low entry level, did not seem to grow past an empirical proof scheme, even though she acknowledged the importance of proof and was dismayed at her inability to produce proofs. Ben, on the other hand, started the program with a well developed PUPA and continued to enhance it during the program, perhaps able to use even the quite sophisticated axiomatizing proof scheme.

Did the case studies reveal particular curricular or instructional elements that seemed to foster (or hinder) students' PUPAs? As an example of a possible hindrance to PUPA development, it could be that the laudable teaching practice of illustrating a theorem with an example has the unfortunate effect of leaving the impression that an example is an acceptable justification, especially if the theorem is given without proof, as is common in beginning courses.

By the time of the interviews--after calculus and during linear algebra and/or discrete mathematics for the students in the case studies--students appear to have learned that giving even several examples is not an acceptable justification in mathematics. Carla felt that she profited from the attention to logic and proof methods in her discrete mathematics course; since she had taken the transition course before the discrete course, the transition course apparently had not been helpful to her, except to alert her that she was missing something important in advanced mathematics. Hence, if a certain sequencing of courses is important for a gradual development of PUPA, we should insist, either through advising or prerequisites, that our students take courses in a recommended order. Despite having several of the same courses as Carla, Ann never progressed to the point that she had any confidence that she could give a proof if one was called for; she would resort to examining specific cases if a result was to be verified. She herself felt that an earlier exposure to proof would have been helpful to her. Fortunately, students with Ann's weak PUPA are fairly rare among the prospective mathematics teachers whom we interviewed. The occasional students like Ben are not a concern, unless in crediting their fine proof performances to our teaching we overlook the needs of the Carlas and the Anns.

How the PUPAs of this variety of prospective teachers can be nurtured, especially in coursework that often serves students from different majors and even mathematics majors at different points in their studies, is a difficult question from the point of view of curriculum. To us, fostering growth in students' PUPAs, through an emphasis on the reasons for results rather than just the results, should be an important consideration in planning any program of study for mathematics majors. Instruction does of course play a crucial role in the development of students' PUPAs, so we turn to what we believe to be helpful guidelines for teaching with PUPA in mind.

Principles for Instructional Treatments That Facilitate PUPA

Perhaps the most disturbing observation about Ann's conception of proof is her belief that proofs must be produced instantly by gathering and applying all the information needed for the proof at once. This can be seen in the following excerpt, with emphases added:

Int: How do you feel about proving?

Ann: Of course I don't like them, but it guess it's because you have to gather so much information in order to be able to prove it, and if you don't know part of the information, or if it doesn't pop out of your head right away, then you don't know how to get it all together. (Emphases added)…

Int: So, you're thinking of high school geometry, or...?

Ann: Basically, I'm looking at calculus and linear algebra. There're just so many things that are out there that you don't think of them right away. And if you don't have all of them, you can't put them together in a proof. So they're very hard, you know. (Emphases added)

Although shocking, this conception should not be surprising: We believe Ann's description here reflects faithfully her experience with proofs. It is likely that Ann has for years been observing her mathematics instructors depicting--not producing--lengthy, complex collections of statements on the blackboard under the title, proof. And it is likely that she has not taken part in the proof production process; nor has she been part of her teachers' thinking process; nor has she been taught how to internalize this process (i.e., make it her own). Ann apparently has not participated in a learning process of how to internalize proofs produced by others.

In the previous section, we discussed one of the PUPA project's goals: to document the progress prospective mathematics teachers make in their conception of mathematical proof, in a typical undergraduate mathematics program. As we have indicated, the progress Ann made in her conception of proof was minimal. Our PUPA project, however, had an additional goal: to seek and test principles for instructional treatments that facilitate proof understanding, production, and appreciation. In this section we will discuss these treatments as we have applied them in a sequence of teaching experiments conducted in a different institution (a large midwestern university). The results of these treatments are very encouraging in that students' use of external and empirical proof schemes gradually diminished as they developed alternative, mathematically adequate proof schemes (i.e., the deductive proof schemes).

In what follows we focus on one instructional principle which aims at facilitating this transition and, in particular, at teaching students how to internalize proofs produced by others. We dubbed this principle, "Make proofs tangible." For a proof to be tangible it must be:

a. Concrete: The proof deals with entities students conceive as mathematical objects (i.e., objects they can handle in the same manner they handle numbers, for example).

b. Convincing: Students understand its underlying idea, not just each of its steps.

c. Essential: Students see the need for the justifications of its steps.

These conditions cannot be determined a priori; they are dependent on the students' stage of mathematical development. For example, consider the proof of Cayley-Hamilton Theorem that is based on Schur's Lemma ("Any square matrix A can be decomposed into the product UTU^*, where U is unitary and T upper triangular," which assumes an algebraically closed field). The proof is likely to be concrete to students in their first or second linear algebra course, because it deals with n-tuples and matrices, and it is likely to be convincing, because it constructively demonstrates the theorem's assertion.

Essentiality too has to be judged relative to the individual student. For example, a ninth-grade student wrote a two-column proof which included the two statements, (a) AB= AB and (b) ABN = 30° and NBR =15° → ABR = 45°, and their reasons, "reflexive property" and "additive property", respectively. Although for a student in an axiomatic geometry course these reasons must be seen as essential, for the ninth-grader who wrote them, they were needless. She had no concept of axiom systems, and so she was neither puzzled by nor doubted the respective assertions; she simply obeyed her teachers' rule: "Put a reason for each step."

The concreteness condition does not result in a loss of knowledge. For example, the restriction of the above proof of Cayley-Hamilton Theorem to matrices over an algebraically closed field is of no limitation to these students, because at this stage in their mathematical education, the complex field is the most advanced (number) system they recognize. Only when these students are brought to see the benefit of matrices over a ring---again, by appealing to their intellectual need---can they appreciate the more general proof.

The convincing condition educates students always to strive to make proofs convincing for themselves. As teachers, we want our students to recognize those situations where one understands each step of the proof yet still remains mystified by its assertion, and we want them to feel cognitively uncomfortable with such situations and to respond to them by seeking a deeper understanding of the proof's underlying idea. But our students seldom do. The reason may be that cognitive discomfort is always judged against one's personal background, and so students who have not been internally convinced by proofs—and our students seldom are—are unlikely to be cognitively disturbed by such situations.

The essentiality condition prevents artificial reasoning. The imposition of justifications such as those appearing in the ninth-grader's proof about angles (above) is a major contributor to the formation of the symbolic and authoritarian proof schemes, which can possibly extinguish all sparks of curiosity. Calculus students, for example, should be mystified by the astonishing assertion that the definite integral of a continuous function from a to b can be found from any one of the function's antiderivatives.

We conclude with a short outline of a teaching episode to demonstrate the implementation of the "make proof tangible" principle. This episode is from the “reading activity.” This is an in-class activity, where students read a proof they have not seen before, discuss it in small groups, and then discuss it again with the participation of the whole class and the instructor.

The Cauchy-Schwarz Inequality theorem was presented in response to a need to generalize the concept of "angle between two vectors in C^n." The complete theorem and its proof as it appeared in the textbook used in this class are given in Table 2.

Table 2. Cauchy-Schwarz Inequality Theorem and Proof.

The majority of the students were unable to understand this proof even at the level of line-by-line comprehension. The entire class session was spent on making this proof tangible to the students.

To make the proof <u>concrete</u>, we reduced the theorem to the context of C^n, rather than a general inner-product vector space. n-tuples are entities students conceive as mathematical objects (as opposed to vector space of functions, for example).

In an effort to make the proof convincing, that is, understanding its underlying idea, not just each of its steps (this is the critical condition in this particular proof), the following questions were raised during the discussion.

a) How did the author know that she needed the inequality $\langle u, e^{i} \; v \rangle \quad 0$? (Line 1)

b) How does one know that a real exists so that $\langle u, e^{i} \; v \rangle \quad 0$? (Line 1)

c) How did the author know to take the function $f(t) = \| tu + e^{i} \; v \|^2$? (Line 2)

d) Why did the author choose a non-negative function? (Line 3)

e) How did the author know to evaluate the function specifically at

$$t = \frac{|\langle u, v \rangle|}{\|u\|^2}$$? (Line 4).

Essentiality was not an issue in this proof because once the students understood the underlying idea of the proof, they saw a need to justify each of the proof's steps.

The conditions, "concreteness," "convincing," and "essentiality," are implications of a more global principle, called the Necessity Principle (Harel, 1998). It states:

> *For students to learn, they must see a need for what we intend to teach them, where "need" refers to intellectual need, rather than social or economic need.*

The challenge for every teacher is to recognize what constitutes an intellectual need for a particular population of students, to present the students with a problematic situation that corresponds to their intellectual need, and then to help them elicit the concept from the problem solution. This general principle proved very effective in teaching specific concepts, as well as in eradicating faulty proof schemes and in helping students construct desirable ones.

Final Comment

In fact, the real challenge for teachers is that their students create problematic situations for themselves. For example, when students read the proof of the Cauchy-Schwarz Inequality theorem presented above, they themselves should pose questions—create problematic situation for themselves, that is—of the kind discussed earlier. As we have seen, our teaching experiments aimed explicitly at achieving this goal. Unfortunately, because the students' initial conception of proof was severely inadequate, this goal turned out to be extremely hard to achieve in a single course. Nevertheless, the conceptual change that our teaching-experiment students made in a one-semester course is in stark contrast to that made by our longitudinal-study students during several semesters of courses. This finding suggests that the instructional treatments we have employed in our teaching experiments are potentially effective and can advance students' conceptions of proof.

[1] The study described in this paper is the result of National Science Foundation Project 9355861; the opinions expressed here, of course, are those of the authors and are not necessarily those of the Foundation.
[2] We wish to acknowledge the contributions of Stacy Brown during the analyses of the transcripts.

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Table 1. Courses Beyond Calculus 1-3 and Computer Programming Enrolled in at Time of Interview.

	Interview 1	Interview 2	Interview 3	Interview 4	Interview 5/6
Ann	Lin. algebra Discrete math (also: Calculus 3	Abstract algebra	Transition History of mathematics Fnds. of geo.	Probability	Adv. calculus
Ben	(completed: Lin. alg. and Discrete) Diff. eqtns. Adv. lin. alg. Adv. calculus	Modeling Probability Adv. calc. 2	Abs. algebra Num. analysis Math. stat.	Complex var.	Non-Euc. geometry Part. diff. eq. Math. stat. 2
Carla	(completed: Calculus 3) Lin. algebra Transition	Discrete math. Prob-stat. (also: comp. programming)	Computers in learning math	Fnds. of geo. Abs. algebra Adv. calculus	5: Non-Euc. geometry 6: Abs. alg.

<u>Table 2. Cauchy-Schwarz Inequality Theorem and Proof.</u>

<u>Theorem</u>:

Let u and v be vectors in n-dimensional inner-product vector space with u non-zero.

If $v = \quad u$ for some scalar , then $|\langle u, v\rangle| = \|u\|\|v\|$

If $v \quad u$ for any scalar , then $|\langle u, v\rangle| < \|u\|\|v\|$

Proof (the proof's lines are numbered to reference them in the later discussion)

1. Choose so that $\langle u, e^{i} v\rangle \quad 0$ and let

2. $$f(t) = \|tu + e^{i} v\|^2 = \|u\|^2 t^2 + 2\mathrm{Re}\langle u, e^{i} v\rangle t + \|e^{i} v\|^2 =$$
$$\|u\|^2 t^2 + 2|\langle u, v\rangle| t + \|v\|^2$$

   for t real.

3. Since $f(t)$ is the square of the norm of a vector, $f(t) \quad 0$.

4. Evaluating f at $t = \dfrac{|\langle u, v\rangle|}{\|u\|^2}$, we get the inequality

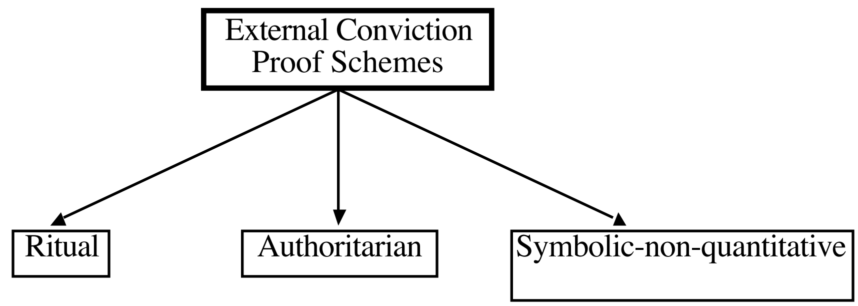
$$\|u\|^2 \quad \frac{|\langle u, v\rangle|}{\|u\|^2} \quad ^2 \quad 2|\langle u, v\rangle| \quad \frac{|\langle u, v\rangle|}{\|u\|^2} \quad + \|v\|^2 \quad 0 \cdot$$

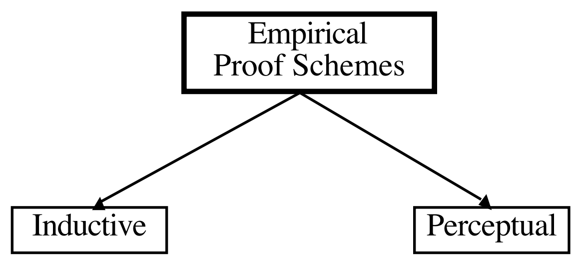
5. Simplifying, we get $\dfrac{|\langle u, v\rangle|^2}{\|u\|^2} + \|v\|^2 \quad 0$,

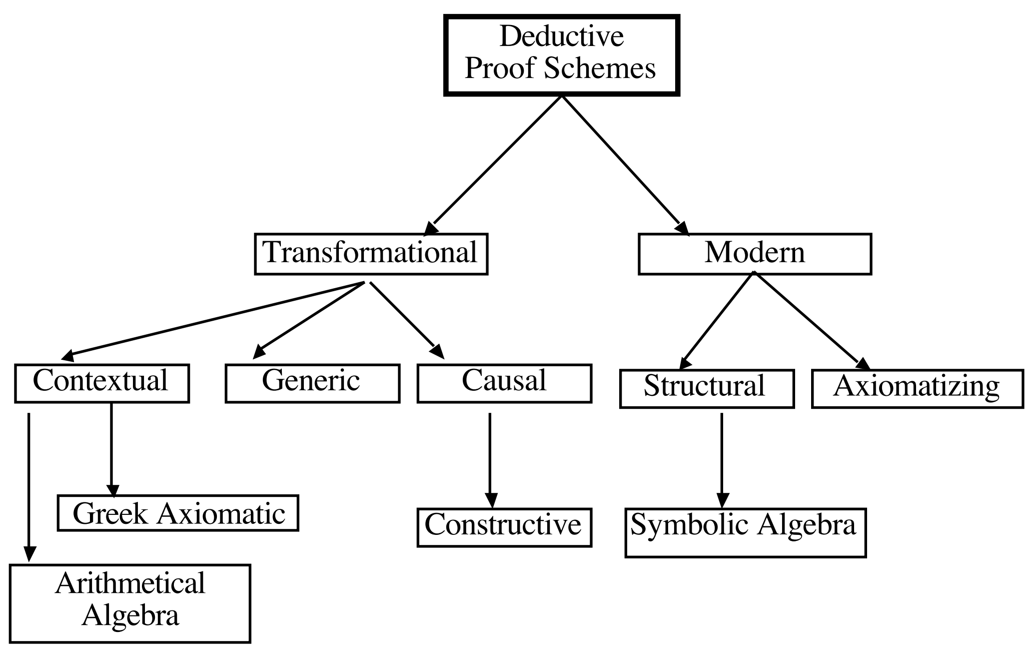
6. which is equivalent to $|\langle u, v\rangle| \quad \|u\|\|v\|$. Now equality occurs in this inequality if and only

   if $f(t) = \|tu + e^{i} v\|^2 = 0$, which implies $v = \quad u$ where $\quad = e^{\ i} \dfrac{|\langle u, v\rangle|}{\|u\|^2}$.

Figure 1. Categories of proof schemes.


External Conviction Proof Schemes
Ritual
Authoritarian
Symbolic-non-quantitative


Empirical Proof Schemes
Inductive
Perceptual


Deductive Proof Schemes
Transformational
Modern
Contextual
Generic
Causal
Structural
Axiomatizing
Greek Axiomatic
Constructive
Symbolic Algebra
Arithmetical Algebra