Mathematical Association of America. (1983). *Recommendations on the mathematical preparation of teachers: MAA Notes # 2.* Washington, DC.

Naisbitt, J. (1982). *Megatrends.* New York: Warner Brooks.

National Council of Teachers of Mathematics. (1981). *Guidelines for the preparation of teachers of mathematics.* Reston, VA.

National Science Board Commission on Precollege Education in Mathematics, Science, and Technology. (1983). *Educating Americans for the 21st century.* Washington, DC: National Science Foundation.

National Science Board Task Committee on Undergraduate Science and Engineering Education. (1986). *Undergraduate science, mathematics and engineering education.* Washington, DC: National Science Board.

National Science Foundation. (1987). *Teacher Preparation and Enhancement: Program announcement and guide.* Washington, DC: Directorate for Science and Engineering Education.

National Science Teachers Association. (1982). *An NSTA position statement: Science—society—science education for the 1980's.* Washington, DC.

Penick, J. E., & Yager, R. E. (1983). Search for excellence in science education. *Phi Delta Kappan, 64*(9), 621-623.

Phillips, E. R., Greabell, L., Bullock, J., & Wanous-Bullock, D. (1987). A collaborative model for non-traditional graduate teacher education. *National Issues in Higher Education, 28.*

Spector, B. S. (1985). Generating a desired state for master's degree programs in science education through grounded theory research. *Journal of Research in Science Teaching, 22*(4), 327-345.

Task Force on Teaching As A Profession (1986). *A Nation Prepared: Teachers for the 21st Century.* New York: Carnegie Forum on Education and the Economy.

The Holmes Group's Executive Board (1986). *Tomorrow's Teachers: A Report of The Holmes Group.* East Lansing: The Holmes Group, Inc.

Applying the Principle of Multiple Embodiments in Teaching Linear Algebra: Aspects of Familiarity and Mode of Representation

Guershon Harel

Department of Mathematical Sciences
Northern Illinois University
De Kalb, Illinois 60115-2888

Dienes (1960, 1964) stated the principle of multiple embodiments as an instructional tool for enhancing the understanding of concepts and for retaining mathematical structures. Several studies (e.g., Beardslee, 1973; Gau, 1973) investigated this principle as applied to concrete representations of concepts in elementary mathematics levels. In higher mathematics, the embodiments usually used are not concrete but abstract; their effects on the understanding of concepts were not investigated. In linear algebra textbooks, the principle of multiple embodiments is found in the process of translating general definitions and theorems to terms of given situations. Showing that the set of continuous function is a vector-space and showing that a set of four directed line-segments is linearly dependent are examples for this process. Most of the linear textbooks we reviewed (Harel, 1985, 1987) use algebraic embodiments rather than geometric ones, and pay scant attention to familiarizing the students with the embodiments used.

Focusing on the understanding of the vector-space concept with university technology students, we initiated our investigation with the following question. Would an emphasis on the embodiment of a familiar geometric system lead students to a better understanding of the vector-space concept than an emphasis on the embodiment of a variety of unfamiliar algebraic systems? By understanding the vector-space concept we mean the ability to determine if, and justify why, a given system is a vector-space; and to solve problems whose solutions require application of the basic properties of vector-space.

Based on the following rationale, we hypothesized a positive answer to that question. In the embodiment process, students are acquainted with instances of concepts they encounter in the abstract system and that has an effect on the formation of mental representations of the abstract concepts. An important component in the mental representation of a concept is an external, physical reference; sources for forming this component are concrete, or at least semi-concrete, embodiments. Since in the abstract system of linear algebra concrete embodiments are complex physical systems which are not in the scope of undergraduate curricula, it is the semi-concrete embodiments, namely the geometric systems, that serve as physical referents. The algebraic systems, on the other hand, do not have easily visualized representation that describe their operations and relations. Moreover, when beginning students are presented with these systems, they encounter, probably for the first time, mathematical systems different in nature from the number system. They have difficulties accepting the idea that a collection of numbers, such as a matrix, or a function, is a mathematical entity; namely, a mathematical object within a system that has its own structure and its own operations—addition and scalar multiplication of vectors. Nevertheless, any system used in the embodiment process would have no constructive cognitive effect if the situations being embodied are not familiar and fully understood by the student.

Method

Subjects and Procedure

The students used in this study were seventy-two second-semester sophomores enrolled in a required linear algebra course designed for the technology program. All students had the same formal education in mathematics. They had taken one calculus course in the first semester and the same mathematics program in academic high school. The calculus course and the high school program did not include linear algebra content. The students were divided randomly into two groups, A and B, consisting of thirty-six students each. All students were taught linear algebra by one instructor for three one-hour lectures each week during one semester. The instructor used mainly the system R^n to illustrate abstract ideas he had taught, without referring to geometric interpretations of these ideas. Weekly, each group was given two separate hours of "recitation" by another instructor, one hour more than what the institutional program requires. Group A was presented in the two hours of "recitation" with a variety of embodiments of the abstract ideas they were taught in the previous lecture. Group B received the same "regular" treatment as Group A for only one hour of "recitation." The other hour was devoted to a "special" treatment of showing how vector-space ideas are represented geometrically.

In accordance with the institutional program requirements, Groups A and B were presented with the spaces of polynomials, matrices, and solutions of linear equations. Group B was presented in the second hour of recitation with the space of the directed line-segments without numerical coordinates. These subjects were previously acquainted with the idea of directed line-segment in a prerequisite physics course, in which they were taught the physical meaning of directed line-segments and their operations, addition and scalar multiplication.

Items

At the end of four weeks, the two groups were given the same test. The items in the test were chosen to be simple problems on the vector-space concept, which can be solved directly by applying the vector-space definition.

Problem 1: Which of the following sets is a vector-space? Justify your answer.
a. $W_1 = \{(x,y): y = 2x\}$
b. $W_2 = \{(x,y): y = -2x + 1\}$.

Problem 2: Let μ and β be vectors in the vector-space V. We define $L_\mu = \{x\mu : x$ is a real number$\}$, and $L = \beta + L_\mu$.
a. Prove that L_μ is a subspace of V.
b. Prove that β-μ is a vector in L.

Problem 3: The square of numbers:

$$\begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}$$

is called a magic square because each row, each column, and each diagonal has the same total sum of entries.
Let W be the set of all magic squares. Define two operations on magic squares so that W will be a vector-space over the real numbers. Justify your answer.

Problem 4: W_1 and W_2 are non-trivial subspaces of a vector-space whose intersection is the zero vector. Is $W_1 \cup W_2$ a vector-space? Justify your answer.

A different setting of Problem 1 (i.e., with different numbers and different equation forms) was discussed during the instruction, all the other problems were not discussed at any time during the instruction. These problems and the instructional material used were submitted to a group of five mathematicians and unanimously judged to be germane to the content taught. Complete agreement by another two experts categorized students' responses into the schema presented in the next section.

Results

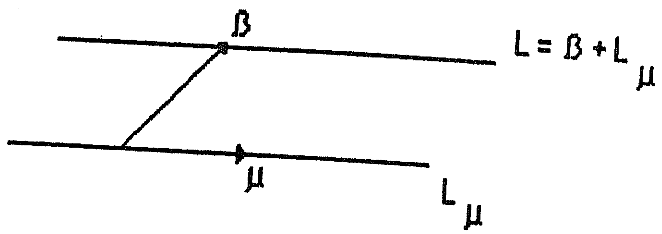
Categories of Answers

We considered three variables in the students' answers: (a) description; (b) final-answer; and (c) justification of answer.

Description. We examined the descriptions that the students used to express the problems and the solutions, by classifying their answers into three categories:

1. Geometric description (GD): The students' answers involved written geometric descriptions, such as: "The elements in W_2 are points on a line going through the origin," or graphical descriptions such as the one shown in Figure 1.

Figure 1. A graphical description in an answer to Problem 2.



2. Algebraic description (AD): The students' answers involved algebraic descriptions without any geometric interpretation. For example, "An element in W_1 is an ordered pair of the form (x,2x);" "L is a subspace of E. I'll show that it satisfies the definition of vector-space."

3. Others(*): This category comprised irrelevant justifications of answers, or no given answer, or unidentifiable answers. Some of the responses in this category included final answers such as, "the given set is a vector-space."

As one can see, all the problems were formulated algebraically, and except for Problem 3, they are amenable to geometric interpretations. As was expected, these "task environments" stimulated geometric and algebraic interpretations. The treatment that students gave to Problem 3 involved mainly matrix arithmetic, which we considered as an algebraic description.

Final-answer. This variable refers to the final answer that students provided to Problems 1 and 4 and to the definition they gave in Problem 3. Problem 2, which is a proof problem, is not relevant here. We classified this variable into three categories: Correct final answer (CF); Incorrect final answer (IF); Absence of final answer (AF).

Justification of answer. In Problems 1, 3, and 4, students were required to justify their (final) answers. Proofs, given in Problem 2, were also considered as justifications. Justifications for *correct final answers* were classified into three categories: Correct justification (CJ); Incorrect justification (IJ); Absence of justification (AJ).

Distributions of the Categories

Description. The distribution of categories across the description variable is given in Table 1. A comparison between the two categories, GD and AD, across the two Groups A and B, showed that in Problems 1a, 1b, 2a, 2b and 4 significantly a geometric mode was used more often in Group B than in Group A and an algebraic mode was used more often in Group A than in Group B.

Table 1

Percents of Categorizations of the Description Variable

Problem	Group	Categories: Geometric Description	Algebraic Description	Others	χ^2(df = 1)
1a	A	11.1	66.6	22.2	7.96*
	B	36.1	36.1	27.7	
1b	A	8.3	69.4	22.2	15.30**
	B	41.6	25.0	33.3	
2a	A	13.8	75.0	11.1	12.24**
	B	47.2	33.3	19.4	
2b	A	11.1	63.8	25.0	9.98**
	B	44.4	36.1	19.4	
4	A	8.3	72.1	19.5	21.09**
	B	52.7	22.2	25.0	

Note. χ^2 computed on GD-AD split.
$*p<.01$ $**p<.001$.

Final-answer. The distribution of the final answer variable is given in Table 2. Within Group B, in Problems 1a, 1b, and 4, about 80% of the students gave a correct final answer and less than 6% gave an incorrect final answer. Within Group A, in the same problems, about 60% of the students gave correct final answers to Problems 1a and 1b and only about 30% gave correct final answers to Problem 4; in either case, more than 27% arrived at incorrect final answers. Within the two groups, in Problem 3, the stability of almost equal number of correct and incorrect final answers was reflected by the non-significant differences. As overall result, a comparison between CF and IF categories across the two Groups A and B showed that more correct final answers were given by students from Group B than by students from Group A. These differences are significant (p<.01) in problems 1a, 1b, and 4, but not significant in Problem 3.

Justification. The distribution of the justification variable is given in Table 3. In this table the results of the following cases are depicted: (a) CF+CJ Category: correct final answer with correct justification (for Problem 2, CJ Category);(b)CF+IJ Category: correct final answer with incorrect justification (for Problem 2, IJ Category);(c)CF+AJ Category: correct final answer with absence of justification (for Problem 2, AJ Category). The overall results of these cases support our hypothesis about the superior effect of the geometrical approach used. However, to substantiate this superiority, we do not need to analyze all these cases; the following analysis will be sufficient.

Table 2

Percents of Categorizations of the Final Answer Variable

Problem	Group	Categories: Correct Final Answer	Incorrect Final Answer	Absence of Final Answer	χ^2(df = 1)
1a	A	69.4	27.7	2.7	6.44* 7.64** 27.19***
	B	83.3	2.7	13.8	
1b	A	63.8	30.5	5.5	4.23* 8.86** 26.19***
	B	80.5	2.7	16.6	
3	A	44.4	33.3	22.2	.57 .049 1.2
	B	50.0	33.3	16.6	
4	A	36.1	30.6	33.3	.16 11.71** 23.63***
	B	80.5	5.4	13.9	

Note. χ^2 computed on CF-IF split.
$^*p<.05$. $^{**}p<.01$. $^{***}p<.001$.

Table 3

Percents of Categorizations of the Three cases in the Justication Variable

Problem	Group	Categories: CF+CJ	CJ	CF+IJ	IJ	CF+AJ	AJ	N%
1a	A	55.5	—	13.8	—	0	—	69.4
	B	44.4	—	13.8	—	25.0	—	83.3
1b	A	58.3	—	2.7	—	2.7	—	63.8
	B	44.4		11.1	—	25.0	—	80.5
2a	A	—	52.7	—	36.1	—	11.1	100
	B	—	77.7	—	2.7	—	19.4	100
2b	A	—	50.0	—	25.0	—	25.0	100
	B	—	74.7	—	5.8	—	19.4	100
3	A	22.2	—	22.2	—	0	—	44.4
	B	33.3	—	16.6	—	0	—	50.0
4	A	5.8	—	19.4	—	11.1	—	36.1
	B	47.1	—	8.4	—	25.0	—	80.5

Note. CF, CJ, IJ, and AJ indicate, respectively, the categories Correct Final-answer, Correct Justification, Incorrect Justification, and Absence of Justification. The + indicates a combination of two categories.
The N in Problem 1a, 1b, 3, and 4 is the percentage of correct final answers.

When dichotomizing the answers as belonging-versus-not-belonging to CF + CJ category and subjecting these dichotomies across Groups A and B to a χ^2 tests, the stability of almost equal distributions of the results belonging to Problems 1a, 1b, and 3 is reflected by the non-significant differences. (The values of χ^2 test are, respectively, .88, 1.39, and 1.1; df = 1). The difference in Problem 4, on the other hand, is significant at the 0.001 level (χ^2 = 15.78, df = 1).

When dichotomizing the answers given to Problems 2a and 2b as belonging-verus-not-belonging to CJ category, almost half of Group A but less than one-fourth of Group B gave incorrect justification. when subjecting these dichotomies to a χ^2 test, the differences are significant at the .05 level. (The values of χ^2 test are, respectively, 4.95, and 4.65, df = 1).

Finally, a comparison between the number of CF + CJ in Problems 1 and the number of CJ in Problem 4 shows a significant ($p<.0001$; Z = −2.76) increase, from 44.4% average to a 76.2% average.

Discussion

Because of the equally low achievement of the two groups in solving Problem 3 (see Tables 2 and 3), the conclusions about the differentiations between Groups A and B in understanding the vector-space concept will be drawn from the responses to the other problems. However, the results in Problem 3 support the viewpoint we expressed in the introduction about the difficulties students have with the embodiment of algebraic systems whose elements are collections of numbers or functions. This viewpoint is deeply related to the understanding of the nature of mathematical objects which requires special research as well.

Our hypothesis about the superior effect of familiar geometric embodiments, as compared to unfamiliar algebraic embodiments, is confirmed by the results we have shown. First, the fact that in the three Problems 1, 2, and 4, a geometric description occurred more often in Group B than in Group A and an algebraic description occurred more often in Group A than in Group B confirms that the instructional approach and the kind of description chosen by the students were not independent (Table 1). Second, in Problems 1 and 4, in which the students were required to determine if a specific model is a vector-space, many students from Group B gave a correct final answer and only a few students gave an incorrect final answer. Within Group A, although relatively many students gave a correct final answer, still many arrived at an incorrect final answer. Nevertheless, more correct final answers were given by students from Group B than from Group A and less incorrect final answers were given by students from Group B than from Group A. Third, in Problem 2, in which the students were required to prove general statements, only a few students from Group B gave incorrect justifications but many more from Group A gave incorrect justifications (Table 3).

The distinction between concept definition and concept image (Vinner, 1983) supports our conclusion that the differences between the two groups achievements are attributed to the different instructional approaches used. A concept definition is "a verbal definition that accurately explains the concept in a non-circular way" (p.293). The concept image of a person, according to Vinner (1983), consists of the set of the concept's properties that the person has in mind and the set of all pictures that have ever been associated with the concept in the person's mind. In order to handle concepts one needs a concept image, not only a concept definition.

Clearly, Problems 1, 2, and 4, can be solved directly by applying the formal definition of vector-space. Despite the fact that the two groups were exposed to this definition under equal instructional conditions, their performances in solving these problems are significantly different. An explanation to this is that though the two groups received the same concept definition of vector-space, they were exposed to different experiences which resulted in forming different concept images. Group B handled the problems more successfully due to its superior concept image of vector-space.

To gain more insight into the two groups' concept image of vector-space, we consider the justifications they provided to their answers (Table 3). As was described earlier, in Problem 1 equal achievement of the two groups occurred in giving a correct final answer with a correct justification. On the other hand, in Problems 2 and 4 a better achievement of Group B occurred in providing correct justification. Moreover, the number of correct justifications that occurred in Group B increased significantly from Problem 1 to Problem 2; this is despite the fact that the latter problem is more abstract than the former one. An explanation to this is that Group B attained a better intuition of the vector-space concept than Group A. The concept image of Group B students included the necessary and the sufficient properties of vector-space. Consequently, they considered their geometric description of the models given in Problem 1 as a sufficient justification to the final answers they provided. This geometric description rendered their answers self-evident in their eyes, and they saw no need for justifying such a self-evident answer. The second problem, on the other hand, was less self-evident to the students because of its high level of generality. As a result, more students from Group B saw the necessity of providing a mathematical justification in addition to the geometric description they provided. It should be noted, however, that this pattern of responses to Problem 1 and 2 did not occur in the distributions of Problem 1 and 4. This can be explained by the fact that a correct justification to Problem 4 is rather complicated to formulate.

General Implications

Implications for those involved in teaching abstract mathematical systems, such as linear algebra or group theory, are that the embodiment process is essential for constructing a desired concept image of abstract concepts. But a special embodiment process is appropriate for this goal: The factors of familiarity and mode of representation must be taken into account in applying the principle of multiple embodiments. Familiar geometric embodiments seem to be a significant contribution to the concept image formation, and the mathematical models whose elements are a collection of numbers or functions must be approached carefully in embodying them in abstract systems.

Finally, we indicate that we are unable to explain why certain students in Group A continued to prefer the geometric mode to which they were not exposed or why certain students in Group B continued to prefer the algebraic mode while others chose the geometric mode. This suggests a further investigation.

References

Beardslee, E. C. (1973). Toward a theory of sequencing: Study 1-7: An exploration of the effect of instructional sequences involving enactive and iconic embodiments on the ability to generalize, Doctoral dissertation, Pennsylvania State University, 1972. *Dissertation Abstracts International, 33*, 6721A, University Microfilms No. 73-13, 953.

Dienes, Z. P. (1960). *Building up mathematics.* New York: Hutchinson Educational Ltd.

Dienes, Z.P. (1964). *The power of mathematics.* London: Hutchinson Educational Ltd.

Gau, G. E. (1973). Toward a theory of sequencing: Study 1-6: An exploration of the effect of instructional sequences involving enactive and iconic embodiments on the attainment of concepts embodied symbolically, Doctoral dissertation, Pennsylvania State University, 1972, *Dissertation Abstracts International, 33*, 6728A, University Microfilms No. 73-13, 980.

Harel, G. (1985). *Teaching linear algebra in high-school.* Unpublished doctoral dissertation, Ben-Gurion University of the Negev, Beer-Sheva, Israel.

Harel, G. (1987). Variations in linear algebra content presentations. *For the Learning of Mathematics*, *7*, 29-32.

Vinner, S. (1983). Concept definition, concept image and the notion of function, *International Journal of Mathematics Education, Science and Technology, 14*, 293-305.