In *Proof in Mathematics Education*, Reid and Knipping attempt to provide a comprehensive review of important perspectives and research on proof. The book “is intended to help teachers, researchers, and students to overcome the difficulty of getting an overview of research on proof and proving” (p. xiii). We were excited by this undertaking and found the book to contain many useful contributions. It is a great step toward furthering the research on proof within mathematics education.

Other sources have attempted to bring together work on proof, but in different ways. *Teaching and Learning Proof across the Grades: A K-16 Perspective* collected together some innovate work on various issues concerning proof at different grade levels, but it was a sampling of different strands of research. The more recent *Proof and Proving in Mathematics Education: the 19th ICMI Study* (Hanna & de Villiers, 2012) brings together a variety of viewpoints on issues such as (1) the potential role of proving in deepening mathematical understanding and practice, (2) the developmental nature of proving across the grades, and (3) curriculum development and teacher education programs to enhance the learning and teaching of proof. Harel and Sowder (2007) attempted to provide comprehensive perspectives on proof, but their scope was much narrower than this book’s. They called for perspectives that would incorporate a broad range of factors: mathematical, historical-epistemological, cognitive, sociological, and instructional. Toward this end, they identified thirteen questions to be addressed within these factors. However, they did not attempt to synthesize most existing research literature on proof, as Reid and Knipping do in this book. Due to differences in purpose and length, *Proof in Mathematics Education* is organized quite differently from these previous works. Reid and Knipping provide a very thorough literature review, including both theoretical and empirical work. They look at different meanings and perspectives, past and current results, and consider additional directions for research on proof. The book is divided into four parts based on these themes.

The first part of the book provides a contextual frame that allows the reader to make sense of findings found in the literature, which may come from quite different perspectives. The book summarizes the so-called standard view of the history of proof (centered around a Euclidean method of proof) and describes limitations and flaws of this view. In particular, it points out that proving occurred in ancient China and India, and that Euclid’s proofs use unstated assumptions and rely on diagrams (thus, these are less rigorous than supposed by some). Moreover, a formalization of most proofs is impossible, and the semiformal proofs that are standard practice must be checked by a qualified reader (not a mere mechanized process). Finally, proofs can serve to locate hidden assumptions or flawed axioms.

Reid and Knipping discuss some of the many uses of the terms “proof” and “proving,” and they suggest that the myriad of perspectives is useful because different researchers focus on different issues—leading them to adopt different definitions. The book describes several research perspectives on proof and characterizes them according to their philosophy (based on the notion of truth that proofs relate to), meaning of proof, and breadth of
what is considered proof. Following Balacheff (2008), they argue that it is neither possible, nor even desirable, to achieve a consensus of perspectives on proof. However, it is possible—and desirable, as part of a goal of this book—for researchers to connect their perspectives and outcomes, a view that served the impetus for Harel and Sowder’s (2007) work on comprehensive perspectives mentioned earlier.

The second part of the book examines the body of research on teaching and learning of proof. It describes studies showing the following accepted results about students’ understanding: many (perhaps most) students accept examples as verification; many students do not accept deductive proofs as verification; many students do not accept counterexamples as refutation; students accept flawed deductive proofs as verification; many students accept arguments on bases other than logical coherence; students offer empirical arguments to verify; most students cannot write correct proofs (p. 57). Some of these results are or can be reinterpreted more subtly. For instance, students may reject a deductive proof for somewhat valid reasons, such as it only proves but does not explain (Hanna, 1990; Harel, 2013; Steiner, 1978), or it gives the argument only for a given diagram or contains unstated or incorrect assumptions. Similarly, students’ use of empirical arguments (even at higher grade levels) may reflect their inability to produce deductive proofs rather than their belief that these arguments are sufficient. Moreover, some empirical reasoning can be quite sophisticated, such as testing extremal cases where a statement is more likely to fail or treating the examples more generically. Based on comparing research over time, it appears that students’ skill at proof writing has declined, although further research would be needed to clarify the causes of this decline. Teachers’ understanding of proof has also been shown to be weak in many cases, and additional research is needed on teacher preparation. Finally, further work should investigate how students think about counterexamples.

The book delineates various roles of proof, including verifying, explaining, exploring, systematization, and communication. Future research could examine what effects giving different roles to proof has on students’ motivation and acceptance of proofs. The book also discusses other types of reasoning beyond deductive. **Inductive reasoning** generalizes merely from specific cases. **Abduction** involves hypothesizing data that might account for a surprising result, and it can be used to generate possible directions that a deductive proof might take. **Reasoning by analogy** involves projecting known ideas onto an unfamiliar domain, and it can be used to discover/conjecture new results that could be deductively verified. Further research could look at the ways in which abduction and reasoning by analogy can be used to explore and discover. In addition, more research is needed on how students formalize their reasoning from any of these types.

Reid and Knipping categorize proofs and arguments into four broad classes, based on the use of representations: Empirical: specific examples are used but do not represent a general class; Generic: specific examples are used as representations; Symbolic: words and symbols are used as representations; Formal: symbols and words do not represent anything (p. 130). Further research is required to examine the ways in which facility with some of these types of reasoning can support or interfere with other types. The book also examines argumentation (e.g. Duval’s 1991 model), which is linked to proof. Further research is warranted to continue looking at the relationship between argumentation and proof.

By reviewing some teaching experiments concerning proof, Reid and Knipping draw out several features of a teaching approach that seem to be beneficial. First, students generate conjectures, then verify or explain these. Second, the form of proof texts arises from extended discussion rather than being prescribed. Third, there is a focus on the nature of the discourse instead of the specific content examined. Fourth, the social context involves expectation for explanations and sufficient time and attention to examine these in depth.

Part 3 of the book looks more generally at processes of argumentation. The book describes four **argumentation structures**, as well as five **patterns of reasoning**, observed in research and compares these. All patterns of reasoning involve testing, but the nature of this differs, and the need motivating the reasoning (verify, explore, explain) also differs. Further research could examine how the common argumentation structure or pattern of reasoning that arises in a classroom might be related to factors such as use of two-column proofs, or an inquiry-oriented approach.

The last part of the book draws conclusions from the research described in the previous parts. First, more practical conclusions are described: some promising possibilities for teaching proof. Given the importance of using established previous knowledge in deduction, teachers should pay attention to the “tool-box” of accepted results that are available, as well as how knowledge is “locally organized” in the classroom (e.g. which property do we take as the definition of an object). Finally, as in any student-centered approach, teachers should take into account students’ current ways of reasoning in trying to move them toward more deductive reasoning.

The book does a good job summarizing current research on proof. Major lines of research are described, although some prominent individual studies or researchers were not included. The literature review provides the reader with a sense of what the field of proof education knows, as well as some of its history. Moreover, it clearly conveys different
meanings of proof, proving, argumentation, and related terms. Correspondingly, it delineates different perspectives on proof and shows how potential confusion arising from differences in perspectives and terminology can be negotiated.

Because of these contributions—providing an extensive literature review and clarification of perspectives—we strongly recommend this book to graduate students and new researchers who are interested in mathematical proof and proving. We find the book to be useful even to more experienced researchers as a reference finder/aid.

The broad conception of proof (clarifying the different foci of studies under different perspectives) makes clear the important role that many types of reasoning, beyond formal deduction, play. In particular, inductive, abductive, and analogical reasoning are fairly well-described. The interplay between these kinds of reasoning, conjecturing, and proving is explored, and we agree that further work is warranted to relate different reasoning.

Overall, the balance of research included is fair, with a couple of exceptions. First, although the literature review seems to be fairly comprehensive for proof in K-12, it mentions fewer of the studies regarding proof at the college/university level. This made the coverage in the book seem less balanced than it otherwise would be. Second, the authors focus on their own work proportionally more than seems necessary; see for example Chapter 10 and the relative number of references to work by the authors. Third, there were some cases where the choice of how much detail to present for different issues seemed imbalanced, in that some sections seemed to give extensive information about a study while others gave too little detail, without clear justification.

Although the book helps researchers find existing gaps in the literature, it could have made an even stronger contribution if more space had been devoted to these gaps. As the authors note, they raise research questions at the end of each chapter in Parts 2 and 3, but these questions are rather local—directly arising from the studies discussed. Part 4 is fairly brief (Chapter 13 occupies only 6 out of the book’s 226 pages) and the last two chapters seem less comprehensive than many of the earlier chapters. We had hoped to see more synthesis of the rest of the book here, so as to unify and make coherent more of the results that had been discussed. Naturally, this is a difficult undertaking, but one that has tremendous value. Chapters 12 and 13 contain primarily summaries. Although useful research questions are raised here, we believe that there are many more existing gaps in research to be identified and research questions that are worth raising. We believe that a follow-up effort that focuses on more of these questions would provide a valuable service to the research community—particularly young researchers who are looking for fruitful questions to pursue.

It seems useful to provide questions that will help the reader think about the organization of a chapter in advance, as this book does. However, we found the questions at the beginning of the chapters to be of limited value. Many of them were more basic and did not serve this role well, so a reader might easily begin skipping over them. Also, Reid and Knipping highlight the potential difficulties that can arise from having a multitude of perspectives on proof and different definitions of key terms. Although we applaud this general effort—these issues definitely can be confusing for a reader—it felt as though these issues were mentioned too frequently, sometimes making the book feel redundant and pedantic. An intelligent reader should be able to keep such issues in mind without being continually reminded.

Abduction is an important type of reasoning that continues to be of interest to mathematics educators, including Reid and one of us. It is mostly well-described in the book, but one example requires clarification. Reid and Knipping present a student who, in thinking about a figure for the Pythagorean Theorem, knows that the inner figure is a rhombus but wants it to be a square. He abducts that it might have a right angle, which would be enough to conclude that it is a square. However, this does not seem to fit the meaning of abduction, in which a fact would be sufficient for the conclusion but not necessary. In this case, if one has a rhombus that must turn out to be a square, it is necessary for it to have a right angle. Moreover, a student at this level could conceive of such a fact—even if he did not in this case.

References


