Math 20B Supplement
Linked to Rogawski, Edition 1

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Bill Helton and John Eggers - The Maintenance Crew
Written By
Ed Bender, Bill Helton and John Eggers
Contributions by Magdelana Musat

Supplement to Chapters 7 and 9 of Rogawski Calculus Edition 1

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1 Complex Numbers

1.1 Introduction

What are numbers? Your first reaction to this question is probably: "What a silly question; everyone knows what numbers are!" However, a moment's thought will reveal that the question is not nearly as trivial as it first appears. It's not so hard to give examples of numbers: natural numbers, integers, rational numbers, and real numbers come to mind. But why are there so many different types of numbers and what distinguishes them? Are there any more types of numbers? A complete answer to these questions would take us too far afield; however, it will be worthwhile to come up with brief answers to these questions.

Historically, the different types of numbers arose to address increasingly sophisticated mathematical problems. The natural numbers provide a mechanism for counting objects. The integers provide the capability to carry out subtraction in order to solve simple equations, such as $x + 5 = 2$. The rational numbers provide the capability to perform division and solve equations such as $7x+3 = 1$. Finally, the real numbers allow one to compute the limit of sequences of rational numbers, such as $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.

The real number system is a very powerful number system and provides the foundation for calculus. However, the real number system does not allow one to solve certain simple equations, such as $x^2 + 1 = 0$[1]. To solve this equation, one must expand the real number system by introducing a new number called i with the property that $i^2 = -1$. Your first reaction to this might be: "What an absurd idea! You can't just make up numbers with imaginary properties!" That's exactly the reaction most people had when the idea was first proposed; in fact, to this day i is still called an *imaginary* number. Despite this, people soon found that the idea of introducing a number i such that $i^2 = -1$ is a very far-reaching idea, the type of idea Albert Einstein had in mind when he said: "If at first the idea is not absurd, then there is no hope for it." In fact, this seemingly absurd idea led to the development of the complex number system: a number system so rich that a whole branch of mathematics, known as complex analysis, grew out of the study of its structure and the properties of its functions.

1.2 Definition and Basic Properties

Definition 1.1. A complex number is a number of the form $a + bi$ (or, equivalently, $a + ib$), where a and b are real numbers and $i^2 = -1$. The real number a is called the real part of $a + bi$ and the real number b is called the imaginary part of $a + bi$.

It is important to notice that the real and imaginary parts of a complex number are *real* numbers.

[1] This is a simplification: it was the unavoidable appearance of complex numbers in the formula for solving the general cubic equation that provided the main initial impetus for accepting complex numbers.

For example, the real part of $3-4i$ is 3 and the imaginary part of $3-4i$ is -4.

The arithmetic of complex numbers is the same as for polynomials where we treat i as an unknown (like x) with the property that $i^2=-1$. Thus, if $\alpha = a+bi$ and $\beta = c+di$ are complex numbers, then

$$\begin{aligned} \alpha+\beta &= (a+c)+(b+d)i \\ \alpha-\beta &= (a-c)+(b-d)i \\ \alpha\,\beta &= (ac-bd)+(ad+bc)i \end{aligned}$$

since $(a+bi)(c+di) = ac + ad\,i + bc\,i + bd\,i^2$ and $i^2=-1$.

Thus addition, subtraction and multiplication of complex numbers is simple enough, but how do we divide complex numbers? The quotient $\frac{\alpha}{\beta} = \frac{a+bi}{c+di}$, so what's the problem? The problem is that $\frac{a+bi}{c+di}$ is not in the form $A+Bi$ where A and B are real numbers. Since we agreed that a complex number is a number of the form $A+Bi$ with A and B real numbers, we must write $\frac{a+bi}{c+di}$ in that form in order to verify that it is also a complex number. But this is easy to do:

$$\frac{a+bi}{c+di} = \left(\frac{a+bi}{c+di}\right)\left(\frac{c-di}{c-di}\right) = \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{-ad+bc}{c^2+d^2}\right)i$$

In order to write the quotient $\frac{a+bi}{c+di}$ as a complex number, we used the fact that $(c+di)(c-di) = c^2+d^2$, a positive real number. This turns out to be an important observation and leads us to define the *complex conjugate* of $c+di$ to be the complex number $c-di$. More formally, we have the following

Definition 1.2. Let $\alpha = a+bi$ be a complex number.
(a) The *complex conjugate* of α is the complex number $\overline{\alpha} = a-bi$.
(b) The *magnitude* of α, written $|\alpha|$, is given by $|\alpha| = \sqrt{\alpha\overline{\alpha}} = \sqrt{a^2+b^2}$.
$|\alpha|$ is also called the *modulus*, *length* or *absolute value* of α.

It is not difficult to verify that conjugation *commutes* with the arithmetic operations; that is, conjugation and the arithmetic operations may be carried out in either order without affecting the result of the computation. More formally, we have the following

Theorem 1.3. *Let z and w be complex variables. Then,*

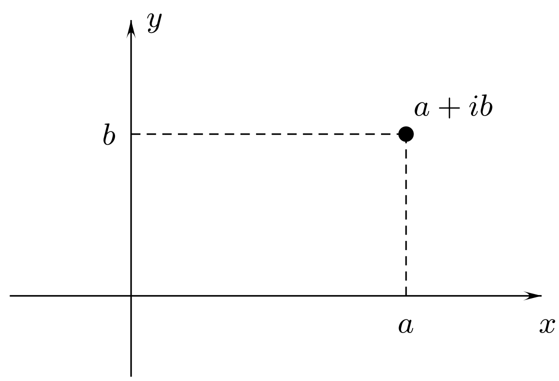
$$\overline{z+w} = \overline{z}+\overline{w}, \qquad \overline{z-w} = \overline{z}-\overline{w}, \qquad \overline{zw} = \overline{z}\,\overline{w} \qquad \textit{and} \qquad \overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}\,.$$

For example, if $z = x+iy$ and $w = u+iv$, then $z+w = (x+u)+i(y+v)$ and the statement that $\overline{z+w} = \overline{z}+\overline{w}$ just says that $(x+u)-i(y+v) = (x-iy)+(u-iv)$. The other three equations can be verified in a similar manner.

It is worth mentioning that there are conventions regarding which letters to use for representing complex numbers. While a and b are commonly used to represent real numbers, you will often see α and β used to represent complex numbers, as we have done here. Also, instead of x and y (used to represent real variables), you will see z and w used to represent complex variables.

1.3 Geometric Properties

Real numbers are geometrically represented by points on a line; we call this the *real number line*. Complex numbers are geometrically represented as points (or, more precisely, *vectors*) in a plane; we call this the *complex plane*.

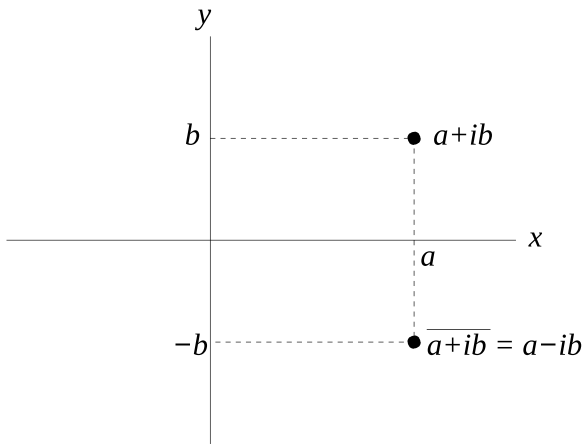


To emphasize this, we could represent the complex numbers $\alpha = a + bi$ and $\beta = c + di$ as ordered pairs by $\alpha = (a, b)$ and $\beta = (c, d)$, and the arithmetic properties could be summarized by

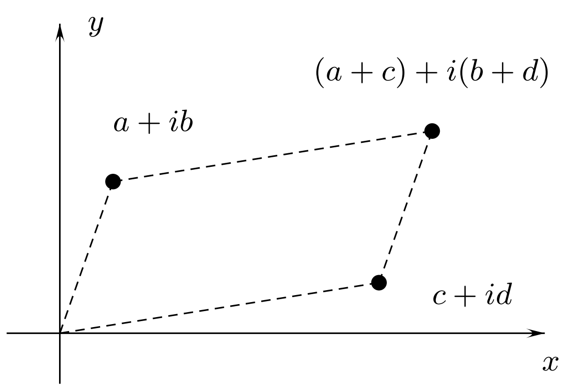
$$\begin{aligned} (a,b) + (c,d) &= (a+c, b+d) \\ (a,b) - (c,d) &= (a-c, b-d) \\ (a,b)(c,d) &= (ac-bd, ad+bc) \\ \frac{(a,b)}{(c,d)} &= \left(\frac{ac+bd}{c^2+d^2}, \frac{-ad+bc}{c^2+d^2}\right) \end{aligned}$$

We will not pursue this method of representing complex numbers, except to mention that it emphasizes the fact that complex numbers provide a way of making the set of points in the plane into a number system. In fact, the realization that complex numbers can be represented in the plane did much to promote their acceptance; moreover, advanced calculators, such as the TI-86, represent complex numbers as ordered pairs. Despite this, we will prefer to represent complex numbers in the form $a + bi$.

It is also worth mentioning that, given $\alpha = a + bi$, $\overline{\alpha} = a - bi$ is the reflection of α about the x-axis and $|\alpha|$ is the distance between from the origin $(0, 0)$ to the point (a, b). Note also that $|\alpha| = |\overline{\alpha}|$.



Addition of complex numbers also has a nice geometric interpretation. Given complex numbers $\alpha = a + bi$ and $\beta = c + di$, the number $\alpha + \beta$ is represented by the fourth point of the parallelogram determined by the points $(0, 0)$, (a, b) and (c, d). For those familiar with vectors, this is just vector addition. Subtraction has a similar geometric interpretation.



What about the geometric interpretation of multiplication? It turns out that multiplication of complex numbers involves rotations and is best described using polar coordinates.

1.4 Polar Form

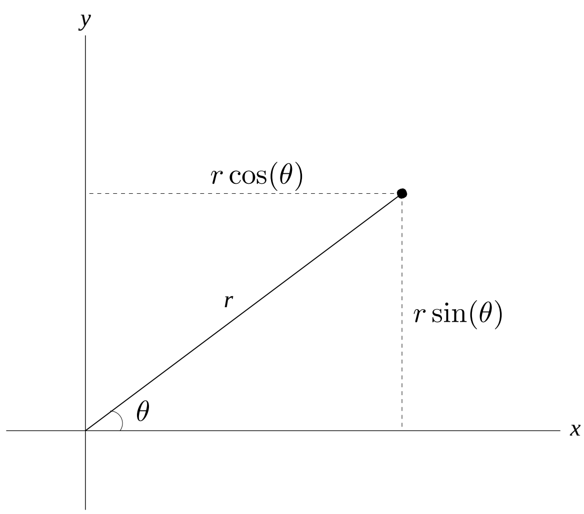
As we have seen, a complex variable $z = x + iy$ is represented geometrically by a point (x, y) in the plane. Any point in the plane can be represented in polar coordinates (r, θ) with $r \geq 0$, and

$$x = r\ \cos(\theta) \qquad y = r\ \sin(\theta).$$

Thus, every complex number z can be written in the form

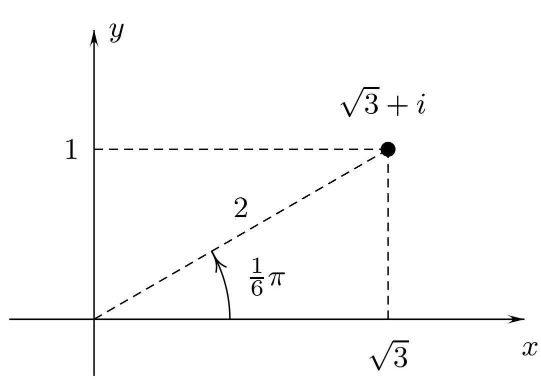
$$z = r\left[\cos(\theta) + i\sin(\theta)\right].$$

This is called the *polar form* of z. Note that $|z| = r$ since $|z| = \sqrt{[r\cos(\theta)]^2 + [r\sin(\theta)]^2} = r$. The angle θ is called the *argument* of z, written $\theta = \arg(z)$. It is important to realize that $\theta = \arg(z)$ is not uniquely determined; for, if θ is an argument of z, then so is $\theta + 2k\pi$ for any integer k. Thus, calling $\theta = \arg(z)$ *the* argument of z is technically not correct: $\theta = \arg(z)$ is an *equivalence class* of arguments of z. However, there is little chance of confusion as long as one understands that θ and $\theta + 2k\pi$ are essentially the "same" angle (more precisely, they are *coterminal* angles).



Example 1.4. Write $\sqrt{3} + i$ in polar form.

Solution: $r = |\sqrt{3} + i| = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$. Thus, $\cos(\theta) = \frac{\sqrt{3}}{2}$ and $\sin(\theta) = \frac{1}{2}$ and we can take $\arg(z) = \frac{\pi}{6}$. Therefore, the polar form of $\sqrt{3} + i$ is $2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$.



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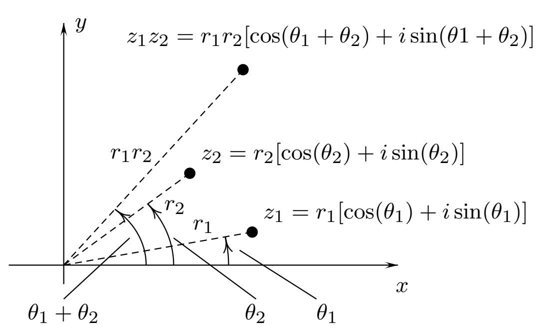
We mentioned earlier that the geometric interpretation of multiplication is best described using polar coordinates. Let's investigate this. Suppose z_1 and z_2 are complex numbers expressed in polar form as $z_1 = r_1\left[\cos(\theta_1) + i\sin(\theta_1)\right]$ and $z_2 = r_2\left[\cos(\theta_2) + i\sin(\theta_2)\right]$. Then,

$$z_1 z_2 = r_1 r_2 \left\{\left[\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\right] + i\left[\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)\right]\right\}$$

Applying the addition formulas for the sine and cosine, we conclude that

$$z_1 z_2 = r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right].$$

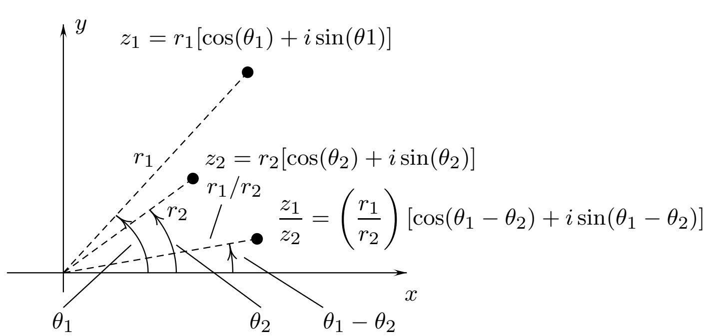
This tells us that multiplying $z_1 = r_1\left[\cos(\theta_1) + i\sin(\theta_1)\right]$ by $z_2 = r_2\left[\cos(\theta_2) + i\sin(\theta_2)\right]$ results in a scaling by a factor of r_2 and a rotation by an angle of θ_2. In other words, magnitudes multiply and arguments add.



A similar computation shows that

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right].$$

This tells us that dividing z_1 by z_2 results in a scaling by a factor of $\frac{1}{r_2}$ and a rotation by an angle of $-\theta$. This should not be a surprise: since division is the reverse of multiplication, we divide magnitudes and subtract arguments.



Example 1.5. Let $\alpha = 1 + i$. By what angle will multiplication by $\beta = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$ rotate α? Compute $\alpha\,\beta$ directly and verify that the result is a complex number with the same length as α and rotated by $\arg(\beta)$.

Solution: $\beta = -\frac{\sqrt{3}}{2} + \frac{1}{2}i = \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)$, thus multiplication by β produces a rotation by $\frac{5\pi}{6}$. Then $\alpha\,\beta = (1+i)\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -\frac{\sqrt{3}+1}{2} - \frac{\sqrt{3}-1}{2}i$, $|\alpha\,\beta| = \frac{1}{2}\sqrt{\left(\sqrt{3}+1\right)^2 + \left(\sqrt{3}-1\right)^2} = \sqrt{2}$ and $|\alpha| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$; thus, $|\alpha\,\beta| = |\alpha|$. To verify that $\alpha\,\beta$ is α rotated by $\frac{5\pi}{6}$, first observe that $\alpha = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$. Since $\frac{\pi}{4} + \frac{5\pi}{6} = \frac{13\pi}{12}$, we are required to verify that $\alpha\,\beta = \sqrt{2}\left[\cos\left(\frac{13\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right)\right]$. By the half angle identities,

$$\cos\left(\frac{13\pi}{12}\right) = -\sqrt{\frac{1+\cos\left(\frac{13\pi}{6}\right)}{2}} = -\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = -\frac{\sqrt{2+\sqrt{3}}}{2}$$

$$\sin\left(\frac{13\pi}{12}\right) = -\sqrt{\frac{1-\cos\left(\frac{13\pi}{6}\right)}{2}} = -\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = -\frac{\sqrt{2-\sqrt{3}}}{2}$$

Thus, $\sqrt{2}\left[\cos\left(\frac{13\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right)\right] = -\frac{\sqrt{2}\sqrt{2+\sqrt{3}}}{2} - \frac{\sqrt{2}\sqrt{2-\sqrt{3}}}{2}i = -\frac{\sqrt{4+2\sqrt{3}}}{2} - \frac{\sqrt{4-2\sqrt{3}}}{2}i$. Since we have $\left(\sqrt{3}+1\right)^2 = 4+2\sqrt{3}$ and $\left(\sqrt{3}-1\right)^2 = 4-2\sqrt{3}$, we conclude that $\alpha\,\beta = \sqrt{2}\left[\cos\left(\frac{13\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right)\right]$.
■

1.5 de Moivre's Theorem

We have seen that when two complex numbers are multiplied together, their magnitudes multiply and their arguments add. Thus, if we raise a complex number to an integral power n, its magnitude will be raised to the power n and its argument will be *multiplied* by n. This is de Moivre's theorem:

Theorem 1.6. *Let $z = r\left[\cos(\theta) + i\sin(\theta)\right]$ be a complex number in polar form and let n be an integer. Then,*

$$z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right].$$

Although a formal proof of de Moivre's theorem would require mathematical induction, it essentially follows from repeated application of the multiplication formula for complex numbers in polar form:

$$\begin{aligned} z &= r\left[\cos(\theta) + i\sin(\theta)\right] \\ z^2 = z\,z &= r^2\left[\cos(2\theta) + i\sin(2\theta)\right] \\ z^3 = z^2\,z &= r^3\left[\cos(3\theta) + i\sin(3\theta)\right] \\ &\cdots \\ z^n = z^{n-1}\,z &= r^n\left[\cos(n\theta) + i\sin(n\theta)\right] \end{aligned}$$

Thus, de Moivre's theorem holds for positive integers. What about *negative* integers? To see that de Moivre's theorem holds for negative integers, observe that de Moivre's theorem holds for $n = 0$

since $z^0 = r^0\,[\cos(0) + i\sin(0)] = 1\,(1 + i\,0) = 1$. Finally, $z^{-n} = r^{-n}\,[\cos(-n\theta) + i\sin(-n\theta)]$ since $z^{-n}\,z^n = r^{-n+n}\,[\cos(-n\theta + n\theta) + i\sin(-n\theta + n\theta)] = 1$, which verifies de Moivre's theorem for all integers.

Although de Moivre's theorem follows easily from the formulas for multiplication and division of complex numbers, it is a powerful result that makes raising a complex number to a power nearly as easy as raising a real number to a power.

Example 1.7. Compute $\left(1 + \sqrt{3}\,i\right)^8$ and write the result in standard $(a + bi)$ form.

Solution: $\left|1 + \sqrt{3}\,i\right| = \sqrt{(1)^2 + (\sqrt{3})^2} = 2$. Thus, $1+\sqrt{3}\,i = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right) = 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$.
Hence, $\left(1 + \sqrt{3}\,i\right)^8 = 2^8\left[\cos\left(\frac{8\pi}{3}\right) + i\sin\left(\frac{8\pi}{3}\right)\right] = 256\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right) = -128 + 128\sqrt{3}\,i$. ■

We can also use de Moivre's theorem to find the roots of a complex number. An n^{th} root of a complex number z is a complex number w such that $w^n = z$. In order to find the solutions to this equation, we write z and w in polar form: $z = r\,[\cos(\theta) + i\sin(\theta)]$ and $w = s\,[\cos(\phi) + i\sin(\phi)]$. By de Moivre's theorem, $w^n = z$ is equivalent to the equation $s^n\,[\cos(n\phi) + i\sin(n\phi)] = r\,[\cos(\theta) + i\sin(\theta)]$. Thus, $s^n = r$ and $n\phi = \theta + 2\pi m$, where m is any integer. Since r and s are positive and since it is sufficient to find the values of ϕ in the interval $[0, 2\pi)$, we have

$$s = r^{\frac{1}{n}}, \quad \text{the positive } n^{\text{th}} \text{ root of } r;$$
$$\phi = \frac{\theta + 2\pi k}{n}, \quad \text{where } k = 0, 1, \ldots, n-1.$$

We have proved the following

Theorem 1.8. *Let $z = r\,[\cos(\theta) + i\sin(\theta)]$ be a complex number in polar form and let n be a positive integer. Then z has the n distinct n^{th} roots*

$$w_k = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right],$$

where $k = 0, 1, \ldots, n-1$.

It is worth observing that $|w_k| = r^{\frac{1}{n}}$ and $\arg(w_{k+1}) - \arg w_k = \frac{2\pi}{n}$ for each k. Thus, the n^{th} roots of z lie equally spaced along the circle of radius $r^{\frac{1}{n}}$.

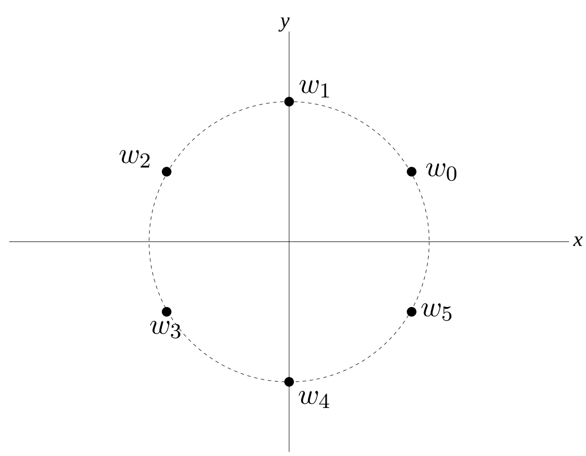
Example 1.9. Find the sixth roots of -64.

Solution: $-64 = 2^6\,[\cos(\pi) + i\sin(\pi)]$. Thus, the 6^{th} roots of -64 are given by

$$w_k = 2\left[\cos\left(\frac{\pi + 2\pi k}{6}\right) + i\sin\left(\frac{\pi + 2\pi k}{6}\right)\right], \quad k = 0, 1, 2, 3, 4 \text{ or } 5.$$

Thus, the explicit list of 6^{th} roots of -64 is as follows:

$$
\begin{aligned}
w_0 &= 2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right] = \sqrt{3} + i \\
w_1 &= 2\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right] = 2i \\
w_2 &= 2\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right] = -\sqrt{3} + i \\
w_3 &= 2\left[\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right] = -\sqrt{3} - i \\
w_4 &= 2\left[\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right] = -2i \\
w_5 &= 2\left[\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right] = \sqrt{3} - i
\end{aligned}
$$



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1.6 Exercises

1. Simplify the following expressions and write the result in rectangular form $a + bi$.
   (a) $5 - \frac{3}{2}i - (8 + \frac{5}{2}i)$
   (b) $(-2 + 7i)(5 - 4i)$
   (c) $\overline{3i(6 - i)}$
   (d) $\frac{1+i}{2-3i}$
2. Let $z = 1 + \sqrt{3}\,i$ and $w = -1 - i$.
   (a) Write z and w in the polar form $r\,e^{i\theta}$.
   (b) Compute zw, $\frac{z}{w}$ and $\frac{1}{z}$, and write the results in polar form.
3. Simplify the following expressions and write the results in the polar form $r\,e^{i\theta}$.
   (a) $(1 + i)^{13}$
   (b) $(-\sqrt{3} + i)^{15}$
   (c) $(32 - 32i)^{-6}$
4. Find each of the following roots and sketch them in the complex plane.
   (a) the cube roots of 1
   (b) the fourth roots of $1 + i$
   (c) the sixth roots of -64

2 Complex Exponentials: For Rogawski Edition 1

How should we define e^{a+bi} where a and b are real numbers? In other words, what is e^{α} when α is a complex number? We would like the nice properties of the exponential to still be true. The most basic properties of an exponential we would like to retain are that for any complex numbers α and β we have

$$e^{\alpha+\beta} = e^{\alpha}\, e^{\beta} \qquad \text{and} \qquad \frac{d}{dx}\, e^{\alpha x} = \alpha e^{\alpha x}. \tag{2.1}$$

It turns out that the following definition produces a function with these properties.

Definition of complex exponential: $e^{a+bi} = e^{a}\left[\cos(b) + i\sin(b)\right] = e^{a}\cos(b) + ie^{a}\sin(b)$

In particular, for any real number x, Euler's formula holds true:

$$e^{ix} = \cos(x) + i\sin(x). \tag{2.2}$$

We now prove the first key property in (2.1).

Theorem 2.1. *If α and w are complex numbers, then*

$$e^{\alpha+\beta} = e^{\alpha}e^{\beta}.$$

Proof. Let $\alpha = a + ib$ and $\beta = h + ik$. Then,

$$\begin{aligned} e^{\alpha}e^{\beta} &= e^{a}\left[\cos(b) + i\sin(b)\right] e^{h}\left[\cos(k) + i\sin(k)\right] \\ &= e^{a}e^{h}\left\{\left[\cos(b)\cos(k) - \sin(b)\sin(k)\right] + i\left[\cos(b)\sin(k) + \sin(b)\cos(k)\right]\right\} \\ &= e^{a+h}\left[\cos(b+k) + i\sin(b+k)\right] \\ &= e^{[a+h+i(b+k)]} \\ &= e^{\alpha+\beta} \end{aligned}$$

∎

We leave checking the second property to the exercises. For those who are interested there is an appendix, Section 7, which discusses what we mean by derivative of a function of complex variables and explains how to obtain the second property as well.

It's easy to get formulas for the trigonometric functions in terms of the exponential. Look at Euler's formula (2.2) with x replaced by $-x$:

$$e^{-ix} = \cos(x) - i\sin(x).$$

We now have two equations in $\cos x$ and $\sin x$, namely

$$\begin{aligned}\cos(x) + i\sin(x) &= e^{ix}\\ \cos(x) - i\sin(x) &= e^{-ix}.\end{aligned}$$

Adding and dividing by 2 gives us $\cos(x)$ whereas subtracting and dividing by $2i$ gives us $\sin(x)$:

Exponential form of sine and cosine: $\cos(x) = \dfrac{e^{ix} + e^{-ix}}{2}$ $\quad \sin(x) = \dfrac{e^{ix} - e^{-ix}}{2i}$

Setting $x = z = a + bi$ gives formulas for the sine and cosine of complex numbers. We can do a variety of things with these formulas. Here are some worth mentioning:

- Since the other trigonometric functions are rational functions of sine and cosine, this gives us formulas for all the trigonometric functions.
- The hyperbolic and trigonometric functions are related:

$$\cos(x) = \cosh(ix) \text{ and } i\sin(x) = \sinh(ix).$$

2.1 Complex Exponentials Yield Trigonometric Identities

The exponential formulas we just derived, together with $e^{z+w} = e^z e^w$ imply the identities

$$\begin{aligned}&\sin^2(\alpha) + \cos^2(\alpha) = 1\\ &\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)\\ &\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).\end{aligned}$$

These three identities are the basis for deriving trigonometric identities; that is, we can derive trigonometric identities by using the exponential formulas and $e^{z+w} = e^z e^w$. We now illustrate this with some examples.

Example 2.2. Show that $\cos^2(x) + \sin^2(x) = 1$. Indeed, we have

$$\begin{aligned}\left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 + \left(\frac{e^{ix} + e^{-ix}}{2i}\right)^2 &= \frac{1}{4}\left[(e^{ix})^2 + 2 + (e^{-ix})^2 + \frac{(e^{ix})^2 - 2 + (e^{-ix})^2}{i^2}\right]\\ &= \frac{1}{4}\left[2+2\right] = 1,\end{aligned}$$

wherein we have used the fact that $i^2 = -1$. ■

Example 2.3.

$$\begin{aligned}\sin(2x) &= \frac{e^{i2x} - e^{-i2x}}{2i} \\ &= \frac{1}{2i}\left[(e^{ix})^2 - (e^{-ix})^2\right] \\ &= 2\,\frac{[e^{ix} - e^{-ix}]}{2i}\,\frac{[e^{ix} + e^{-ix}]}{2} \\ &= 2\sin(x)\cos(x)\end{aligned}$$

■

2.2 Exercises

1. We know that $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$. Use the relationship between the sine, cosine and exponential functions to express $\cos^3(x)$ as a sum of sines and cosines.

2. Show that $e^{\pi i}+1 = 0$. This uses several basic concepts in mathematics, such as π, e, addition, multiplication and exponentiation of complex numbers, all in one compact equation.

3. What are the cartesian coordinates x and y of the complex number $x + iy = e^{2+3i}$?

4. Use the fact that
$$\frac{d}{dx}\left[\cos(bx) + i\sin(bx)\right] = b[-\sin(bx) + i\cos(bx)]$$
and the product rule to show that
$$\frac{d}{dx}\left[e^{(a+ib)x}\right] = (a+ib)e^{(a+ib)x}.$$
This is the key differentiation property for complex exponentials.

5. The trigonometric functions given by
$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$
are defined for all complex numbers z.
   (a) Compute $\cos\left(i\ln(2)\right)$.
   (b) Solve the equation $\cos(z) = \frac{5}{3}$. (Hint: write $w = e^{iz}$; then, $\frac{e^{iz}+e^{-iz}}{2} = \frac{5}{3}$ becomes $\frac{1}{2}\left(w + \frac{1}{w}\right) = \frac{5}{3}$.)

3 Integration of Functions which Take Complex Values

This supplements Section 7.3 and 7.5 of Rogawski Ed. 1.

Now we turn to the problem of integrating functions which take complex values. Of course this is bound up with what we mean by antiderivatives of complex functions. A function, such as $f(x) = (1+2i)x + i3x^2$, may have complex values but the variable x is only allowed to take on real values and we only define definite integrals for this type of function. In this case nothing differs from what we already learned about integrals of real-valued functions.

- The Riemann sum definition of an integral still applies.
- The Fundamental Theorem of Calculus is still true.
- The properties of integrals, including substitution and integration by parts, still work.
- The formulas we have derived and those in the table at the back of the book are all still valid when the complex functions are defined appropriately (as we did for e^{a+bi}), with one important exception: all absolute values must be removed from logarithms. For example, $\int \frac{dx}{x+i} = \ln(x+i) + C$ (and *not* $\ln|x+i| + C$).[2]

As an example,

$$\int_0^2 \left[(1+2i)\,x + 3ix^2\right]\,dx = \int_0^2 x\,dx + 2i\int_0^2 x\,dx + 3i\int_0^2 x^2\,dx = \frac{(1+2i)x^2}{2} + ix^3\Big]_0^2$$
$$= (1+2i)2 + 8i \;=\; 2+12i.$$

3.1 Integrating Products of Sines, Cosines and Exponentials

In Section 7.3 products of sines and cosines were integrated using trigonometric identities. There are other ways to do this now that we have complex exponentials. Examples will make this clearer.

Example 3.1. Let's integrate $8\cos(3x)\sin(x)$.

$$8\cos(3x)\sin(x) = 8\left(\frac{e^{3ix}+e^{-3ix}}{2}\right)\left(\frac{e^{ix}-e^{-ix}}{2i}\right)$$
$$= \frac{2}{i}\left(e^{4ix}+e^{-2ix}-e^{2ix}-e^{-4ix}\right).$$

[2] You can see that $\ln|x+i|$ will not work because it is always a real number so it's derivative can't be a complex number. So how is the logarithm of a complex number defined? We won't discuss this here, but if you want to think about it and ask your professor or TA, here's a hint: remember that the logarithm is the inverse of the exponential and that $x+iy = re^{i\theta}$.

It is not difficult to integrate this:

$$\begin{aligned}
\int 8\cos(3x)\sin(x)\,dx &= \frac{2}{i}\int e^{4ix} + e^{-2ix} - e^{2ix} - e^{-4ix}\,dx \\
&= \frac{2}{i}\left[\frac{e^{4ix}}{4i} - \frac{e^{-2ix}}{2i} - \frac{e^{2ix}}{2i} + \frac{e^{-4ix}}{4i}\right] + C \\
&= \frac{2}{i}\left[\frac{e^{4ix}}{4i} + \frac{e^{-4ix}}{4i} - \frac{e^{-2ix}}{2i} - \frac{e^{2ix}}{2i}\right] + C \\
&= -\left(\frac{e^{4ix} + e^{-4ix}}{2}\right) + 2\left(\frac{e^{2ix} + e^{-2ix}}{2}\right) + C \\
&= -\cos(4x) + 2\cos(2x) + C
\end{aligned}$$

■

Example 3.2. Let's integrate $e^{2x}\sin(x)$. Problems like this were solved in Section 7.2 by using integration by parts twice. Here is another way. Using the formula for sine and integrating we have

$$\begin{aligned}
\int e^{2x}\sin(x)\,dx &= \frac{1}{2i}\int e^{2x}(e^{ix} - e^{-ix})\,dx \\
&= \frac{1}{2i}\int (e^{(2+i)x} - e^{(2-i)x})\,dx \\
&= \frac{1}{2i}\left(\frac{e^{(2+i)x}}{2+i} - \frac{e^{(2-i)x}}{2-i}\right) + C \\
&= -\frac{e^{2x}}{2}\left(\frac{e^{ix}}{1-2i} + \frac{e^{-ix}}{1+2i}\right) + C \\
&= -\frac{e^{2x}}{2}\left[\frac{(1+2i)e^{ix}}{5} + \frac{(1-2i)e^{-ix}}{5}\right] + C \\
&= -\frac{e^{2x}}{5}\left[\left(\frac{e^{ix} + e^{-ix}}{2}\right) - 4\left(\frac{e^{ix} - e^{-ix}}{2i}\right)\right] + C \\
&= -\frac{e^{2x}}{5}\left[\cos(x) - 4\sin(x)\right]
\end{aligned}$$

■

This method works for integrals of products of sines, cosines and exponentials, and often for quotients of them, (though this requires more advanced methods, such as partial fractions). The advantage of using complex exponentials is that it takes the guess out of computing such integrals. The method, however, could be messier than the one presented in the book, though it is often simpler.

3.2 Exercises

Compute the following integrals using complex exponentials.

1. $\int_{-\pi}^{\pi} 7\sin(5x)\cos(3x)\,dx$
2. $\int e^{i7x}\cos(2x)\,dx$
3. $\int \cos^2(x)\;e^{-3x}\,dx$
4. $\int \cos^3(x)\cos(7x)\,dx$
5. $\int \sin^2(x)e^{\sqrt{5}x}\,dx$
6. $\int x\cos^3(x)\,dx$
7. $\int \sin^3(x)\cos(10x)\,dx$

4 The Fundamental Theorem of Algebra

This supplements Section 7.6 of Rogawski Ed. 1.

A polynomial p of degree n is a function of the form

$$Q(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n. \tag{4.1}$$

where the coefficients a_j can be either real or complex numbers. The following is a basic fact which is hard to prove (and we shall not attempt a proof here).

Fundamental Theorem of Algebra: Any nonconstant polynomial can be factored as a complex constant times a product of linear factors of the form $x - \beta$, where β is a complex number, that is,

$$Q(x) = c\,(x - \beta_1)^{n_1}\,(x - \beta_2)^{n_2} \cdots (x - \beta_k)^{n_k}, \tag{4.2}$$

where the β_j are k distinct complex numbers.

This tells us that we can factor a polynomial of degree n into a product of n linear factors. For example,

- $3x^2 + 2x - 1 = 3(x - \frac{1}{3})(x + 1)$ ($n = 2$ here),
- $x^3 - 8 = (x - 2)(x + \alpha)(x + \overline{\alpha})$ where $\alpha = 1 \pm i\sqrt{3}$ ($n = 3$ here),
- $(x^2 + 1)^2 = (x + i)^2 (x - i)^2$ ($n = 4$ here).

4.1 Zeroes and their multiplicity

Notice that a very important feature of the factorization is:

> *Each factor $x - \beta$ of p corresponds to a number β which is a* ***zero of the polynomial*** *p, that is,*
>
> $$Q(\beta) = 0.$$

To see this, just consider the factorization of p evaluated at β, namely

$$Q(\beta) = c(\beta - \beta_1)(\beta - \beta_2)\ldots(\beta - \beta_n).$$

This is equal to zero if and only if one of the factors is 0; say the *jth* factor is zero, which gives $\beta - \beta_j = 0$. Thus $\beta = \beta_j$ for some j.

For some polynomials a factor $x - \beta_j$ will appear more than once, for example, in

$$Q(x) = 7(x-2)^5(x-3)(x-8)^2$$

the $x-2$ factor appears 5 times, the $x-3$ factor appears once, the $x-8$ factor appears twice. The terminology for this is

2 is a zero of p of multiplicity 5

3 is a zero of p of multiplicity 1

8 is a zero of p of multiplicity 2.

The general form for a factored polynomial is

$$Q(x) = k(x-\beta_1)^{m_1}(x-\beta_2)^{m_2}\cdots(x-\beta_\ell)^{m_\ell} \tag{4.3}$$

where β_j is called **a zero of Q of multiplicity** m_j and k is a constant.

4.2 Real Coefficients

All polynomials which you see in math 20B have real coefficients. So it is useful to give a version of the Fundamental Theorem of Algebra all numbers in the factoring are real.

Fundamental Theorem of Algebra: real factors Any nonconstant polynomial p with real coefficients can be factored as constant times a product of linear factors and quadratic factors all having real coefficients, that is,

$$Q(x) = c(x-r_1)^{m_1}\cdots(x-r_\ell)^{m_\ell}\,(x^2+b_1x+c_1)^{n_1}\cdots(x^2+b_kx+c_k)^{n_k}, \tag{4.4}$$

where the r_j are l distinct real numbers and the $x^2+b_jx+c_j$ are k distinct real polynomials with no real roots.

Later we shall study rational functions $f = \frac{P}{Q}$. The partial fraction expansions in Sec. 7.6 of Rogawski are based on this version of the Fundamental Theorem of Algebra. Thus, if we allow complex numbers, partial fractions can be done with only linear factors. When we only allowed real numbers as coefficients of the factors, we obtained both linear and quadratic factors, as does Rogawski.

Proof. A useful fact is:

If all the coefficients Q_j of the polynomial Q are real numbers, then

$$Q(\beta) = 0 \quad \text{implies} \quad Q(\overline{\beta}) = 0.$$

To see this, think of x as a real number and suppose $(x-\alpha)^k$ is a factor of Q; then, $(x-\overline{\alpha})^k$ is also a factor:

(a) Since $(x-\alpha)^k$ is a factor of $Q(x)$, we have $Q(x) = (x-\alpha)^k r(x)$ for some polynomial $r(x)$.

(b) Taking complex conjugates, $\overline{Q(x)} = (x-\overline{\alpha})^k \overline{r(x)}$.

(c) Since $Q(x)$ has real coefficients, $\overline{Q(x)} = Q(x)$ and so by (b), $(x-\overline{\alpha})^k$ is a factor of $Q(x)$.

This and the Fundamental Theorem of Algebra (with complex factors) implies a polynomial Q with real coefficients has a factorization

$$Q(x) = (x-\beta_1)(x-\overline{\beta}_1)\cdots(x-\beta_k)(x-\overline{\beta}_k)(x-r_1)\cdots(x-r_\ell) \tag{4.5}$$

or, equivalently,

$$Q(x) = (x^2+b_1x+c_1)\cdots(x^2+b_kx+c_k)(x-r_1)\cdots(x-r_\ell) \tag{4.6}$$

where $b_1,\ldots,b_k$ and $c_1,\cdots,c_k$ and $r_1,\ldots,r_\ell$ are real numbers. In fact, you can check that $b_j = 2\operatorname{Re}\beta_j$ and $c_j = |\beta_j|^2$. ■

The advantage of the first version of the Fundamental Theorem Algebra is that all terms in the factorization are linear in x; the disadvantage is that some of them may contain numbers β_j which are not real. The advantage of the second version of the Fundamental Theorem Algebra is that all numbers in the factorization are real.

4.3 Rational Functions and Poles

The quotient of two polynomials $\frac{P}{Q}$ is called a **rational function**. For a rational function f we call any point β for which $\lim_{x\to\beta}|f(x)| = \infty$ a **pole** of f. For example,

$$f(x) = \frac{x^7}{(x-1)^2(x-9)^3}$$

has poles at 1, 9 and ∞. You might think calling ∞ a pole peculiar, but $\lim_{x\to\infty}|f(x)| = \infty$, as the definition requires. This should *not* be interpreted as saying that ∞ is a number (notice that we used the word "point" in the definition of a pole: a justification for calling ∞ a "point" is made in in a course on complex analysis). Poles have multiplicity; in this case

$$\begin{aligned} &1 \ \text{ is a pole of } p \text{ of multiplicity } 2\\ &9 \ \text{ is a pole of } p \text{ of multiplicity } 3\\ &\infty \ \text{ is a pole of } p \text{ of multiplicity } 2, \text{ since } f(x)\sim x^2 \text{ as } x\to\infty. \end{aligned}$$

A rational function is called **proper** if $\lim_{x\to\infty} |f(x)| = 0$. This occurs when the degree of the polynomial in the denominator exceeds the degree of the polynomial in the numerator of the rational function. For example, $f(x) = \dfrac{x+1}{x^2+x+1}$ is a proper rational function; whereas $g(x) = \dfrac{x^2+3x+5}{x^2+x+1}$ is not proper.

The growth rate of f near a high multiplicity pole exceeds that of f near a low multiplicity pole. For example, $\lim_{x\to 1} \left|\dfrac{1}{x-1}\right|$ and $\lim_{x\to 1} \left|\dfrac{1}{(x-1)^4}\right|$ are both infinite; however, $\lim_{x\to 1} \left|\dfrac{\frac{1}{x-1}}{\frac{1}{(x-1)^4}}\right| = \lim_{x\to 1} |x-1|^3 = 0$.

4.4 Exercises

1. Expand $Q(x) = (x-2)(x-3)(x^2+1)$ in the form (4.1).

2. Show that if P is a polynomial and $P(5) = 0$, then $\frac{P(x)}{x-5}$ is a polynomial.

3. (a) How many poles does the rational function $r(x) = \frac{3}{5x^3+x+6}$ have? Does it have a "pole at ∞"?

   (b) What are the pole locations and their multiplicities for $r(x) = \frac{3-2x}{(x-2)(x^2+5x+7)}$?

4. The following is the simplest mathematical model used for a building hit by an earthquake. If the bottom of the building is displaced horizontally from rest a distance $b(t)$ at time t, then the roof of the building is displaced from vertical by a distance $r(t)$. The problem is how to describe the relationship between b and r in a simple way. Fortunately, there is a rational function $\mathcal{T}(s)$ called the **transfer function** of the building with the property that when b is a pure sine wave

   $$b(t) = \sin(wt)$$

   at frequency $\frac{w}{2\pi}$, then r is a sine wave of *the same frequency*[3] and with amplitude $|\mathcal{T}(iw)|$.

   While earthquakes are not pure sine waves, they can be modeled by combinations of sine waves.

   If

   $$\mathcal{T}(s) = \frac{s^2}{(s+3i+.01)(s-3i+.01)(s+7i+.1)(s-7i+.1)},$$

   then at approximately what frequency does the building shake the most? At approximately what frequency does the building shake the *second* most?

5. Electric circuits behave similarly and are typically described by their transfer function $\mathcal{T}$. If $c(t)$, a sinusoidal current of frequency $w/2\pi$ is imposed, and $v(t)$ is the voltage one measures it is a sine wave of the same frequency with amplitude $|\mathcal{T}(iw)|$. If

   $$\mathcal{T}(s) = \frac{1}{(s+3i+.01)(s-3i+.01)} + \frac{2}{s-10},$$

   then approximately how much accuracy do we lose in predicting the amplitude for our output with the simpler mathematical model

   $$\widetilde{\mathcal{T}}(s) = \frac{1}{(s+3i+.01)(s-3i+.01)}$$

   When a sine wave at frequency $\frac{w}{2\pi}$ is put in?

   Hint: You may use the fact that $\left||\widetilde{\mathcal{T}}(s)| - |\mathcal{T}(s)|\right| \leq |\widetilde{\mathcal{T}}(s) - \mathcal{T}(s)|$, even though we have not proved it.

[3] r has the form $r(t) = |\mathcal{T}(iw)|\sin(wt + \psi(iw))$

5 Partial Fraction Expansions (PFE)

This supplements Section 7.6 of Rogawski Ed. 1.

The method we use for computing the required constants in the partial fraction expansion (PFE) of a rational function is similar to the one found in the text; we expand that discussion here. Later, in Section 5.4, we discuss the form of a partial fraction expansion. This should help you determine the appropriate form of the PFE in the problems you come across.

5.1 A Shortcut when there are no Repeated Factors

Finding the PFE is easiest when there are no repeated factors in the denominator. In what follows, we discuss the general principle behind the PFE and then exhibit some specific examples.

Theorem 5.1. *Suppose that the n numbers $\alpha_1, \ldots, \alpha_n$ are pairwise distinct and that $P(x)$ is a polynomial with degree less than n. Then, there are constants $C_1, \ldots, C_n$ such that*

$$\frac{P(x)}{(x-\alpha_1)\cdots(x-\alpha_n)} = \frac{C_1}{x-\alpha_1} + \cdots + \frac{C_n}{x-\alpha_n}. \tag{5.1}$$

■

To determine the constants $C_1, \ldots, C_n$; we carry out the following steps:

- *Multiply both sides of (5.1) by $x-\alpha_j$ and then set $x = -\alpha_j$. The left side will evaluate to a number $\mathcal{N}$.*
- *The right side evaluates to C_j, since the other terms have a factor of $x - \alpha_j$ which is 0 when $x = \alpha_j$.*
- *We conclude that $\mathcal{N} = C_j$.*

Now for some illustrations.

Example 5.2. (Proper rational functions with distinct linear factors)

Let's expand $f(x) := \frac{x^2+2}{(x-1)(x+2)(x+3)}$ by partial fractions. By Theorem 5.1,

$$f(x) = \frac{x^2+2}{(x-1)(x+2)(x+3)} = \frac{C_1}{x-1} + \frac{C_2}{x+2} + \frac{C_3}{x+3}$$

Multiply by $x-1$ to eliminate the pole at $x=1$ and get

$$(x-1)f(x) = \frac{x^2+2}{(x+2)(x+3)} = C_1 + \frac{C_2(x-1)}{x+2} + \frac{C_3(x-1)}{x+3}.$$

Set $x = 1$ and obtain

$$\frac{1+2}{(1+2)(1+3)} = C_1.$$

and so $C_1 = \frac{1}{4}$. Similarly,

$$C_2 = (x+2)f(x)\Big]_{x=-2} = \frac{x^2+2}{(x-1)(x+3)}\Big]_{x=-2} = \frac{4+2}{(-3)1} = -2.$$

and

$$C_3 = (x+3)f(x)\Big]_{x=-3} = \frac{x^2+2}{(x-1)(x+2)}\Big]_{x=-3} = \frac{9+2}{(-4)(-1)} = \frac{11}{4}.$$

We conclude that

$$f(x) = \frac{x^2+2}{(x-1)(x+2)(x+3)} = \frac{1}{4(x-1)} - \frac{2}{(x+2)} + \frac{11}{4(x+3)}.$$

A cultural aside is that the numbers C_1, C_2, C_3 are often (though not in Rogawski) called the *residues* of the poles at $1, -2, -3$; many of you will see them later in your career under that name.

If we wish to find the antiderivatives of f from this, we immediately get

$$\int f(x)\,dx = \frac{1}{4}\ln|x-1| + 2\,\ln|x+2| + \frac{11}{4}\ln|x+3| + K$$

■

5.2 The Difficulty with Repeated Factors

Let us apply the previous method to

$$f(x) = \frac{1}{(x-1)^2(x-3)}$$

whose partial fraction expansion we know (by Rogawski's book) has the form

$$f(x) = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{x-3}. \tag{5.2}$$

We can find C quickly from

$$C = (x-3)f(x)\Big]_{x=3} = \frac{1}{(3-1)^2} = \frac{1}{4}$$

and A from

$$A = (x-1)^2 f(x)\Big]_{x=1} = \frac{1}{1-3} = -\frac{1}{2}.$$

However, B does not succumb to this technique; we must use other means to find it. What we have gotten from our method are just the coefficients of the "highest terms" at each pole.

There are several ways to find B; in fact, Rogawski shows two methods. One of these is to plug in a convenient value of x, say $x = 0$ and obtain

$$f(0) = \frac{1}{(-1)^2(-3)} = -\frac{1}{2} - B - \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)$$
$$B = \frac{1}{3} - \frac{1}{2} - \frac{1}{12} = -\frac{1}{4}.$$

To summarize

$$f(x) = -\frac{1}{2}\frac{1}{(x-1)^2} - \frac{1}{4}\frac{1}{(x-1)} + \frac{1}{4}\frac{1}{(x-3)}.$$

The antiderivative of f is

$$\int f(x)\,dx = \frac{1}{2}\frac{1}{(x-1)} - \frac{1}{4}\ln|x-1| + \frac{1}{4}\ln|x-3| + K.$$

5.3 Every Rational Function has a Partial Fraction Expansion

Now we mention a pleasant fact.

Theorem 5.3. *Every rational function $f = \frac{P}{Q}$ has a partial fraction expansion.*

The core of the reason is the Fundamental Theorem of Algebra, which can be used to factor Q as in formula (4.3). This produces

$$f(x) = \frac{P(x)}{(x-\beta_1)^{m_1}(x-\beta_2)^{m_2}(x-\beta_\ell)^{m_\ell}}.$$

If the numerator and denominator polynomials defining f have real coefficients, then f can always be written

$$f(x) = \frac{P(x)}{(x-r_1)^{m_1}\cdots(x-r_\ell)^{m_\ell}\ (x^2+b_1x+c_1)^{n_1}\cdots(x^2+b_kx+c_k)^{n_1}}$$

with all the coefficients in the factors real numbers. This is the factoring behind the various cases treated in Rogawski Section 7.6 Ed 1. One then needs to write out the appropriate form for the PFE and then identify the coefficients as has been explained in Rogawski Ed.1 Section 7.6 and in these notes for cases where all factors are linear (even with high multiplicity) and where there is a multiplicity one quadratic factor.

5.4 The Form of the Partial Fraction Expansion

We have seen that if a proper rational function has the form $f(x) = \frac{P(x)}{(x-a)^k Q(x)}$, then the PFE has the form

$$f(x) = \frac{C_1}{(x-a)} + \frac{C_2}{(x-a)^2} + \cdots + \frac{C_k}{(x-a)^k} + g(x),$$

where $g(x) = \frac{r(x)}{Q(x)}$ does not depend on $(x-a)$. It is natural to ask why the PFE has this particular form: why, in general, can't the PFE be written in the form

$$f(x) = \frac{C_k}{(x-a)^k} + g(x)?$$

Specifically, why do we need to include the lower order terms (lower multiplicity poles)? Here is one way to view the form of the PFE. Recall that a high multiplicity pole has a "faster growth rate" than a lower multiplicity pole: it can "overshadow" the lower multiplicity pole.

Example 5.4. The function $f(x) = \frac{1}{(x-1)^2(x-3)}$ has a multiplicity 2 pole at 1 and a multiplicity 1 pole at 3. Thus the PFE has the form

$$f(x) = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{x-3}.$$

The role of the multiplicity 1 pole at 3 is obvious. Let us turn to the multiplicity 2 pole at 1. Its coefficient $A = \left.\frac{1}{(x-3)}\right]_{x=1} = -\frac{1}{2}$ Thus, the "strength" of the multiplicity 2 pole at 1 is $-\frac{1}{2}\frac{1}{(x-1)^2}$. However, when we subtract this pole from f, we obtain

$$\begin{aligned} e(x) &= f(x) - \left[-\frac{1}{2}\frac{1}{(x-1)^2}\right] \\ &= \frac{1}{(x-1)^2}\left[\frac{1}{(x-3)} + \frac{1}{2}\right] \\ &= \frac{1}{(x-1)^2}\left[\frac{1}{2}\frac{(x-1)}{(x-3)}\right] \\ &= \frac{1}{2}\frac{1}{(x-1)(x-3)} \end{aligned}$$

which still has a pole at 1, though now it is a pole of multiplicity 1. This illustrates why both $\frac{A}{(x-1)^2}$ and $\frac{B}{(x-1)}$ must be included in the PFE and what we meant earlier by saying that a high multiplicity pole can "overshadow" a low multiplicity pole. ■

Similar intuition tells us that $f(x) = \frac{x^7}{(x-1)^2(x-9)^3}$ has a PFE of the form

$$f(x) = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{(x-9)^3} + \frac{D}{(x-9)^2} + \frac{E}{x-9} + Fx^2 + Gx + H,$$

since it has poles at 1 of multiplicity 2, at 9 of multiplicity 3 and at ∞ of multiplicity 2.

5.5 More Examples: Non-proper Rational Functions and Quadratic Factors

Example 5.5. (Rational functions with a pole at infinity and all linear factors with none repeated.) Let's expand $f(x) = \frac{x^3+2}{(x-1)(x+2)}$ by partial fractions. Clearly, f has a pole at 1, -2 and ∞, each with multiplicity one. The form of the PFE is

$$f(x) = \frac{x^3+2}{(x-1)(x+2)} = \frac{C_1}{x-1} + \frac{C_2}{x+2} + Ax + B \tag{5.3}$$

where $Ax + B$ is included to represent the pole at ∞. Indeed, $Ax + B$ is the simplest form of a rational function with a general multiplicity one pole at infinity. Beware: you must include the constant term B.

Now solve for A, B, C_1 and C_2: multiply by $x - 1$ to eliminate the pole at $x = 1$ and obtain

$$(x-1)f(x)\Big]_{x=1} = \frac{x^3+2}{(x+2)}\Big]_{x=1} = C_1.$$

That is,

$$C_1 = \frac{1+2}{(1+2)} = 1$$

Similarly,

$$C_2 = (x+2)f(x)\Big]_{x=-2} = \frac{x^3+2}{(x-1)}\Big]_{x=-2} = \frac{-8+2}{(-3)} = 2.$$

Finding A is easy since it is the "highest order term" at infinity. First observe that

$$\lim_{x\to\infty} \frac{f(x)}{x} = A.$$

Then,

$$A = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^2 + 2/x}{(x-1)(x+2)} = \lim_{x\to\infty} \frac{x^2}{x^2} = 1$$

Now we must only find B. As usual there are several ways to do this. For example, plug $x = 0$ into (5.3) and obtain

$$\frac{2}{(-1)(2)} = f(0) = \frac{C_1}{-1} + \frac{C_2}{2} + B = \frac{1}{-1} + \frac{2}{2} + B = B$$

Thus, $B = -1$. ■

Example 5.6. (Proper rational functions with an irreducible quadratic factor but no repeated factors) Let's find the PFE for $\frac{x+1}{x^3+x}$. Note that $f(x) = \frac{x+1}{x^3+x}$ has two natural forms of partial fraction expansions corresponding to whether we factor the denominator $x^3 + x$ in the form (4.6) with real coefficients or (4.5) with complex coefficients. Rogawski Sec. 7.6 Ed. 1 uses (4.6) so we emphasize and recommend that one, namely

$$f(x) = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}.$$

We proceed like Rogawski and obtain

$$A = xf(x)\Big]_{x=0} = \frac{1}{1} = 1.$$

Next, multiply by $x(x^2+1)$ to get $x+1 = x(x^2+1)f(x) = x^2+1+x(Bx+C)$. Cancel ones and divide by x, to get

$$1 = x + Bx + C.$$

Set $x = 0$ to get $C = 1$ and so $B = -1$. Thus, the PFE is

$$f(x) = \frac{x+1}{x^3+x} = \frac{1}{x} + \frac{-x+1}{x^2+1}.$$

If we want antiderivatives, this gives

$$\int f(x)\,dx = \ln|x| - \frac{1}{2}\ln|x^2+1| + \arctan(x) + K$$

This solves the problem completely.

Although using the (4.6) form of expansion suffices to solve the problem, for the sake of completeness (and the curious), we show how to use the (4.5) form of expansion.

$$f(x) = \frac{x+1}{(x^3+x)} = \frac{x+1}{x(x-i)(x+i)} = \frac{C_1}{x} + \frac{C_2}{x-i} + \frac{C_3}{x+i}.$$

Since $x = x - 0$,

$$xf(x)\Big]_{x=0} = C_1 \; = \; \frac{1}{(-i)i} = 1.$$

Also

$$(x-i)f(x)\Big]_{x=i} = C_2 = \frac{i+1}{i(2i)} = \frac{-1-i}{2} \quad \text{and} \quad (x+i)f(x)\Big]_{x=-i} = C_3 = \frac{-i+1}{(-i)(-2i)} = \frac{-1+i}{2}.$$

Note that $C_3 = \bar{C}_2$ and we can get the first PFE from this PFE by

$$f(x) = \frac{1}{x} + \frac{C_2}{x-i} + \frac{C_3}{x+i} = \frac{1}{x} + \frac{C_2(x+i)+C_3(x-i)}{x^2+1} = \frac{1}{x} + \frac{2\mathrm{Re}C_2 x + (-2)\mathrm{Im}C_2}{x^2+1}$$

$$f(x) = \frac{1}{x} + \frac{-x+1}{x^2+1}.$$

which is what we got before. ■

We will not consider higher multiplicity quadratic factors here.

5.6 Exercises

1. Find the partial fraction expansion of $\frac{2x+1}{(x-1)^2(x+2)}$.
2. Given $f(x) = \frac{3}{(x-1)(x-2)^2}$. What value of A makes $f(x) - \frac{A}{x-1}$ have its only pole located at 2?
3. Find the partial fraction expansion of $\frac{x^3+2}{x(x^2+1)(x^2+4)}$.
4. Find the partial fraction expansion of $\frac{x^3+2}{x(x^2-1)(x^2-4)}$.
5. Consider the PFE of r in (5.2). We claim that

$$\frac{d}{dx}\left[(x-1)^2 f(x)\right]\Big]_{x=1}$$

is either A B, or C in the partial fraction expansion.

   (a) Which one is it?

   (b) Does such a formula hold for any rational function with a second order pole? Justify your answer.

   (c) Find a similar formula for a rational function with a third order pole.

6 Improving on Euler's Method

This supplements Section 9.3 Rogawski Edition 1.

Suppose we are given the differential equation $y' = F(x, y)$ with initial condition $y(x_0) = y_0$. Euler's method, discussed in Section 9.2, produces a sequence of approximations $y_1, y_2, \dots$ to $y(x_1), y(x_2), \dots$ where $x_n = x_0 + nh$ are equally spaced points.

This is almost the left endpoint approximation in numerical integration (Chapter 7 of Rogawski Ed. 1). To see this, suppose that we have an approximation y_{n-1} for $y(x_{n-1})$, and that we want an approximation for $y(x_n)$. Integrate $y' = F(x, y)$ from x_{n-1} to x_n and use the left endpoint approximation:

$$y(x_n) - y(x_{n-1}) = \int_{x_{n-1}}^{x_n} F(x, y)\, dx \approx hF(x_{n-1}, y(x_{n-1})).$$

Now we have a problem that did not arise in numerical integration: We don't know $y(x_{n-1})$. What can we do? We replace $y(x_{n-1})$ with the approximation y_{n-1} to obtain

$$y(x_n) - y_{n-1} \approx hF(x_{n-1}, y_{n-1}).$$

Rearranging and calling the approximation to $y(x_n)$ thus obtained y_n we have Euler's method:

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}). \tag{6.1}$$

We know that the left endpoint approximation is a poor way to estimate integrals and that the Trapezoidal Rule is better. Can we use it here? Adapting the argument that led to (6.1) for use with the Trapezoidal Rule gives us

$$y_n = y_{n-1} + \frac{h}{2}\Big(F(x_{n-1}, y_{n-1}) + F(x_n, y_n)\Big). \tag{6.2}$$

You should carry out the steps. Unfortunately, (6.2) can't be used: We need y_n on the right side in order to compute it on the left!

Here is a way around this problem: First, use (6.1) to estimate ("predict") the value of y_n and call this prediction y_n^*. Second, use y_n^* in place of y_n in the right side of (6.2) to obtain a better estimate, called the "correction". The formulas are

$$\begin{aligned} &\text{(predictor)} \quad y_n^* = y_{n-1} + hF(x_{n-1}, y_{n-1}) \\ &\text{(corrector)} \quad y_n = y_{n-1} + \frac{h}{2}\Big(F(x_{n-1}, y_{n-1}) + F(x_n, y_n^*)\Big). \end{aligned} \tag{6.3}$$

This is an example of a *predictor-corrector* method for differential equations. Here are results for Example 9.2.3, the differential equation $y' = x + y$ with initial condition $y(0) = 1$:

step size	$y(1)$ by (6.1)	$y(1)$ by (6.3)
0.50	2.500000	3.281250
0.20	2.976640	3.405416
0.10	3.187485	3.428162
0.05	3.306595	3.434382
0.02	3.383176	3.436207
0.01	3.409628	3.436474

The correct value is 3.436564, so (6.3) is much better than Euler's method for this problem.

6.1 Exercises

1. Write down a predictor-corrector method based on Simpson's Rule for numerical integration. Hint: a bit tricky is that we consider not two, but three grid points x_{n-2}, x_{n-1}, x_n and assume we know f_{n-2} and f_{n-1}. The problem for you is to give an algorithm for producing f_n.

7 Appendix: Differentiation of Complex Functions

Suppose we have a function $f(z)$ whose values are complex numbers and whose variable z may also be a complex number. We can define limits and derivatives as Rogawski did for real numbers. Just as for real numbers, we say the complex numbers z and w are "close" if $|z - w|$ is small, where $|z - w|$ is the absolute value of a complex number.[4]

- We say that $\lim_{z \to \alpha} f(z) = L$ if, for every real number $\epsilon > 0$ there is a corresponding real number $\delta > 0$ such that

$$|f(z) - L| < \epsilon \qquad \text{whenever} \qquad 0 < |z - \alpha| < \delta.$$

- The derivative is defined by $f'(\alpha) = \lim_{z \to \alpha} \dfrac{f(z) - f(\alpha)}{z - \alpha}$.

Our variables will usually be real numbers, in which case z and α are real numbers. Nevertheless the value of a function can still be a complex number because our functions contain complex constants; for example, $f(x) = (1 + 2i)x + 3ix^2$.

Since our definitions are the same, the formulas for the derivative of the sum, product, quotient and composition of functions still hold. Of course, before we can begin to calculate the derivative of a particular function, we have to know how to calculate the function.

What functions can we calculate? Of course, we still have all the functions that we studied with real numbers. So far, all we know how to do with complex numbers is basic arithmetic. Thus we can differentiate a function like $f(x) = \dfrac{1 + ix}{x^2 + 2i}$ or a function like $g(x) = \sqrt{1 + i}\, e^x$ since $f(x)$ involves only the basic arithmetic operations and $g(x)$ involves a (complex) constant times a real function, e^x, that we know how to differentiate. On the other hand, we cannot differentiate a function like e^{ix} because we don't even know how to calculate them.

7.1 Deriving the Formula for e^z Using Differentiation

Two questions left dangling in Section 2 were

- How did you come up with the definition of complex exponential?
- How do you know it satisfies the simple differential equation properties?

[4]The definitions are nearly copies of Rogawski Sections 2.2 and 2.8. We have used z and α instead of x and a to emphasize the fact that they are complex numbers and have called attention to the fact that δ and ϵ are real numbers.

We consider each of these in turn.

From the first of formula in (2.1) with $\alpha = a$ and $\beta = b$, e^{a+bi} should equal $e^a\, e^{bi}$. Thus we only need to know how to compute e^{bi} when b is a real number.

Think of b as a variable and write $f(x) = e^{xi} = e^{ix}$. By the second property in (2.1) with $\alpha = i$, we have $f'(x) = if(x)$ and $f''(x) = if'(x) = i^2 f(x) = -f(x)$. It may not seem like we're getting anywhere, but we are!

Look at the equation $f''(x) = -f(x)$. There's not a complex number in sight, so let's forget about them for a moment. Do you know of any real functions $f(x)$ with $f''(x) = -f(x)$? Yes. Two such functions are $\cos(x)$ and $\sin(x)$. In fact,

$$\text{If } f(x) = A\cos(x) + B\sin(x), \text{ then } f''(x) = -f(x).$$

We need constants (probably complex) so that it's reasonable to let $e^{ix} = A\cos(x) + B\sin(x)$. How can we find A and B? When $x = 0$, $e^{ix} = e^0 = 1$. Since

$$A\cos(x) + B\sin(x) = A\cos(0) + B\sin(0) = A,$$

we want $A = 1$. We can get B by looking at $(e^{ix})'$ at $x = 0$. You should check that this gives $B = i$. (Remember that we want the derivative of e^{ix} to equal ie^{ix}.) Thus we get

$$\boxed{\textbf{Euler's formula:}\quad e^{ix} = \cos(x) + i\sin(x)}$$

Putting it all together we finally have our definition for e^{a+bi}.

We still need to verify that our definition for e^z satisfies (2.1). The verification that $e^{\alpha+\beta} = e^\alpha e^\beta$ is left as an exercise. We will prove that $(e^z)' = e^z$ for complex numbers. Then, by the Chain Rule, $(e^{\alpha x})' = (e^{\alpha x})(\alpha x)' = \alpha e^{\alpha x}$, which is what we wanted to prove.

Example 7.1. (A proof that $(e^z)' = e^z$)

By the definition of derivative and the fact that $e^{\alpha+\beta} = e^\alpha e^\beta$ with $\alpha = z$ and $\beta = w$, we have

$$(e^z)' = \lim_{w\to 0} \frac{e^{z+w} - e^z}{w} = \lim_{w\to 0} \frac{e^z(e^w - 1)}{w} = e^z \lim_{w\to 0} \frac{e^w - 1}{w}.$$

Let $w = x + iy$ where x and y are small real numbers. Then, using the definition of complex exponential, we get

$$\frac{e^w - 1}{w} = \frac{e^x(\cos(y) + i\sin(y)) - 1}{x + iy}.$$

Since x and y are small, we can use linear approximations [5] for e^x, $\cos(y)$ and $\sin(y)$, namely $1 + x$, 1 and y. (The approximation $\cos(y) \approx 1$ comes from $(\cos(y))' = 0$ at $y = 0$.) Thus $\frac{(e^w-1)}{w}$ is

[5] Linear approximations are discussed in Section 4.1 of Rogawski.

approximately equal to

$$\frac{(1+x)(1+iy)-1}{x+iy} = \frac{(1+x)+i(1+x)y-1}{x+iy} = \frac{(x+iy)+ixy}{x+iy} = 1 + \frac{ixy}{x+iy}$$

When x and y are very small, their product is much smaller than either one of them. Thus $\lim\limits_{w\to 0} \frac{ixy}{x+iy} = 0$ and so $\lim\limits_{w\to 0} \frac{(e^w-1)}{w} = 1$. This shows that $(e^z)' = e^z$. ■