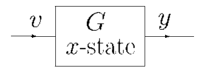
HOW NONCOMMUTING ALGEBRA ARISES IN SYSTEMS THEORY

Bill at UC San Diego

helton@ucsd.edu



$$\frac{dx(t)}{dt} = Ax(t) + Bv(t)$$
$$y(t) = Cx(t) + Dv(t)$$
A, B, C, D are matrices
x, v, y are vectors

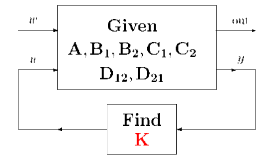
Asymptotically stable || $\mathrm{Re}(\mathrm{eigvals}(A)) \prec 0 \iff$ $A^T\mathbf{E} + \mathbf{E}A \prec 0 \quad \mathbf{E} \succ 0$

Energy dissipating $G : L^2 \to L^2$, $\int_0^T |v|^2 dt \geq \int_0^T |Gv|^2 dt$, $x(0) = 0$ || $\exists \quad \mathbf{E} = \mathbf{E}^T \succeq 0$
$$H := A^T\mathbf{E} + \mathbf{E}A + \mathbf{E}BB^T\mathbf{E} + C^TC = 0$$
$\mathbf{E}$ is called a storage function

Two minimal systems $[A, B, C, D]$ and $[a, b, c, d]$ with the same input to output map. || $\exists$ $\mathbf{M}$ invertible, so that
$$\mathbf{M}A\mathbf{M}^{-1} = a$$
$$\mathbf{M}B = b$$
$$C\mathbf{M}^{-1} = c$$

Every state is reachable from $x = 0$ || $(B\ AB\ A^2B \cdots) : \ell^2 \to X$ is onto

H^∞ Control Problem



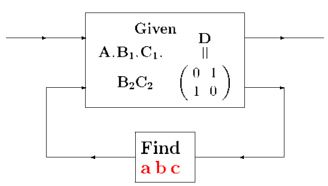
$$\frac{dx}{dt} = Ax + B_1w + B_2u$$
$$\text{out} = C_1x + D_{12}u + D_{11}w$$
$$y = C_2x + D_{21}w$$

$D_{21} = I \qquad D_{12}D_{12}' = I \qquad D_{12}'D_{12} = I \qquad D_{11} = 0$

PROBLEM: Find a control law $\mathbf{K} : y \to u$ which makes the system dissipative over every finite horizon:
$$\int_0^T |out(t)|^2 dt \leq \int_0^T |w(t)|^2 dt$$
The unknown $\mathbf{K}$ is the system
$$\frac{d\xi}{dt} = \mathbf{a}\xi + \mathbf{b} \qquad u = \mathbf{c}\xi$$
So $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the critical unknowns.

CONVERSION TO ALGEBRA

Engineering Problem: Make a given system dissipative by designing a feedback law.



DYNAMICS of "closed loop" system: BLOCK matrices

$$\mathcal{A} \quad \mathcal{B} \quad \mathcal{C} \quad \mathcal{D}$$

ENERGY DISSIPATION:
$$H := \mathcal{A}^T\mathbf{E} + \mathbf{E}\mathcal{A} + \mathbf{E}\mathcal{B}\mathcal{B}^T\mathbf{E} + \mathcal{C}^T\mathcal{C} = 0$$

$$\mathbf{E} = \begin{pmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{pmatrix} \qquad \mathbf{E}_{12} = \mathbf{E}_{21}{}^T$$

$$H = \begin{pmatrix} H_{xx} & H_{xy} \\ H_{yx} & E_{yy} \end{pmatrix} \qquad H_{xy} = H_{yx}^T$$

H^∞ Control Problem

ALGEBRA PROBLEM:

Given the polynomials:

$$H_{xx} = \mathbf{E}_{11}A + A^T\mathbf{E}_{11} + C_1^TC_1 + \mathbf{E}_{12}{}^T\mathbf{b}C_2 + C_2^T\mathbf{b}^T\mathbf{E}_{12}{}^T + \mathbf{E}_{11}B_1\mathbf{b}^T\mathbf{E}_{12}{}^T + \mathbf{E}_{11}B_1B_1^T\mathbf{E}_{11} + \mathbf{E}_{12}\mathbf{b}\mathbf{b}^T\mathbf{E}_{12}{}^T + \mathbf{E}_{12}\mathbf{b}B_1^T\mathbf{E}_{11}$$

$$H_{xz} = \mathbf{E}_{21}A + \frac{a^T(\mathbf{E}_{21} - \mathbf{E}_{12}{}^T)}{2} + \mathbf{c}^TC_1 + \mathbf{E}_{22}\mathbf{b}C_2 + \mathbf{c}^TB_2^T\mathbf{E}_{11}{}^T + \frac{\mathbf{E}_{21}B_1\mathbf{b}^T(\mathbf{E}_{21} + \mathbf{E}_{12}{}^T)}{2} + \mathbf{E}_{21}B_1B_1^T\mathbf{E}_{11}{}^T + \frac{\mathbf{E}_{22}\mathbf{b}\mathbf{b}^T(\mathbf{E}_{21} + \mathbf{E}_{12}{}^T)}{2} + \mathbf{E}_{22}\mathbf{b}B_1^T\mathbf{E}_{11}{}^T$$

$$H_{zx} = A^T\mathbf{E}_{21}{}^T + C_1^T\mathbf{c} + \frac{(\mathbf{E}_{12} + \mathbf{E}_{21}{}^T)\mathbf{a}}{2} + \mathbf{E}_{11}B_2\mathbf{c} + C_2^T\mathbf{b}^T\mathbf{E}_{22}{}^T + \mathbf{E}_{11}B_1\mathbf{b}^T\mathbf{E}_{22}{}^T + \mathbf{E}_{11}B_1B_1^T\mathbf{E}_{21}{}^T + \frac{(\mathbf{E}_{12} + \mathbf{E}_{21}{}^T)\mathbf{b}\mathbf{b}^T\mathbf{E}_{22}{}^T}{2} + \frac{(\mathbf{E}_{12} + \mathbf{E}_{21}{}^T)\mathbf{b}B_1^T\mathbf{E}_{21}{}^T}{2}$$

$$H_{zz} = \mathbf{E}_{22}\mathbf{a} + \mathbf{a}^T\mathbf{E}_{22}{}^T + \mathbf{c}^T\mathbf{c} + \mathbf{E}_{21}B_2\mathbf{c} + \mathbf{c}^TB_2^T\mathbf{E}_{21}{}^T + \mathbf{E}_{21}B_1\mathbf{b}^T\mathbf{E}_{22}{}^T + \mathbf{E}_{21}B_1B_1^T\mathbf{E}_{21}{}^T + \mathbf{E}_{22}\mathbf{b}\mathbf{b}^T\mathbf{E}_{22}{}^T + \mathbf{E}_{22}\mathbf{b}B_1^T\mathbf{E}_{21}{}^T$$

(HGRAIL) A, B_1, B_2, C_1, C_2 are knowns.

Solve the inequality $\begin{pmatrix} H_{xx} & H_{xz} \\ H_{zx} & H_{zz} \end{pmatrix} \succeq 0$ for unknowns

$\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and for $\mathbf{E}_{11}$, $\mathbf{E}_{12}$, $\mathbf{E}_{21}$ and $\mathbf{E}_{22}$

When can they be solved?

If these equations can be solved, find formulas for the solution.

TEXTBOOK SOLUTION TO THE H^∞ PROB

DGKF = Doyle-Glover Kargonekar - Francis 1989 ish

KEY Riccatis

$$DGKF_X := (A - B_2C_1)'\mathbf{X} + \mathbf{X}(A - B_2C_1) + \mathbf{X}(\gamma^{-2}B_1B_1' - B_2^{-1}B_2')\mathbf{X}$$

$$DGKF_Y := A^\times\mathbf{Y} + \mathbf{Y}A^{\times\prime} + \mathbf{Y}(\gamma^{-2}C_1'C_1 - C_2'C_2)\mathbf{Y}$$

here $A^\times := A - B_1C_2$.

THM DGKF There is a system $\mathbf{K}$ solving the control problem if there exist solutions

$$\mathbf{X} \succeq 0 \quad \textit{and} \quad \mathbf{Y} \succ 0$$

to inequalities the

$$\mathbf{DGKF_Y \preceq 0} \quad \text{and} \quad \mathbf{DGKF_X \preceq 0}$$

which satisfy the coupling condition

$$\mathbf{X} - \mathbf{Y}^{-1} \prec 0.$$

This is iff provided $\mathbf{Y} \succeq 0$ and $\mathbf{Y}^{-1}$ is interpreted correctly.

BIG TECHNIQUE IN LIN SYS

MATRIX INEQUALITIES

Riccati Inequalities

$$A_1'\mathbf{X} + \mathbf{X}A_1 + \mathbf{X}G_1^T G_1 \mathbf{X} + R_1 \preceq 0$$
$$A_2'\mathbf{X} + \mathbf{X}A_2 + \mathbf{X}G_2^T G_2 \mathbf{X} + R_2 \preceq 0$$
$$\mathbf{X} \succeq 0$$

These are "matrix convex" in the unknown $\mathbf{X}$. If such an $\mathbf{X}$ exists, then can simultaneously control or stablize several systems.

Riccati Inequalities

$$A_1'\mathbf{X} + \mathbf{X}A_1 + \mathbf{X}G_1^T G_1 \mathbf{X} + R_1 \preceq 0$$
$$A_2'\mathbf{X} + \mathbf{X}A_2 + \mathbf{X}G_2^T G_2 \mathbf{X} + R_2 \preceq 0$$
$$\mathbf{X} \succeq 0$$

Equivalent to Linear matrix inequality. LMI.

$$\begin{pmatrix} -[A_1'\mathbf{X} + \mathbf{X}A_1 + R_1] & G_1\mathbf{X} \\ \mathbf{X}G_1^T & -I \end{pmatrix} \succ 0$$
$$\begin{pmatrix} -[A_1'\mathbf{X} + \mathbf{X}A_1 + R_1] & G_2\mathbf{X} \\ \mathbf{X}G_2^T & -I \end{pmatrix} \succ 0$$
$$\mathbf{X} \succ 0$$

Numerical Solution Can solve convex (especially linear) matrix inequalities numerically with $\mathbf{X}$ smaller than 50×50 matrices using interior point optimization methods - called **semidefinite programming**.

Main Algebra Problem in Linear Systems Engineering
"Convert" your engineering problem to a set of equivalent'linear matrix inequalities, if possible. Is it possible? .

More Flexible Goal
Converting your engineering problem to a set of equivalent "convex" matrix inequalities will do fine in practice. .

QUESTION: How much more general are CONVEX MIs, than LINEAR MIs?