On Certainty Equivalence Measurement Feedback Power Gain Control of Nonlinear Systems

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Abstract

Substantial classes of systems (eg. ones with dead zone nonlinearities (like gears) or hysteresis) have the property that controlling them to an equilibrium point O is impractical, but that controlling into some modest size R region about O is natural (eg. arguably compressor surge and stall). That is, a realistic goal is to design a closed loop system whose trajectories oscillate around O, but do not converge to it.

H^∞ control is unsuited to such problems since achieveing a prescribed ENERGY GAIN is so demanding as to force the closed loop sysem to be asymptotically stable. This paper presents a theory for the design of measurement feedback controllers which achieve a prescribed POWER GAIN for the closed loop system. The theory seems at least as implementable as nonlinear H^∞ control and maybe more so because the specifications are considerably less demanding. This weaker power gain design criteria appears to be very natural for designing systems with limit cycles or oscillatory behavior. Such issues were discussed for state feedback power gain analysis and control in [2, 3] and the results here are based on that work and upon broadening the notion of "information state" or "cost to come" in existing H^∞ theory [5]. While our theory assumes certainty equivalence, theoretical work on more general controllers which do not require the certainty equivalence property is currently in progress.

1 Introduction

There are important classes of nonlinear plants where no $\mathcal{H}_\infty$ controller exists for any gain. Certain limit cycle systems and chaotic systems can exhibit this property due to their fundamentally oscillatory behaviour. However, such systems commonly can be controlled to exhibit an attracting set R (which which depends on which controller is used). At the beginning of a design one does not know exactly which regions R are reasonable and what these are should emerge in the course of the design. Under certain technical conditions regarding the dynamics outside this attracting set, such systems can be shown to exhibit *power dissipation*, as discussed in [3, 4]. Conversely, closed loop power dissipation plus heavy assumptions implies that unperturbed trajectories converge to and are eventually contained by a modest size region R. In practice, the assumptions will present an overly pessimistic picture, whereas the actual power dissipating controllers may be very well suited to oscillatory problems.

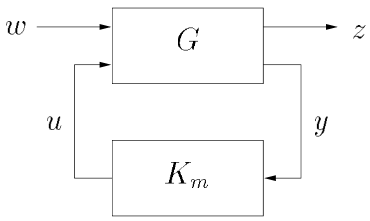

Figure 1: Measurement Feedback Closed Loop (G, K_m)

Now we describe the problem we treat more formally. The basic problem is to find a causal controller K_m for plant G such that the closed loop system (G, K_m) as shown in Figure 1 is dissipative in the sense:

Definition 1.1 *Closed loop system (G, K_m) has* **finite power ($\mathcal{FP}$-) gain** $\leq \gamma$ *with* **available power** $\leq \lambda$ *if there exists a finite nonnegative pair $(\lambda, \beta(x_0))$ such that*

$$\tfrac{1}{2}\int_0^t |z(s)|^2 ds \;\leq\; \tfrac{\gamma^2}{2}\int_0^t |w(s)|^2 ds + \tfrac{\lambda}{2} t + \beta(x_0) \qquad (1)$$

for all $w \in \mathcal{L}_2[0,t]$, all $t \geq 0$, and all $x_0 \in \mathbf{R}^n$, where x_0 is the initial state of the plant G. Here, the parameter λ is called the **power bias**.

A conspicuous special case is when $\lambda = 0$. Then the power dissipation inequality (1) becomes the classical energy dissipation inequality, and we get the conventional H^∞ control problem. When $\lambda \neq 0$, we can divide by $t > 0$ and take an infinite time horizon to obtain that the system shown in Figure 1 has the property

$$(\text{power of } z)^2 \;\;\leq\;\; \gamma^2(\text{power of } w)^2 + \lambda,$$

giving rise to the name power gain.

Observe that if the input w to (G, K_m) is in $\mathcal{L}_{2,loc}$ with power $P(w)$ and the output is periodic with power $P(z)$ (as would be the case if the system trajectory is on is on a limit cycle), then the power bias λ of (G, K_m) is

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no less than $\lambda_{max}(w) := P(z) - \gamma^2 P(w)$. Since this must be true for any input w, the power bias λ must be no less than $\overline{\lambda}_{max} := \sup_{w \in \mathcal{L}_{2,loc}} \{\lambda_{max}(w)\}$. That is, $\overline{\lambda}_{max}$ represents the maximum power generation of (G, K_m), giving rise to the name available power for $\overline{\lambda}_{max}$. Clearly, H^∞ control is unsuited to oscillitory situations since it requires $\overline{\lambda}_{max}$ to be 0. For further details on this and the interpretation of (1), see [3].

In this paper we give a recipe for a CE power gain controller and prove that the corresponding closed loop measurement feedback system is power dissipative. This is done in terms of the *information state* framework of [5, 1]. Conventionally, the information state may be interpreted as the worst case energy retrievable from a plant, given the applied controls and the observed measurements. An elegant tool in our solution adapts this notion to the power gain case by allowing the power generated "internally" by the plant (i.e., the oscillatory behaviour) to be accounted for in the calculations. We find that as for standard $\mathcal{H}_\infty$ control, the CE property in power gain control has the effect of separating the measurement feedback controller into an estimator part and a state feedback controller part.

Finally, CE controller implementation requires the computation of a state feedback controller which is typically based on a power gain storage function and we present new results showing that quadratic power gain storage functions can be computed even though the system is highly nonlinear. Note that this is not possible in nonlinear H^∞ control.

2 Problem Formulation and Assumptions

Throughout the paper, the following class of plants are considered:

$$G : \begin{cases} \dot{x}(t) &= A(x(t)) + B_1(x(t))w(t) + B_2(x(t))u(t), \\ z(t) &= C_1(x(t)) + D_{12}(x(t))u(t), \\ y(t) &= C_2(x(t)) + D_{21}(x(t))w(t), \end{cases} \tag{2}$$

where $x_0 = x(0) \in \mathbf{R}^n$ is the initial state, $x(t)$ is the state at time t, $w(t) \in \mathbf{R}^s$ is the disturbance, $u(t) \in \mathbf{R}^m$ is the control, $y(t) \in \mathbf{R}^p$ is the measurement, and $z(t) \in \mathbf{R}^r$ is the output. For simplicity, we consider the *two-block* case. Standard assumptions for this case are as follows:

(A1) $E_1(x) := D_{12}(x)' D_{12}(x)$ is invertible $\forall\ x \in \mathbf{R}^n$.

(A2) $D_{21}(x) = I\ \forall\ x \in \mathbf{R}^n$.

We make the additional following assumptions (where necessary) regarding the system G.

(A3) A, B_1, B_2, C_1, D_{12}, and C_2 are continous functions.

(A4) $A(x) - B_2(x)E_1(x)^{-1}D_{12}(x)'C_1(x) = \bar{A}x + \tilde{A}(x)$, where $|\tilde{A}(x)| \leq K_0$ for all $x \in \mathbf{R}^n$.

(A5) $B_1(x)B_1(x)' = \bar{M}_1 + \tilde{M}_1(x)$, where $\bar{M}_1 \geq 0$ and $|\tilde{M}(x)| \leq K_1$ for all $x \in \mathbf{R}^n$.

(A6) $B_2(x)E_1(x)^{-1}B_1(x)' = \bar{M}_2 + \tilde{M}_2(x)$, where $\bar{M}_2 \geq 0$ and $|\tilde{M}_2(x)| \leq K_2$ for all $x \in \mathbf{R}^n$.

(A7) $C_1(x)'(I - D_{12}(x)E_1(x)^{-1}D_{12}(x)')C_1(x) = x'\bar{R}'\bar{R}x + \tilde{R}(x)$, where $|\tilde{R}(x)| \leq K_3$ for all $x \in \mathbf{R}^n$.

Assumption (A3) is required for the state feedback necessity results of Section 4.3. Assumptions (A4)- (A7) are required for proving that quadratic storage functions can be used for the state feedback controller.

With regard to notation, ∇_x is used to denote the partial derivative $\frac{\partial}{\partial x}$ with respect to x.

The measurement feedback power gain problem is:

Problem 2.1 (Measurement Feedback Power Gain Control) *Find a causal measurement feedback control K_m which maps, for each time $t \geq 0$, measurements $y \in \mathcal{L}_2(\mathbf{R}^p, [0,t])$ to controls $u \in \mathcal{L}_2(\mathbf{R}^m, [0,t])$ such that the closed loop system (G, K_m) has power gain (1) from disturbance $w \in \mathcal{L}_2(\mathbf{R}^s, [0,t])$ to output $z \in \mathcal{L}_2(\mathbf{R}^r, [0,t])$.*

3 The Information State

The extension of $\mathcal{H}_\infty$ control to the nonlinear state feedback case is well known, see [7]. Recent work [5] has focused on generalizing the measurement feedback case. An essential tool in this generalization is the notion of *information state*, or "cost to come", where the "cost" refers to the accumulated $\mathcal{H}_\infty$ running cost of the plant G with control $u \in \mathcal{L}_2[0,t]$ applied yielding measurement $y \in \mathcal{L}_2[0,t]$ and final state $x \in \mathbf{R}^n$.

Definition 3.1 *The two-block information state $p_t^{u,y}(x)$ for plant G (2) is defined to be a function of time $t \geq 0$, state $x \in \mathbf{R}^n$, control $u \in \mathcal{L}_2[0,t]$, and measurement $y \in \mathcal{L}_2[0,t]$. Explicitly,*

$$\begin{aligned} p_t^{u,y}(x) &= p_0(\xi(0)) + \tfrac{1}{2}\int_0^t \big[|C_1(\xi(s)) + D_{12}(\xi(s))u(s)|^2 \\ &\qquad -\gamma^2|y(s) - C_2(\xi(s))|^2\big]\ ds \end{aligned} \tag{3}$$

where p_0 is the initial information state

$$\begin{aligned} \dot{\xi}(s) &= A(\xi(s)) + B_1(\xi(s))w(s) + B_2(\xi(s))u(s), \\ \xi(t) &= x. \end{aligned} \tag{4}$$

for all $s \in [0,t]$,

Inspection of (3) and (4) reveals that the information state characterizes the worst case cost that would be accumulated in the interval $[0,t]$ if a fixed control $u \in \mathcal{L}_2[0,t]$ was applied to plant G (2) resulting in the (fixed) measurement $y \in \mathcal{L}_2[0,t]$ and the final state $\xi(t) = x$. That is, the information state carries with it information about the worst case state estimate.

An important distinction between power gain and $\mathcal{H}_\infty$ control is that in the power gain case, the accumulated cost can grow asymptotically linearly with t. Consistent with [3], it is reasonable in the power gain case to account for this growth by subtracting off a linear term in t (in the same way that the $\frac{\lambda}{2}t$ term is added to the supply rate $\gamma^2|w(s)|^2 - |z(s)|^2$ in the power gain inequality (1)).

A revised definition of the information state for the two-block power gain problem (consistent with assumptions (A1) and (A2)) then follows.

Definition 3.2 *The two-block power gain information state $\tilde{p}_t^{\lambda,u,y}(x)$ for plant G (2) is defined to be a function of time $t \geq 0$, state $x \in \mathbf{R}^n$, power bias $\lambda \geq 0$, control $u \in \mathcal{L}_2[0,t]$, and measurement $y \in \mathcal{L}_2[0,t]$. Explicitly,*

$$\tilde{p}_t^{\lambda,u,y}(x) = p_t^{u,y}(x) - \tfrac{\lambda}{2}t. \tag{5}$$

Theorem 3.3 *The power gain information state $\tilde{p}_t^{\lambda,u,y}(x)$ satisfies the PDE*

$$\tfrac{\lambda}{2} + \frac{\partial \tilde{p}_t^{\lambda,u,y}}{\partial t}(x) = F(\tilde{p}_t^{\lambda,u,y}, u, y)(x), \tag{6}$$

where

$$\begin{aligned} F(p,u,y)(x) = &-\nabla_x p(x)\cdot[A(x) + B_1(x)w + B_2 u] \\ &+\tfrac{1}{2}\left|C_1(x) + D_{12}(x)u\right|^2 - \gamma^2\tfrac{1}{2}\left|y - C_2(x)\right|^2 \end{aligned} \tag{7}$$

Proof: The information state $p_t^{u,y}(x)$ (3) satisfies the PDE (see [5])

$$\frac{\partial p_t^{u,y}}{\partial t}(x) = F(p_t^{u,y}, u, y)(x), \tag{8}$$

with F given by (7). However, rearranging (5),

$$p_t^{u,y}(x) = \tilde{p}_t^{\lambda,u,y}(x) + \tfrac{\lambda}{2}t. \tag{9}$$

Substituting (9) in (8) then yields the required PDE (6). ∎

4 Certainty Equivalence Measurement Feedback Power Gain Controllers

4.1 The Certainty Equivalence Assumption

Fundamental to the controllers presented in this section is the *certainty equivalence* (**CE**) assumption. This assumption is stated as follows.

(A8) Given information state $\tilde{p}_t^{\lambda,u,y}$ and a state feedback storage function V^λ, there exists for each $t \geq 0$ a unique maximizer $\hat{x}^{\lambda,u,y}(t)$ such that

$$\begin{aligned} &\tilde{p}_t^{\lambda,u,y}(\hat{x}^{\lambda,u,y}(t)) + V^\lambda(\hat{x}^{\lambda,u,y}(t)) \\ &\quad = \sup_{x\in\mathbf{R}^n}\left\{\tilde{p}_t^{\lambda,u,y}(x) + V^\lambda(x)\right\}. \end{aligned} \tag{10}$$

Here, V^λ is a storage function for the state feedback system (G, K_{st}^λ) in the sense of the following definition:

Definition 4.1 *A system is power dissipative if there exists a finite nonnegative power bias and storage function pair $(\lambda, V^\lambda(x))$ satisfying the dissipation inequality*

$$V^\lambda(x_0) + \tfrac{1}{2}\int_0^t\left[\gamma^2|w(s)|^2 - |z(s)|^2 + \lambda\right]ds \geq V^\lambda(x(t)) \tag{11}$$

for all $w \in \mathcal{L}_2[0,t]$, all $t \geq 0$, and all $x_0 \in \mathbf{R}^n$.

4.2 An Information State Power Gain Controller

By accounting for the power generation of the closed loop system in the revised definition of information state, it is possible to derive a class of measurement feedback controllers which render the closed loop system power dissipative.

The CE assumption (A8) plays a critical role in the application of such controllers. Under the CE assumption, an information state power gain controller is:

$$\tilde{K}_m^{\tilde{p}_0,\lambda} : \begin{cases} u(t) &= K_{st}^\lambda(\hat{x}^{\lambda,u,y}(t)) \\ \hat{x}^{\lambda,u,y}(t) &= \operatorname{argmax}_{x\in\mathbf{R}^n}\left\{\tilde{p}_t^{\lambda,u,y}(x) + V^\lambda(x)\right\} \\ \frac{\lambda}{2} &= -\frac{\partial \tilde{p}_t^{\lambda,u,y}}{\partial t}(x) + F(\tilde{p}_t^{\lambda,u,y}, u, y)(x) \end{cases} \tag{12}$$

where $\tilde{p}_0 = \tilde{p}_0^{\lambda,u,y}$ is the initial power gain information state, and F is given by (7).

It is useful to recognize that since λ is independent of the state, the maximization in the definition of $\hat{x}^{\lambda,u,y}$ in (12) ensures that the power gain information state $\tilde{p}_t^{\lambda,u,y}$ (5) may be replaced by the conventional $\mathcal{H}_\infty$ information state $p_t^{u,y}$ (3) in the definition of the information state controller $\tilde{K}_m^{\tilde{p}_0,\lambda}$ (12). That is, an equivalent information state power gain controller is:

$$K_m^{p_0,\lambda} : \begin{cases} u(t) &= K_{st}^\lambda(\hat{x}^{\lambda,u,y}(t)) \\ \hat{x}^{\lambda,u,y}(t) &= \operatorname{argmax}_{x\in\mathbf{R}^n}\left\{p_t^{u,y}(x) + V^\lambda(x)\right\} \\ 0 &= -\frac{\partial p_t^{u,y}}{\partial t}(x) + F(p_t^{u,y}, u, y)(x), \end{cases} \tag{13}$$

where $p_0 = p_0^{u,y} = \tilde{p}_0^{\lambda,u,y}$ is the initial information state. Equivalence of the initial power gain information state and the initial $\mathcal{H}_\infty$ information state follows from (5).

Remark 4.2 Equivalence of the two controllers $\tilde{K}_m^{p_0,\lambda}$ (12) and $K_m^{p_0,\lambda}$ (13) implies that the dependence of the controllers on the power bias λ arises from the dependence of the state feedback controller K_{st}^λ on the power bias λ. This follows since the additional $\frac{\lambda t}{2}$ term in the power gain information state $\tilde{p}_t^{\lambda,u,y}$ (5) does not affect the location of the maximum which defines the state estimate $\hat{x}^{\lambda,u,y}(t)$. The dependence of this state estimate on λ arises from the dependence of the state feedback storage function V^λ on λ.

4.3 State Feedback Power Gain Control

A typical certainty equivalence controller of the form (13) consists of the state feedback control policy K^λ_{st} evaluated at the state estimate obtained via the certainty equivalence assumption (A8). Proving dissipativity for such a certainty equivalence controller requires that the state feedback controller K^λ_{st} renders the closed loop (G, K^λ_{st}) power dissipative. Hence, it is necessary to consider the state feedback power gain control problem.

Problem 4.3 (State Feedback Power Gain Control) *Find a causal state feedback control K^λ_{st} which maps, for each time $t \geq 0$, states $x \in \mathcal{L}_2(\mathbf{R}^n, [0,t])$ to controls $u \in \mathcal{L}_2(\mathbf{R}^m, [0,t])$ such that the closed loop system (G, K^λ_{st}) has power gain (1) from disturbance $w \in \mathcal{L}_2(\mathbf{R}^s, [0,t])$ to output $z \in \mathcal{L}_2(\mathbf{R}^r, [0,t])$ with power bias λ.*

Definition 4.4 (Admissible Controller) *A state feedback controller $K^\lambda_{st} : \mathbf{R}^n \to \mathbf{R}^m$ is admissible if the trajectory $x(t)$ of the resulting closed loop system (G, K^λ_{st}) exists, is unique, and is continuous for every $t \geq 0$ and every initial state $x(0) \in \mathbf{R}^n$.*

The following result from [3] is vital in finding a solution to Problem 4.3.

Theorem 4.5 *The closed loop system (G, K^λ_{st}) has $\mathcal{FP}$-gain $\leq \gamma$ (1) with power bias λ iff the system is power dissipative as per (11) with the same power bias λ.*

Since power gain as per (1) corresponds to existence of a finite nonnegative solution pair (λ, V^λ) of the dissipation inequality (11), necessary and sufficient conditions for existence of a solution of Problem 4.3 follow.

Theorem 4.6 (Necessity) *Suppose there exists an admissible controller K^λ_{st} which solves the state feedback power gain control Problem 4.3 with gain γ and power bias λ. Then, there exists a viscosity supersolution V^λ of the HJBI PDI*

$$H_\gamma(x, \nabla_x V^\lambda(x)) \leq \tfrac{\lambda}{2}, \tag{14}$$

where

$$\begin{aligned} H_\gamma(x,p) = \inf_{u\in\mathbf{R}^m} \sup_{w\in\mathbf{R}^s} \{ & \\ & p\cdot[A(x) + B_1(x)w + B_2(x)u] \\ & + \tfrac{1}{2}\left[|C_1(x) + D_{12}(x)u|^2 - \gamma^2|w|^2\right] \} \end{aligned} \tag{15}$$

Proof: Since K^λ_{st} solves Problem 4.3 with gain γ and power bias λ, the closed loop system (G, K^λ_{st}) has by definition power gain $\leq \gamma$ with power bias γ, as per (1). Theorem 4.5 then implies that (G, K^λ_{st}) is power dissipative with gain γ and power bias λ. Applying Definition 4.1, (11), there exists a finite nonnegative storage function V^λ such that for any $t > 0$, $w \in \mathcal{L}_2[0,t]$, and $x \in \mathbf{R}^n$,

$$\begin{aligned} \tfrac{V^\lambda(x(t)) - V^\lambda(x)}{t} + \tfrac{1}{2t}\int_0^t \big[& |C_1(x(s)) + D_{12}(x(s))K^\lambda_{st}(x(s))|^2 \\ & -\gamma^2|w(s)|^2\big]ds \leq \tfrac{\lambda}{2} \end{aligned} \tag{16}$$

Fix $w \in \mathcal{L}_2[0,t]$ to be continuous and define $K^{\lambda-}_{st}(s)$ to be the right hand lower semicontinuous envelope of $K^\lambda_{st}(x(s))$. That is,

$$K^{\lambda-}_{st}(s) = \liminf_{\sigma\downarrow s}\left\{K^\lambda_{st}(x(\sigma))\right\}. \tag{17}$$

Additionally, let $\phi \in \mathbf{C}^1(\mathbf{R}^n)$ be any function such that $V^\lambda - \phi$ has a local minimum at $x \in \mathbf{R}^n$. For sufficiently small $t > 0$, this implies that

$$V^\lambda(x(t)) - \phi(x(t)) \geq V^\lambda(x) - \phi(x)$$

or equivalently,

$$\tfrac{\phi(x(t)) - \phi(x)}{t} \leq \tfrac{V^\lambda(x(t)) - V^\lambda(x)}{t}. \tag{18}$$

Combining (16), (17), and (18) implies that

$$\begin{aligned} \tfrac{\phi(x(t))-\phi(x)}{t} + \tfrac{1}{2t}\int_0^t \big[& |C_1(x(s)) + D_{12}(x(s))K^{\lambda-}_{st}(s)|^2 \\ & -\gamma^2|w(s)|^2\big]ds \leq \tfrac{\lambda}{2}. \end{aligned}$$

Since C_1, D_{12}, and the trajectory $x(s)$ are continuous functions, $K^{\lambda-}_{st}(s)$ is continuous on the right, and ϕ is differentiable, the Fundamental Theorem of Calculus implies that by sending $t \downarrow 0$,

$$\begin{aligned} & \nabla_x\phi(x)\cdot\left[A(x) + B_1(x)w + B_2(x)K^{\lambda-}_{st}(0)\right] \\ & \quad + \tfrac{1}{2}\left[|C_1(x) + D_{12}K^{\lambda-}_{st}(0)|^2 - \gamma^2|w|^2\right] \leq \tfrac{\lambda}{2} \end{aligned}$$

for all $w \in \mathbf{R}^s$. Hence,

$$\begin{aligned} \sup_{w\in\mathbf{R}^s} \big\{ & \nabla_x\phi(x)\cdot\left[A(x) + B_1(x)w + B_2(x)K^{\lambda-}_{st}(0)\right] \\ & + \tfrac{1}{2}\left[|C_1(x) + D_{12}K^{\lambda-}_{st}(0)|^2 - \gamma^2|w|^2\right]\big\} \leq \tfrac{\lambda}{2}. \end{aligned}$$

Since $K^{\lambda-}_{st}(0)$ is suboptimal in the definition of the Hamiltonian (15), the HJBI PDI (14) follows. That is,

$$H_\gamma(x, \nabla_x\phi) \;\leq\; \tfrac{\lambda}{2}.$$

The definition of viscosity supersolution implies that V^λ must be a supersolution of the same inequality. ∎

Theorem 4.7 (Sufficiency) *Suppose there exists a finite nonnegative solution pair (λ, V^λ), with $V^\lambda \in C^1(\mathbf{R}^n)$, of the HJBI PDI (14)*

$$H_\gamma(x, \nabla_x V^\lambda) \;\leq\; \tfrac{\lambda}{2}. \tag{14}$$

Then, the controller defined by

$$K^\lambda_{st}(x) = -E_1(x)^{-1}\left\{D_{12}(x)'C_1(x) + B_2(x)'\nabla_x V^\lambda(x)'\right\} \tag{19}$$

solves the state feedback power gain control Problem 4.3 with gain γ and power bias λ. That is, the closed loop state feedback system (G, K^λ_{st}) has power gain $\leq \gamma$ with power bias λ.

Proof: Completion of squares in $H_\gamma(x, \nabla_x V^\lambda)$ of (14) yields that

$$\begin{aligned} \tfrac{\lambda}{2} & \geq H_\gamma(x, \nabla_x V^\lambda) \\ & = \sup_{w\in\mathbf{R}^s}\big\{\nabla_x V^\lambda(x)\cdot\left[\left(A(x) + B_2(x)K^\lambda_{st}(x)\right)\right. \\ & \quad \left.+ B_1(x)w\right] + \tfrac{1}{2}\left[|C_1(x) + D_{12}(x)K^\lambda_{st}(x)|^2 - \gamma^2|w|^2\right]\big\} \end{aligned}$$

Integrating yields the dissipation inequality (11) for the closed loop system (G, K_{st}^{λ}) with gain γ and power bias and storage function pair (λ, V^{λ}). Since λ and V^{λ} are finite and nonnegative, (G, K_{st}^{λ}) is power dissipative. Theorem 4.5 implies that (G, K_{st}^{λ}) has power gain $\leq \gamma$ with power bias λ. Hence, K_{st}^{λ} solves the state feedback power gain control Problem 4.3. ∎

4.4 Quadratic Storage Functions for Power Dissipative State Feedback Systems

Typically the state feedback controller K_{st}^{λ} and the corresponding storage function V^{λ} are computed using the PDI (14). However, solving such PDIs present the designer with a difficult computational problem. As an alternative to solving this PDI, we show that it is possible (under suitable assumptions) to use a quadratic storage function and a simple state feedback controller as part of the CE measurement feedback control design.

This result relies on linearizing the plant at $|x| = \infty$. By solving an explicit algebraic Riccati inequality at $|x| = \infty$, it can be shown that the proposed state feedback power gain controller yields a power dissipative closed loop.

Theorem 4.8 *Consider the nonlinear plant G given by (2). Suppose that assumptions (A4), (A5), (A6), and (A7) hold. Suppose additionally that given $\varepsilon > 0$, there exists a gain $\bar{\gamma}_{\varepsilon} < \infty$ such that there exists a finite nonnegative definite solution P_{ε} to the algebraic Riccati equation*

$$\bar{A}' P_{\varepsilon} + P_{\varepsilon} \bar{A} + P_{\varepsilon} \left(\frac{1}{\bar{\gamma}_{\varepsilon}^2} \bar{M}_1 - \bar{M}_2 \right) P_{\varepsilon} + \bar{R}' \bar{R} + \varepsilon^2 I = 0. \tag{20}$$

Then, there exists a $\hat{\gamma}_{\varepsilon} \geq \bar{\gamma}_{\varepsilon}$ such that for any $\gamma > \hat{\gamma}_{\varepsilon}$, the state feedback controller

$$K_{\varepsilon}(x) = -E_1(x)^{-1} \left\{ D_{12}(x)' C_1(x) + B_2(x)' P_{\varepsilon} x \right\} \tag{21}$$

solves the state feedback power gain control Problem 4.3 with gain $\leq \gamma$ and power bias / storage function pair $(\lambda_{\varepsilon}, Q_{\varepsilon})$, where

$$\lambda_{\varepsilon} = \frac{1}{2} \left\{ \frac{K_0^2 |P_{\varepsilon}|^2}{\Gamma_{\varepsilon}(\gamma)} + K_3 \right\}, \quad Q_{\varepsilon}(x) = x' P_{\varepsilon} x \tag{22}$$

and

$$\hat{\gamma}_{\varepsilon} = \max \left(\bar{\gamma}_{\varepsilon}, \sqrt{\frac{K_1}{K_2 + \frac{\varepsilon^2}{|P_{\varepsilon}|^2}}} \right), \tag{23}$$

$$\Gamma_{\varepsilon}(\gamma) = \varepsilon^2 - |P_{\varepsilon}|^2 \left(\frac{1}{\gamma^2} K_1 - K_2 \right). \tag{24}$$

The proof can be found in Appendix A.

4.5 The Information State Controller is Power Dissipative

Theorem 4.9 *Suppose that the certainty equivalence assumption (A8) holds. Given an admissible state feedback controller K_{st}^{λ}, corresponding storage function V^{λ}, and initial information state p_0, then the measurement feedback closed loop system $(G, \tilde{K}_m^{p_0, \lambda})$ given by (2) and (13) is power dissipative, as per Definition 4.1, with power bias and storage function pair $(\lambda, e^{\lambda}(x,p))$, where*

$$\begin{aligned} e^{\lambda}(x, \tilde{p}) &= -\tilde{p}(x) + \Psi_{V^{\lambda}}(\tilde{p}), && (25) \\ \Psi_{V^{\lambda}}(\tilde{p}) &= \sup_{x \in \mathbf{R}^n} \left\{ \tilde{p}(x) + V^{\lambda}(x) \right\}. && (26) \end{aligned}$$

Before proving this theorem, we require the following lemma.

Lemma 4.10 *Suppose that the certainty equivalence assumption (A8) holds, given the state feedback power bias / storage function pair (λ, V^{λ}). Then, the function $\Psi_{V^{\lambda}}(\tilde{p}_t^{\lambda,u,y})$ is nonincreasing in t.*

Proof: [Lemma 4.10] Since certainty equivalence holds, following ideas of Chapter 10 of [5],

$$\begin{aligned} \limsup_{t \downarrow 0} & \left\{ \frac{\Psi_{V^{\lambda}}(\tilde{p}_t^{\lambda,u,y}) - \Psi_{V^{\lambda}}(\tilde{p}_0^{\lambda,u,y})}{t} \right\} \\ &= F(-V^{\lambda}, u, y)(\hat{x}^{\lambda,u,y}(t)) - \frac{\lambda}{2}. \end{aligned} \tag{27}$$

But, (λ, V^{λ}) is a power bias and storage function pair for the state feedback power closed loop system (G, K_{st}^{λ}). That is,

$$\begin{aligned} \sup_{w \in \mathbf{R}^s} & \left\{ \nabla_x V^{\lambda}(x) \cdot \left[A(x) + B_1(x) w + B_2(x) K_{st}^{\lambda}(x) \right] \right. \\ & \left. + \frac{1}{2} |C_1(x) + D_{12}(x) K_{st}^{\lambda}(x)|^2 - \gamma^2 \frac{1}{2} |w|^2 \right\} \leq \frac{\lambda}{2} \end{aligned} \tag{28}$$

Furthermore, choosing suboptimal $w = y - C_2(x)$ in the left hand side of (28) yields precisely $F(-V^{\lambda}, u, y)$. Hence, combining (27) and (28) yields that

$$\begin{aligned} \limsup_{t \downarrow 0} & \left\{ \frac{\Psi_{V^{\lambda}}(\tilde{p}_t^{\lambda,u,y}) - \Psi_{V^{\lambda}}(\tilde{p}_0^{\lambda,u,y})}{t} \right\} \\ &= F(-V^{\lambda}, u, y)(\hat{x}^{\lambda,u,y}(t)) - \frac{\lambda}{2} \\ &\leq \frac{\lambda}{2} - \frac{\lambda}{2} \\ &= 0. \end{aligned}$$

That is, $\Psi_{V^{\lambda}}(\tilde{p}_t^{\lambda,u,y})$ is nonincreasing in t. ∎

Proof: [Theorem 4.9] Applying (25) for plant state $x(t)$ and power gain information state $\tilde{p}_t^{\lambda,u,y}$, and the definition of power gain information state (5),

$$\begin{aligned} e^{\lambda}(x(t), \tilde{p}_t^{\lambda,u,y}) &= -\tilde{p}_t^{\lambda,u,y}(x(t)) + \Psi_{V^{\lambda}}(\tilde{p}_t^{\lambda,u,y}) \\ &= -\frac{1}{2} \int_0^t \left[|C_1(x(s)) + D_{12}(x(s)) u(s)|^2 \right. \\ &\qquad \left. - \gamma^2 |y - C_2(x(s))|^2 - \lambda \right] ds \\ &\qquad + \left(-\tilde{p}_0^{\lambda,u,y} + \Psi_{V^{\lambda}}(\tilde{p}_0^{\lambda,u,y}) \right) \\ &\qquad + \left(\Psi_{V^{\lambda}}(\tilde{p}_t^{\lambda,u,y}) - \Psi_{V^{\lambda}}(\tilde{p}_0^{\lambda,u,y}) \right) \\ &= \frac{1}{2} \int_0^t \left[\gamma^2 |w(s)|^2 - |z(s)|^2 + \lambda \right] ds + e^{\lambda}(x_0, \tilde{p}_0^{\lambda,u,y}) \\ &\qquad + \left(\Psi_{V^{\lambda}}(\tilde{p}_t^{\lambda,u,y}) - \Psi_{V^{\lambda}}(\tilde{p}_0^{\lambda,u,y}) \right) \end{aligned} \tag{29}$$

Applying Lemma 4.10 yields

$$\begin{aligned} e^{\lambda}(x(t), \tilde{p}_t^{\lambda,u,y}) \leq & \frac{1}{2} \int_0^t \left[\gamma^2 |w(s)|^2 - |z(s)|^2 + \lambda \right] ds \\ & + e^{\lambda}(x_0, \tilde{p}_0^{\lambda,u,y}), \end{aligned} \tag{30}$$

which is precisely the dissipation inequality (11) with power bias and storage function pair $(\lambda, e^{\lambda}(x,p))$. ∎

Remark 4.11 Note that Theorem 4.9 may also be proved for the closed loop system (G, K_m^λ) (that is, using the $\mathcal{H}_\infty$ information state $p_t^{u,y}$ rather than the power gain information state $\tilde{p}_t^{\lambda,u,y}$. In the proof, the $\frac{\lambda t}{2}$ term in the dissipation inequality (30) does not then follow from the information state. Instead, this term is contributed from the fact that the function $\Psi_{V^\lambda}(p_t^{u,y})$ is *almost nonincreasing* rather than nonincreasing as in Lemma 4.10. That is,

$$\limsup_{t\downarrow 0}\left\{\frac{\Psi_{V^\lambda}(\tilde{p}_t^{\lambda,u,y})-\Psi_{V^\lambda}(\tilde{p}_0^{\lambda,u,y})}{t}\right\} \leq \frac{\lambda}{2},$$

where u is the control $K_{st}(\hat{x}^{\lambda,u,y}(t))$ in (13).

A Proof of Theorem 4.8

Proof: [Theorem 4.8] Let $\ell_V(x)$ denote the left hand side of the HJBI PDI (14). By completion of squares,

$$\begin{aligned}\ell_V(x) = {} & \nabla_x V(x)\big[A(x) - B_2(x)E_1(x)^{-1}D_{12}(x)'C_1(x)\big] \\ & +\tfrac{1}{2}\nabla_x V(x)\big[\tfrac{1}{\gamma^2}B_1(x)B_1(x)' \\ & \quad -B_2(x)E_1(x)^{-1}B_2(x)'\big]\nabla_x V(x)' \\ & +\tfrac{1}{2}C_1(x)'\big[I - D_{12}(x)E_1(x)^{-1}D_{12}(x)'\big]C_1(x).\end{aligned} \tag{31}$$

By choosing $V(x) = Q_\varepsilon(x)$ as given by (22) and applying assumptions (A4), (A5), (A6), and (A7), equation (31) implies that

$$\begin{aligned}\ell_{Q_\varepsilon}(x) = {} & \tfrac{1}{2}x'\big\{\bar{A}'P_\varepsilon + P_\varepsilon\bar{A} + P_\varepsilon\big(\tfrac{1}{\gamma^2}\bar{M}_1 - \bar{M}_2\big)P_\varepsilon \\ & +\bar{R}'\bar{R}\big\}x + x'P_\varepsilon\tilde{A}(x) \\ & +\tfrac{1}{2}x'P_\varepsilon\big(\tfrac{1}{\gamma^2}\tilde{M}_1(x) - \tilde{M}_2(x)\big)P_\varepsilon x + \tfrac{1}{2}\tilde{R}(x).\end{aligned} \tag{32}$$

But, since $P_\varepsilon \geq 0$ is a solution of the ARE (20), and $\bar{M}_1 \geq 0$ by assumption (A5), $P_\varepsilon \geq 0$ is also a solution of the algebraic Riccati inequality

$$\bar{A}'P_\varepsilon + P_\varepsilon\bar{A} + P_\varepsilon\left(\tfrac{1}{\gamma^2}\bar{M}_1 - \bar{M}_2\right)P_\varepsilon + \bar{R}'\bar{R} \leq -\varepsilon^2 I \tag{33}$$

for all $\gamma \geq \bar{\gamma}_\varepsilon$. Also, assumptions (A5), (A6), and (A7) imply that

$$x'P_\varepsilon\tilde{A}(x) \leq K_0|P_\varepsilon||x|, \tag{34}$$

$$\begin{aligned}& x'P_\varepsilon\left(\tfrac{1}{\gamma^2}\tilde{M}_1(x) - \tilde{M}_2(x)\right)P_\varepsilon x \\ & \qquad \leq |P_\varepsilon|^2\left(\tfrac{1}{\gamma^2}K_1 - K_2\right)|x|^2,\end{aligned} \tag{35}$$

$$\tilde{R}(x) \leq K_3 \tag{36}$$

for all $x \in \mathbf{R}^n$. Hence, choosing $\gamma \geq \bar{\gamma}_\varepsilon$ and substituting (33), (34), (35), and (36) in equation (32),

$$\begin{aligned}\ell_{Q_\varepsilon}(x) \leq {} & -\tfrac{1}{2}\big\{\varepsilon^2 - |P_\varepsilon|^2\big(\tfrac{1}{\gamma^2}K_1 - K_2\big)\big\}|x|^2 \\ & +K_0|P_\varepsilon||x| + \tfrac{1}{2}K_3\end{aligned} \tag{37}$$

for all $x \in \mathbf{R}^n$. For $\ell_{Q_\varepsilon}(x)$ to be bounded above for all $x \in \mathbf{R}^n$, clearly we require that

$$\Gamma_\varepsilon(\gamma) := \varepsilon^2 - |P_\varepsilon|^2\big(\tfrac{1}{\gamma^2}K_1 - K_2\big) > 0.$$

Rearranging yields that

$$\gamma > \sqrt{\frac{K_1}{K_2 + \frac{\varepsilon^2}{|P_\varepsilon|^2}}}.$$

Combining this with the assumed minimal gain $\bar{\gamma}_\varepsilon$ yields (23). With gain $\gamma > \hat{\gamma}_\varepsilon$ ($\hat{\gamma}_\varepsilon$ given by (23)), the right hand side of (37) has a maximum with respect to $|x|$. Differentiating and setting equal to zero yields a maximizing value of $|x^*| = \frac{K_0|P_\varepsilon|}{\Gamma_\varepsilon(\gamma)}$. Substituting into the right hand side of (37) yields that for all $x \in \mathbf{R}^n$,

$$\ell_{Q_\varepsilon}(x) \leq \tfrac{1}{2}\big\{\tfrac{K_0^2|P_\varepsilon|^2}{\Gamma_\varepsilon(\gamma)} + K_3\big\},$$

the right hand side of which is precisely λ_ε as given by (22). Finally, since $\ell_{Q_\varepsilon}(x)$ is the left hand side of the HJBI PDI (14), this implies that $(\lambda_\varepsilon, Q_\varepsilon)$ is a finite nonnegative solution pair (with $Q_\varepsilon \in C_1(\mathbf{R}^n)$) of the HJBI PDI (14). Theorem 4.7 then implies that the state feedback controller K_ε given by (21) solves the state feedback power gain control Problem 4.3 with gain γ and power bias and storage function pair $(\lambda_\varepsilon, Q_\varepsilon)$. ∎

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