MANIPULATING MATRIX INEQUALITIES AUTOMATICALLY

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Abstract. Matrix inequalities have come to be extremely important in systems engineering in the past decade. This is because many systems problems convert directly into matrix inequalities.

Matrix inequalities take the form of a list of requirements that polynomials or rational functions of matrices be positive semidefinite. Of course while some engineering problems present rational functions which are well behaved, many other problems present rational functions which are badly behaved. Thus taking the list of functions which a design problem presents and converting these to a nice form, or at least checking if they already have or do not have a nice form is a major enterprise. Since matrix multiplication is not commutative, one sees much effort going into calculations (by hand) on noncommutative rational functions. A major goal in systems engineering is to convert, if possible, "noncommutative inequalities" to equivalent Linear Noncommutative Inequalities (effectively to Linear Matrix Inequalities, to LMI's).

This survey concerns efforts to process "noncommutative inequalities" using computer algebra. The most basic efforts, such as determining when noncommutative polynomials are positive, convex, convertible to noncommutative LMI's, transformable to convex inequalities, etc., force one to a rich area of undeveloped mathematics.

OUTLINE

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1. Introduction. Many different types of matrix inequalities have come up in the mathematics of the previous century, but the ones which predominate in engineering systems usually take the form of a polynomial or rational function of matrices being positive semidefinite, (**PSD**). For example, there are Riccati expressions like

$$AX + XA^T - XBB^TX + C^TC \succeq 0 \tag{1.1}$$

or Linear Matrix Inequalities LMI's like

$$\begin{pmatrix} AX + XA^T + C^TC & XB \\ B^TX & I \end{pmatrix} \succeq 0 \tag{1.2}$$

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which is equivalent to the Riccati inequality (has an equivalent set of solutions). The matrices A, B, C are typically given, and $X = X^T$ is a symmetric matrix which must be found by some numerical method. Here all matrices are assumed to have compatible dimension. References on applications and numerics for LMI's are [BEFB94] and [ESLSL00].

Inequalities involving polynomials and rational functions also occur, as well as lists of them. Indeed there are rich mathematical issues which bear on matrix inequalities. The focus of this article is on algebraic (rather than numerical) ones.

This paper is dedicated to the memory of my very devoted mother, Maxene Helton.

2. To Commute or Not Commute: An Homage to Formulas which Scale Elegantly with System Size. This section discusses two different ways of writing matrix inequalities. As an example, we could consider either the Riccati inequality (1.1) or the equivalent LMI in (1.2). Let us focus on this LMI, and discuss the various ways one could write this linear matrix inequality.

The LMI in (1.2) has the same form regardless of the dimension of the system and its defining matrices A, B, C. In other words, if we take the matrices A, B, C and X to have compatible dimension, (regardless of what those dimensions are), then the inequality (1.2) is meaningful and substantive and its form does not change.

When the dimensions of the matrices A, B, C and X are specified it is common to write (1.2) as a linear combination of known matrices $L_0, L_1, \ldots, L_m$ of dimension $d \times d$ in unknown real numbers $x_1, \ldots, x_m$:

$$L_0 + \sum_{j=1}^{m} L_j x_j \succeq 0. \tag{2.1}$$

For example, in the inequality (2.1) if $A \in R^{2\times 2}, B \in R^{2\times 1}, C \in R^{1\times 2}$, then $X^T = X \in R^{2\times 2}$ and we would take $m = 3$ and the numbers x_i in

$$X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$$

as unknowns in the inequality (2.1). The unpleasant part is that the L_i are

$$L_0 := \begin{pmatrix} C^T C & 0 \\ 0 & I \end{pmatrix} \tag{2.2}$$

$$L_1 := \begin{pmatrix} 2a_{11} & a_{21} & b_{11} & b_{12} \\ a_{21} & 0 & 0 & 0 \\ b_{11} & 0 & 0 & 0 \\ b_{12} & 0 & 0 & 0 \end{pmatrix}$$

$$L_2 := \begin{pmatrix} 2a_{12} & a_{11}+a_{22} & b_{21} & b_{22} \\ a_{22}+a_{11} & 2a_{21} & b_{11} & b_{12} \\ b_{21} & b_{11} & 0 & 0 \\ b_{22} & b_{12} & 0 & 0 \end{pmatrix}$$

$$L_3 := \begin{pmatrix} 0 & 0 & 0 & a_{12} \\ 0 & 0 & 0 & a_{22} \\ 0 & 0 & 0 & b_{21} \\ a_{12} & a_{22} & b_{21} & 2b_{22} \end{pmatrix}$$

Now consider $A \in R^{3\times 3}, B \in R^{3\times 2}, C \in R^{2\times 3}, X \in R^{3\times 3}$. This gives a messier formula than (2.1) whose relationship to (2.1) takes a little while to figure out. The point is that the formula (2.1) (2.2) does not scale simply with dimension of the matrices or of the system producing them, while formula (1.2) does.

I vastly prefer scalable formulas to ones of the second type. It would be good to write down lists of rational reasons for this, but the real reason is that I find unscalable formulas are usually UGLY. A more rational reason is that formula (1.1) and (1.2) are easier for a specialist to manipulate with a pencil and paper (or with NCAlgebra) than formula (2.1) (2.2). Also scalable formulas tend to keep physical quantities nicely grouped. Moreover, there are situations where one wants to build a network out of simple pieces and then build a bigger network out of the same simple pieces. Scalable formulas lend themselves to this approach more than do badly scalable ones, like inequality (1.1) .

Advantages of unscalable formulas like (2.1) (2.2) are: they hold more generally than scalable ones, and the formula (2.1) (2.2) loses (or at least scrambles) a lot of special structure which means a student deriving the formula can ignore a lot of special structure. Also a disadvantage of formula (1.1) and (1.2) is intrinsically noncommutative, so to take advantage of their simplicity a person must have skill with noncommutative calculations. On balance, I think it is fine to focus on formulas like (1.2) if one is teaching Masters degree students in systems and control, or maybe people in other areas where similar formulas do not easily scale, or people who want a quick introduction, or numerical analysts who only want to solve such equations. However, Ph.D. students in systems and control should be encouraged to strive for formulas which automatically scale with dimension. Even though such formulas are typically noncommutative, they have the advantages just described.

3. Noncommutative Inequalities Behave Better than Commutative Ones. To develop a basis for computer algebra packages which could assist engineers in manipulating matrix inequalities, (that is, ones which scale), we first need to say what a noncommutative inequality is.

3.1. Noncommutative polynomials. We consider polynomials, which are, weighted sums of words on $2n$ variables (letters) together with an involution on the words, denoted by ^{T}, somewhat loosely called **transpose**. An involution satisfies properties $(X_iX_j)^T = X_j^T X_i^T$ and $(X_j^T)^T = X_j$ for all i, j. We denote the variables (often called indeterminates) by $X_1, \cdots, X_n, X_1^T, \cdots, X_n^T$ and abbreviate them with the notation

$$\vec{X} = \{X_1, X_2, \ldots, X_n\} \text{ and } \vec{X}^T = \{X_1^T, X_2^T, \ldots, X_n^T\}.$$

Call Q a **symmetric polynomial** provided $Q(\vec{X})^T = Q(\vec{X}^T)$.

3.2. Statements Must be True for Matrices. One who designs a Matlab Toolbox wants formulas in it which have the advertised property whenever matrices are of any dimension plugged into their formulas. For example, what would one mean by a polynomial being positive? There is an obvious answer, if we bear in mind our insistence that properties survive matrix substitution. We call a symmetric polynomial Q in $\vec{X}$ and $\vec{X}^T$ **matrix-positive** provided that for any r when we substitute into Q any real matrices $\mathcal{X}_1, \cdots, \mathcal{X}_n$ of dimension $r \times r$ for $X_1, \cdots, X_n$ and we substitute their transposes, $\mathcal{X}_1^\tau, \cdots, \mathcal{X}_n^\tau$ for $X_1^T, \cdots, X_n^T$, then the resulting matrix $Q(\mathcal{X}_1, \cdots, \mathcal{X}_n, \mathcal{X}_1^\tau, \cdots, \mathcal{X}_n^\tau)$ is positive semidefinite.

Consider the following example in two indeterminates

$$Q(\vec{X}) = X_1^2 + (X_1^2)^T + X_2^T X_2. \tag{3.1}$$

If $\mathcal{X}_1$ and $\mathcal{X}_2$ are one dimensional, then

$$Q(\vec{\mathcal{X}}) = \mathcal{X}_1^2 + (\mathcal{X}_1^2)^T + \mathcal{X}_2^T \mathcal{X}_2 = 2\mathcal{X}_1^2 + \mathcal{X}_2^2.$$

This is a sum of squares of numbers and so is positive semidefinite. However, if we substitute $\mathcal{X}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for X_1 and $\mathcal{X}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for X_2, then we get

$$Q(\vec{\mathcal{X}}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.2}$$

which is not positive semidefinite. Thus $Q(\vec{X})$ is not matrix positive.

3.3. Characterizing Positive Polynomials. We say a polynomial Q is a **Sum of Squares (SoS)** provided Q can be put in the form

$$Q(\vec{X}) = \sum_{i=1}^{k} h_i(\vec{X})^T h_i(\vec{X}) \tag{3.3}$$

where each h_i is a polynomial in $\vec{X}$ and $\vec{X}^T$.

Clearly, any polynomial which is a SoS is matrix positive, but what about the converse? The commutative case, that is, the case of polynomials on R^n is very classical. Hilbert one hundred years ago knew that not all positive polynomials are sums of squares. In fact his famous 17^{th} problem is devoted to this sort of problem: which polynomials are sums of squares of rational functions. By now there is a lively area of mathematics devoted to understanding which polynomials are and which are not SoS, along with related phenomena. See [R00] [deAprept] for excellent surveys. Now we return to the seemingly more complicated situation of noncommutative polynomials.

THEOREM 3.1. *Suppose $\mathcal{Q}$ is a non-commutative symmetric polynomial. If Q is a SoS, then Q is matrix-positive. If Q is matrix-positive, then Q is a SoS.*

The paper [Hprept] is devoted to this Theorem and its proof. The important case where all operators are complex unitary was proved in [Mprept].

3.4. Computing the SoS Decomposition of a Polynomial.

3.4.1. Representing Symmetric Polynomials. In this section we give a standard "Gram" representation for a polynomial. Also we characterize the non-uniqueness in the representation.

LEMMA 3.1. *If $Q(\vec{X})$ is a symmetric polynomial, then there exists a symmetric matrix M_Q with real entries, not dependent on $\vec{X}$, and a vector $V(\vec{X})$ of monomials in $\vec{X}$ such that*

$$Q(\vec{X}) = V(\vec{X})^T M_Q \, V(\vec{X}). \tag{3.4}$$

Furthermore, the vector $V(\vec{X})$ can always be chosen to be $V^d(\vec{X})$ where d is the least integer bigger than $\frac{1}{2}$ (degree of $\mathcal{Q}$).

Here we let V^d denote the column vector of all monic monomials of degree less than or equal to d in X_j and X_j^T for $j = 1, \ldots, n$, listed in graded lexicographic order. The length of V^d is $\nu(d) := 1+(2n)+(2n)^2+\cdots+(2n)^d$, since that is the number of monomials in $\vec{X}, \vec{X}^T$ of length $= d$. For example, if $\vec{X} = \{X_1, X_2\}$, then $V^2(\vec{X})$ is the column vector with entries

$$\{I, X_1, X_2, X_1^T, X_2^T, X_1^2, X_1X_2, X_1X_1^T, X_1X_2^T, X_2X_1, \ldots, (X_2^T)^2\}.$$

We think of $V(\vec{X})$ as a vector of monomials which often will be denoted

$$V(\vec{X}) = \begin{pmatrix} V(\vec{X})_0 \\ \vdots \\ V(\vec{X})_{p-1} \end{pmatrix}. \tag{3.5}$$

Examples of the representation (3.5) are

$$\mathcal{Q} = X_1X_1^T + X_1X_2 + X_2^TX_1^T + 2 + 2X_2^TX_1^TX_1X_2$$

$$= \begin{pmatrix} I \\ X_1^T \\ X_1X_2 \end{pmatrix}^T \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} I \\ X_1^T \\ X_1X_2 \end{pmatrix}.$$

Here

$$V(\vec{X}) = \begin{pmatrix} I \\ X_1^T \\ X_1X_2 \end{pmatrix} \quad \text{and } M_{\mathcal{Q}} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}. \tag{3.6}$$

Another example is

$$\mathcal{Q} = 2X_2^TX_1^TX_2^TX_2X_1X_2 = \left(X_2X_1X_2\right)^T (2) \left(X_2X_1X_2\right).$$

3.4.2. The Non-uniqueness in the Represention. Define $SR^{p\times p}$ to be the symmetric $p \times p$ matrices. A non-commutative polynomial $\mathcal{Q}$ with a representation

$$\mathcal{Q}(\vec{X}) = V(\vec{X})^T M_{\mathcal{Q}}^0 V(\vec{X}) \tag{3.7}$$

with $M_{\mathcal{Q}}^0$ a symmetric matrix in $SR^{p\times p}$. $\mathcal{Q}$ may also be represented by different M's and the same V. These representations of $\mathcal{Q}$ all have M of the form

$$M = M_{\mathcal{Q}}^0 + B \tag{3.8}$$

where $B \in \mathcal{B}_V$ with the subspace of matrices $\mathcal{B}_V \subset SR^{p\times p}$ defined by

$$\mathcal{B}_V := \{B : V(\vec{X})^T BV(\vec{X}) = 0 \qquad \text{and} \qquad B = B^T\}. \tag{3.9}$$

Here $M_{\mathcal{Q}}^0 \in SR^{p\times p}$ is one fixed matrix.

3.4.3. Computing SoS Decomposition is a LMI Problem.

If one matrix M of the form (3.8) is PSD, then the Cholesky decomposition of M constructs a SoS decomposition of $\mathcal{Q}$. Conversely, if $\mathcal{Q}$ has a SoS decomposition, then there is a positive semidefinite matrix M of the form (3.8).

It is this well known structure which was exploited quite successfully by N. Z. Shor [S87], Powers-Wormann [PW98] for computational purposes in attempting to express commutative polynomials as a SoS. Pablo Parrilo observed that the critical problem here is an LMI, namely, the matrix B enters the matrix valued expression

$$M_{\mathcal{Q}}^0 + B$$

linearly, and we wish to find B making this PSD. Software for solving LMIs abounds, see

http://plato.la.asu.edu/dimacs.html

for results of benchmark comparisons of numerical packages for solving LMIs. There is a rich area of research devoted to finding better numerical algorithms. Although we will not deal with numerical issues, LMIs will be the focus of Sections 4 and 5 which appear later in this article.

While this SoS algorithm, which was just outlined, appears in the literature for ordinary commutative polynomials, it also works without modification for noncommutative polynomials.

Often the point in computing a SoS decomposition for a polynomial p is just to affirm that p is everywhere positive. We can not help but emphasize that the algorithm, by determining if a SoS decomposition for noncommutative p exists, determines definitively if p is matrix positive.

3.5. Applications of SoS Decompositions. Parrilo and Sturmfels started experiments on commutative polynomials which ultimately suggested that representing polynomials as SoS is an effective way in practice to check if they are positive. While not all positive polynomials are SoS, his experiments show a dramatic disposition for positive polynomials to be represented as SoS, see [PSprept]. Parrilo also uses [P00] a theorem of Reznick to the effect that if p is a positive polynomial, then there is a integer $n \geq 0$ such that,

$$\frac{p(\vec{x})}{(1+|\vec{x}|^2)^n}$$

is a SoS of rational functions. With this, by stepping thru successively higher n, one can determine if p is positive. Using these techniques Parrilo has found many applications to areas such as combinatorial optimization, dynamical systems and quantum entanglement.

Convexity is a very important property of a function one wishes to optimize. Now this paper is promoting formulas which scale and so are noncommutative. For these we need a theory of "noncommutative convexity". That is the subject of Section 6. There we describe a noncommutative second derivative and then use a theory of matrix positivity like the one described above to determine automatically using a computer algebra "the region of convexity" of a noncommutative rational function.

4. Which Sets have LMI Representations? We say a set $\mathcal{C} \subset \mathbf{R}^m$ has a **Linear Matrix Inequality (LMI) Representation** provided that there are symmetric matrices $L_0, L_1, L_2, \cdots, L_m$ for which the set

$$\{\vec{x} = (x_1, x_2, \cdots x_m) : L_0 + L_1 x_1 + \cdots + L_m x_m \quad \text{is PosSemiDef}\}$$

equals the set $\mathcal{C}$. We shall use the term **linear pencil** to refer to a m real parameter family of symmetric matrices

$$L(\vec{x}) := L_0 + L_1 x_1 + \cdots + L_m x_m$$

where $\vec{x} = (x_1, \cdots, x_m)$ are m real parameters. We shall require by our use of the term **linear pencil** that $L_0, L_1, L_2, \cdots L_m$ are symmetric real entried matrices, and often we refer to the linear pencil simply by listing the $(m+1)-$tuple $L = \{L_0, L_1, L_2, \cdots, L_m\}$ of matrices. A **monic** pencil is one with $L_0 = I$.

In the many applications which LMIs have found there is no systematic way to produce LMIs for general classes of problems. Each area has a few special tricks which convert "lucky problems" to LMIs. Before there is any hope of producing LMIs systematically one must have a good idea of which types of constraint sets convert to LMIs and which do not.

This seems like fundamental issue regarding LMI's and I am grateful to Stephen Boyd and Berndt Sturmfels for mentioning it to me. The question is formally stated as an open problem in [PSprept]. In this survey I shall sketch briefly some theory of this question developed with Victor Vinnikov in [HVprept] .

4.1. Obvious Necessary Conditions. Suppose we are given a pencil L which represents a set $\mathcal{C}$. Clearly

$\mathcal{C}$ is a convex set.

Without loss of generality take $0 \in \mathcal{C}$. We have L_0 is PSD. It is easy (see [HVprept]) to reduce general pencils with L_0 a PSD matrix to monic pencils, so we assume that L is a monic pencil. Define a polynomial $\check{p}$ by

$$\check{p}(\vec{x}) := \det[I + L_1 x_1 + \cdots + L_m x_m]. \tag{4.1}$$

where the L_j are symmetric $d \times d$ matrices.

4.1.1. $\mathcal{C}$ is an Algebraic Interior. Clearly, the boundary of $\mathcal{C}$ is contained in an algebraic curve, namely, the zero set of $\check{p}$. In fact, $\mathcal{C}$ is what we call an **Algebraic Interior**, that, is a set $\mathcal{C}$ in R^m for which there is a polynomial p in m variables (normalized by $p(0) = 1$), such that $\mathcal{C}$ equals the set $\mathcal{C}_p$, defined to be the closed connected set containing 0, with $p(\vec{x}) > 0$ on the interior of $\mathcal{C}_p$, and $p(\vec{x}) = 0$ on the boundary of $\mathcal{C}_p$. Note that the set $\mathcal{C}_p$ is uniquely determined by the polynomial p as the closure of the connected component of the origin in

$$\{\vec{x} \in \mathbf{R}^m : p(\vec{x}) \neq 0\}.$$

4.1.2. p Satisfies the Real Zeroes Condition . For an LMI representation to exist there is another obvious condition. Observe from (4.1) that

$$\check{p}(\mu\vec{x}) := \mu^d \det[\frac{I}{\mu} + L_1 x_1 + \cdots + L_m x_m].$$

Since (for real numbers x_j) all eigenvalues of the symmetric matrix $L_1 x_1 + \cdots + L_m x_m$ are real we see that, while $p(\mu\vec{x})$ is a complex valued function of the complex number μ, it vanishes only at μ which are real numbers. This condition is critical enough that we formalize it in a definition.

A **Real Zero polynomial** (**RZ** polynomial) is defined to be a polynomial in m variables satisfying, for each $\vec{x} \in R^m$,

$$(\mathbf{RZ}). \qquad p(\mu \vec{x}) = 0 \quad \text{implies} \quad \mu \quad \text{is real}$$

It is shown in [HVprept] that for an Algebraic Interior $\mathcal{C}$, the minimal degree defining polynomial p is unique; of course p has some degree d and we say that $\mathcal{C}$ is an **Algebraic Interior of Degree** d.

Of course the RZ condition is biased with respect to the point zero, so if we wish to study a set $\mathcal{C}$ which does not contain 0 but contains a point O, then we would need to check that

$$(\mathbf{RZ_O}). \qquad p(O + \mu \vec{x}) = 0 \quad \text{implies} \quad \mu \quad \text{is real}$$

4.2. Geometrical Version of the Necessary Conditions: Rigid Convexity. In this section we give a geometric characterization of the Real Zeroes Condition.

4.2.1. Rigid Convexity. An algebraic interior $\mathcal{C}$ of degree d in R^m with minimal defining polynomial p will be called **rigidly convex** provided that for every point $\vec{x}^0$ in $\mathcal{C}$ and almost every line ℓ through $\vec{x}^0$ (i.e. all but a finite number), ℓ intersects the (affine) real algebraic surface $p(\vec{x}) = 0$ in exactly d points [1]. In this counting one ignores lines which go thru $\vec{x}^0$ and hit the boundary of $\mathcal{C}$ at ∞.

PROPOSITION 4.1. *(See [HVprept]) If the line test which defines rigid convexity holds for one point $\vec{x}^0$ inside $\mathcal{C}$ then it holds at all points inside $\mathcal{C}$, thereby implying rigid convexity.*

Obviously this proposition simplifies testing for rigid convexity, since we just need to pass lines through one point, not all points. Although we do not prove it here, rigid convexity implies convexity.

THEOREM 4.1. *Rigid convexity of an Algebraic Interior $\mathcal{C}$ is the same as its minimal degree defining polynomial p having the RZ Property. As a consequence we obtain that if $\mathcal{C}$ has an LMI representation, then it is rigidly convex.*

We prove this later after turning to some examples.

4.2.2. Example 1. The polynomial

$$p(x_1, x_2) = x_1^3 - 3x_2^2 x_1 - (x_1^2 + x_2^2)^2$$

has zero set shown in Figure 1.

[1] One can replace here "almost every line" by "every line" if one takes multiplicities into account when counting the number of intersections, and also counts the intersections at infinity, i.e., replaces the affine real algebraic hypersurface $p(\vec{x}) = 0$ by the projective real algebraic hypersurface $p(\vec{x}) = 0$.

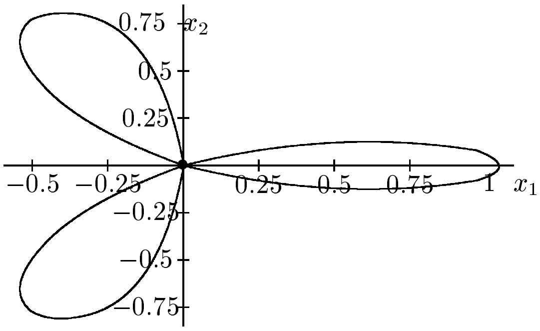

FIG. 1. $p(x_1, x_2) = x_1^3 - 3x_2^2x_1 - (x_1^2 + x_2^2)^2$

The complement to $p = 0$ in $\mathbf{R}^2$ consists of 4 components, 3 bounded convex components where $p > 0$ and an unbounded component where $p < 0$. Let us analyze one of the bounded components, say the one in the right half plane, $\mathcal{C}$ is the closure of

$$\{\vec{x} : p(\vec{x}) \succ 0, x_1 > 0\}.$$

Does $\mathcal{C}$ have an LMI representation? To check this: fix a point O inside $\mathcal{C}$, e.g. $O = (.7, 0)$.

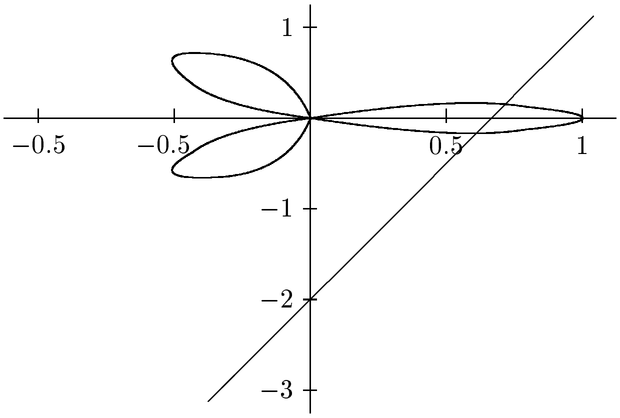

FIG. 2. *A line thru* $p = x_1^3 - 3x_2^2x_1 - (x_1^2 + x_2^2)^2$ *hitting* Z_p *in only 2 points.*

By Theorem 4.1 almost every line l (as in Figure 2) must intersect $p = 0$ in 4 real points or the RZ_O condition is violated. We can see from the picture in $\mathbf{R}^2$ that there is a continuum of real lines ℓ through O intersecting $p = 0$ in exactly two real points. Thus by Theorem 4.1 the set $\mathcal{C}$ does not have an LMI representation. (Since p is irreducible it is the minimum defining polynomial for $\mathcal{C}$.)

4.2.3. Example 2.

$$p(x_1, x_2) = 1 - x_1^4 - x_2^4$$

Clearly, $\mathcal{C}_p := \{x : p(\vec{x}) \geq 0\}$ has degree 4 but all lines in $\mathbf{R}^2$ through it intersect the set $p = 0$ in exactly two places. Thus $\mathcal{C}_p$ is not rigidly convex.

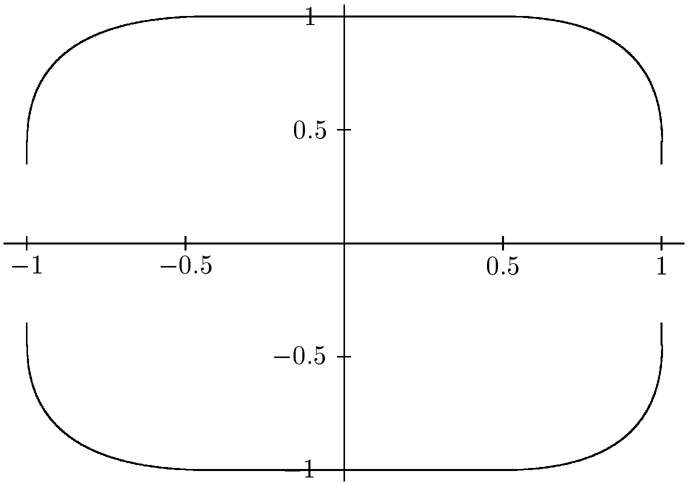

FIG. 3. $p(x_1, x_2) = 1 - x_1^4 - x_2^4$

4.2.4. Idea of the Proof of Theorem 4.1. Let $\ell := \{\vec{x} \in R^m : \vec{x} = \mu\vec{v} + \vec{x}^0,\ \mu \in R\}$, be a parameterization of a line through $\vec{x}^0$ in the interior of $\mathcal{C}$; its direction is $\vec{v} \in R^m$. The points of intersection of the real algebraic hypersurface $p(\vec{x}) = 0$ with the line ℓ are parameterized by exactly those $\mu \in \mathbf{R}$ at which

$$f(\mu) := p(\mu\vec{v} + \vec{x}^0) = 0,$$

and rigid convexity of $\mathcal{C}_p$ says (for all but finitely many ℓ) there are exactly d such distinct μ. However, the degree of p is d, so for all but finitely many directions $\vec{v} \in R^m$, the degree of the polynomial f equals d. Thus these f, by the Fundamental Theorem of Algebra, have exactly d zeroes $\mu \in \mathbf{C}$, counting multiplicities. Thus rigid convexity says precisely that all zeroes of f are at real μ, for f arising from almost all $\vec{v}$. That this is equivalent to the RZ Property for p follows from an elementary continuity argument which implies that all f have only real zeroes. ••

4.3. Rigid Convexity Suffices in 2 Dimensions . Now we give a definitive charachterization for LMI representations in 2 dimensions.

THEOREM 4.2. *If $\mathcal{C}$ is a closed convex set in R^m with an LMI representation, then $\mathcal{C}$ is rigidly convex. When $m = 2$, the converse is true, namely, a rigidly convex degree d set, whose interior contains 0, has a monic LMI representation with symmetric matrices $L_j \in R^{d \times d}$.*

The proof is not at all elementary and involves a considerable amount of Riemann Surface theory. For related topics the reader is referred to the Multi-dimensional systems workshop at this MTNS 2002 conference, organized by Krzysztof Galkowski, Eric Rogers, and Victor Vinnikov. The proof is based on earlier work of V. Vinnikov [V89], [V93] and improved by more recent work with J. Ball [BV96] [BV99]. The proof itself can be found in [HVprept]. There we give two proofs. One of them is a construction of the L_0, L_1, L_2 based on Riemann surface theory, θ functions and such, which does not bode well for dimensions higher than 2.

Nobody, has any idea at this point if in dimension higher than $m = 2$, what conditions beside rigid convexity of $\mathcal{C}$ are required to ensure that $\mathcal{C}$ has an LMI representation.

5. LMI Representations for Sets which Automatically Scale. There seems to a be disconnect between the results of the previous section and much of the control theory LMI literature. In the literature one typically sees methods for construction which depend on Schur complement formulas and which clearly represent only very very special sets with LMI's. In the previous section we saw that the rather weak RZ condition was all that was required for there to exist an LMI representation at least in 2 dimensions. Maybe the reason is that most formulas obtained in the systems literature scale automatically with the dimension of the system being analyzed, as opposed to what was studied in the previous section. In this section we turn to scalable versions of LMI representations and simply ask questions.

5.1. Questions. First we state some questions of this genre. We begin with the easiest to state, although it is not the closest to our theme.

QUESTION 5.1. *For which symmetric polynomials p in symmetric noncommuting variables X_j is there a monic linear pencil*

$$L(\vec{X}) := I + L_1 \otimes X_1 + L_2 \otimes X_2 + \cdots + L_m \otimes X_m$$

with m matrices $L_j \in S\mathbf{R}^{d \times d}$, such that

$$det\ p(\vec{\mathcal{X}}) = \ det[I + L_1 \otimes \mathcal{X}_1 + L_2 \otimes \mathcal{X}_2 + \cdots + L_m \otimes \mathcal{X}_m]$$

for all m- tuples $\vec{\mathcal{X}}$ of $S\mathbf{R}^{r \times r}$ matrices.

The notation $L_j \otimes \mathcal{X}_j$ denotes tensor product but might be puzzling, so we give an example.
Example 1. Take $m = 2$ and d $= 2$ and

$$L_1 := \begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix} \qquad L_2 := \begin{pmatrix} 3 & 5 \\ 5 & 0 \end{pmatrix} \tag{5.1}$$

Then

$$L_1 \otimes X_1 := \begin{pmatrix} 2X_1 & 3X_1 \\ 3X_1 & 0 \end{pmatrix}$$

the pencil $L(\vec{X})$ is

$$L(\vec{X}) = \begin{pmatrix} 1 + 2X_1 + 3X_2 & 3X_1 + 5X_2 \\ 3X_1 + 2X_2 & 1 \end{pmatrix}.$$

Clearly, if we were to replace $\vec{X}$ by 2-tuples of 3×3 matrices, then $L(\vec{\mathcal{X}})$ would be a 6×6 matrix.

Our main interest is in sets of matrices on which p is PSD (rather than in Question 5.1 about determinants). This requires a definition. Given a symmetric polynomial p in symmetric noncommutative variables as in Question 5.1, we call a subset $\mathcal{C}_p^r$ of m-tuples of symmetric $r \times r$ matrices the $\mathbf{r^{th}}$ **- Algebraic Interior** for p provided that the set $\mathcal{C}_p^r$ is arcwise connected and, most important, we require $p(\vec{\mathcal{X}})$ is positive definite for each m-tuple $\vec{\mathcal{X}}$ of symmetric $r \times r$ matrices. The **Noncommutative Algebraic Interior** for p is the union $\mathcal{C}_p$ of the $\mathcal{C}_p^r$ for all r. The definition also applies to a symmetric pencil L, since L is a polynomial in $\vec{X}$, associate to a pencil L a Noncommutative Algebraic Interior, denoted $\mathcal{C}_L$. Here we emphasize that, for $\vec{\mathcal{X}}$ a tuple of $r \times r$ matrices, $L(\vec{\mathcal{X}})$ is a $R^{dr \times dr}$ matrix given by

$$L(\vec{\mathcal{X}}) := L_0 \otimes I_r + L_1 \otimes \mathcal{X}_1 + L_2 \otimes \mathcal{X}_2 + \cdots + L_m \otimes \mathcal{X}_m.$$

Also if we assume the normalization the zero matrix tuple is in the interior of $\mathcal{C}_L$, then we may without loss of generality take L to be monic.

QUESTION 5.2. *Which Noncommutative Algebraic Interiors $\mathcal{C}$ have a monic noncommutative LMI representation? In other words, for which $\mathcal{C}$ is there a symmetric linear pencil $L(\vec{X})$ having the same Noncommutative Algebraic Interior, that is $\mathcal{C}_L = \mathcal{C}$?*

QUESTION 5.3. *There are many obvious variants on these questions centering on what is symmetric and what is not.*

Example 2 Given

$$p(\vec{X}) := 1 + 2X_1 + 3X_2 - (3X_1 + 2X_2)(3X_1 + 2X_2).$$

It is easy to show that the answer to Question 5.2 is yes for this polynomial p. Indeed, the monic linear pencil $L(\vec{X})$ given in equation (5.1) and $p(\vec{X})$ have the same Noncommutative Algebraic Interior.

To see this note that $p(\vec{X})$ is a Schur complement for the 2×2 matrix function $L(\vec{X})$ and that $p(\vec{\mathfrak{X}})$ being PSD implies $1+2\mathcal{X}_1+3\mathcal{X}_2$ is also PSD, which together are equivalent to $L(\vec{\mathfrak{X}})$ being PSD. Note that h does not factor, so is irreducible. ••

There is progress on these questions by Scott McCullough and me, but findings are far from complete. The interested reader can look at Helton's web page where preprints are posted.

6. Noncommutative Convexity. For numerical purposes convexity goes a long way; indeed an LMI representation for many problems might well be overkill. This section concerns noncommutative convexity and how one in practice can test if a given noncommutative function is "convex". This section will describe work with Juan Camino, Bob Skelton and Jeiping Ye, see [CHSYprept].

We shall be investigating noncommutative rational functions, namely functions Γ which are polynomial or rational in noncommutative variables. Examples of noncommutative symmetric polynomials are

$$\Gamma(A, B, X) = AX + XA^T - \frac{3}{4}XBB^TX, \qquad X = X^T,$$

$$\Gamma(A, D, X, Y) = X^TAX + DYD^T + XYX^T, \ \ Y = Y^T \text{ and } A = A^T. \tag{6.1}$$

They have coefficients which are real numbers. Noncommutative rational functions of $\vec{X}$ are polynomials in $\vec{X}$ and inverses of polynomials in $\vec{X}$. An example of a symmetric rational function is

$$\Gamma(A, D, E, X, Y) = A(I + DXD^T)^{-1}A^T + E(YXY^T)E^T, \qquad X = X^T. \tag{6.2}$$

We also assume there is an involution on these rational functions which we denote superscript T, and which will play the role of transpose later when we substitute matrices for the indeterminates. The setup in this section is more general than in Section 3 in that here we are willing to restrict some variables to be symmetric and let others not be symmetric. In Section 3 the variables were unconstrained. That is symmetry was not (and could not be) imposed on variables.

6.1. Notational Tedium. Often we shall think of some variables as knowns and others as unknowns. We shall be concerned primarily with a function's properties with respect to unknowns. For example, in function (6.2) when we are mainly concerned about behavior such as convexity of Γ in X, Y we often write $\Gamma(A, D, E, X, Y)$ simply as $\Gamma(X, Y)$. Often we use $\vec{Z}$ to abbreviate all indeterminates which appear, for example, in (6.2) we have $\vec{Z} = \{A, D, E, X, Y\}$. Often we distinguish knowns $\vec{A} = \{A_1, \ldots, A_m\}$ from unknowns $\vec{X} = \{X_1, \ldots, X_k\}$ by writing $\vec{Z} = \{\vec{A}, \vec{X}\}$. Throughout this exposition, letters near the beginning of the alphabet denote knowns, while the letters X, Y stand for unknowns.

We call a noncommutative function $\Gamma(\vec{A}, \vec{X})$ **symmetric** provided that

$$\Gamma(\vec{A}, \vec{X})^T = \Gamma(\vec{A}, \vec{X}) \ \text{ if } \ \vec{A}^T = \vec{A}, \vec{X}^T = \vec{X},$$

or provided that

$$\Gamma(\vec{A}, \vec{X})^T = \Gamma(\vec{A}^T, \vec{X}) \quad \text{if} \quad \vec{X}^T = \vec{X},$$

etc.

6.2. The Geometric Definition of Matrix Convexity. A noncommutative rational symmetric function Γ of $\vec{X} = \{X_1, \ldots, X_k\}$ will be called **geometrically matrix convex** provided that whenever the noncommutative variables $\vec{X}$ are taken to be any matrices of compatible dimensions, then for all scalars $0 \le \alpha \le 1$ we have that

$$\Gamma(\alpha \vec{\mathcal{X}}^1 + (1-\alpha)\vec{\mathcal{X}}^2) \preceq \alpha\Gamma(\vec{\mathcal{X}}^1) + (1-\alpha)\Gamma(\vec{\mathcal{X}}^2).$$

Here $\vec{\mathcal{X}}^1 = \{\mathcal{X}_1^1, \ldots, \mathcal{X}_k^1\}$ are k-tuples and $\vec{\mathcal{X}}^2 = \{\mathcal{X}_1^2, \ldots, \mathcal{X}_k^2\}$ of matrices of compatible dimensions. The function Γ is **strictly geometrically matrix convex** if the inequality is strict for $0 < \alpha < 1$. The reverse inequality characterizes **geometrically matrix concave**.

6.3. The Algebraic Definition of Matrix Convexity. We are all familiar with the fact that for an ordinary (commutative) function f convexity corresponds to positivity of the second derivatives of f. Now we describe a quite practical noncommutative version of this. To emphasize the concreteness of this formalism we list the NCAlgebra command which implements it.

6.3.1. First Derivatives. Conventional convexity of a function can be characterized by the second derivative being positive. As we shall see in Section 6.3.3, this is also the case with "noncommutative convex functions" and so we review a notion of second derivative which is suitable for symbolic computation. We begin with first rather than second derivatives.

Later we study convexity tests, and these are based on derivatives of Γ and their transposes. Directional derivatives of noncommutative rational $\Gamma(\vec{A}, \vec{X})$ with respect to $\vec{X}$ in the direction $\vec{H}$ are defined in the usual way

$$\begin{aligned} D\Gamma(\vec{X})[\vec{H}] &:= \lim_{t \to 0} \frac{1}{t} \left(\Gamma(\vec{X} + t\vec{H}) - \Gamma(\vec{X}) \right) \\ &= \frac{d}{dt} \Gamma(\vec{X} + t\vec{H}) \bigg|_{t=0} . \end{aligned} \tag{6.3}$$

For example, the derivative of Γ in (6.1) with respect to X is

$$D_X \Gamma(X, Y)[H] = H^T A X + X^T A H + H Y X^T + X Y H^T .$$

and the derivative of Γ in (6.2) with respect to Y is

$$D_Y \Gamma(X, Y)[K] = E(K X Y^T + Y X K^T) E^T .$$

It is easy to check that derivatives of symmetric noncommutative rational functions always have the form

$$D\Gamma(X)[H] = sym \left[\sum_{\ell=1}^{k} A_\ell H B_\ell \right]$$

where $sym[R] := R + R^T$.

The noncommutative algebra command to generate the directional derivative $D_X \Gamma(X, Y)[H]$ is:

NCAlgebra Command: DirectionalD[Function Γ, X, H].

6.3.2. Second Derivatives. To obtain sufficient conditions for optimization we must use the second order terms of a Taylor expansion of $\Gamma(\vec{X} + t\vec{H})$ about $t = 0 \in R$.

$$\Gamma(\vec{X} + t\vec{H}) = \Gamma(\vec{X}) + D\Gamma(\vec{X})[\vec{H}]\, t + \vec{\mathcal{H}}\Gamma(\vec{X})[\vec{H}]\, t^2 + \dots$$

Here $\mathcal{H}\Gamma$ denotes the Hessian of Γ and is defined by

$$\mathcal{H}\Gamma(\vec{X})[\vec{H}] := \frac{d^2}{dt^2} \Gamma(\vec{X} + t\vec{H}) \bigg|_{t=0} .$$

One can easily show that the second derivative of a hereditary symmetric noncommutative rational function Γ with respect to one variable X has the form

$$\mathcal{H}\Gamma(X)[H] = sym \left[\sum_{\ell=1}^{k} A_\ell H^T B_\ell H C_\ell \right] .$$

An analogous more general expression holds for more variables. For example, the second derivative of Γ in (6.2) with respect to X is

$$\mathcal{H}_X\Gamma(X,Y)[H] =$$
$$2\,(A(I+DXD^T)^{-1}DHD^T(I+DXD^T)^{-1}DHD^T(I+DXD^T)^{-1}A^T).$$

Once the Hessian $\mathcal{H}\Gamma(\vec{X})[\vec{H}]$ is computed, the only variable of interest is $\vec{H}$. Thus, for convenience, the variables $\vec{X}$ and $\vec{A}$ are gathered in $\vec{Z}$, producing a function $\mathcal{Q}$,

$$\mathcal{Q}(\vec{Z})[\vec{H}] := \mathcal{H}\Gamma(\vec{X})[\vec{H}],$$

which is quadratic in $\vec{H}$. Here of course, a noncommutative polynomial in variables $H_1, H_2, \ldots, H_k$ is said to be **quadratic** if each monomial in the polynomial expression is of order two in the variables $H_1, H_2, \ldots, H_k$.

We emphasize that for our convexity considerations, once the Hessian is computed, the fact that $\vec{X}$ played a special role has no influence.

NCAlgebra Command: Hessian[function Γ, $\{X_1, H_1\}, \ldots, \{X_k, H_k\}$].

6.3.3. The Second Derivative Definition of Matrix Convexity. The function $\Gamma(\vec{A},\vec{X})$ is said to be **matrix convex** with respect to variable $\vec{X}$ provided its Hessian $\mathcal{H}\Gamma(\vec{\mathfrak{X}})[\vec{\mathcal{H}}]$ is a positive semidefinite matrix for all square matrices $\vec{\mathcal{A}}, \vec{\mathfrak{X}}$ and all square matrices $\vec{\mathcal{H}}$.

6.4. Matrix Convex and Geometrically Matrix Convex are Equivalent. Both the definitions, matrix convex and geometrically matrix convex, are equivalent provided that the domain of the function Γ is a convex set, as stated by the following lemma.

LEMMA 6.1. *Suppose Γ is a noncommutative rational symmetric function. Then it is geometrically matrix convex (respectively geometrically matrix concave) on a convex region Ω of matrices of fixed sizes if and only if*

$$\mathcal{H}\Gamma(\vec{\mathfrak{X}})[\vec{\mathcal{H}}] \geq 0$$

(respectively ≤ 0) for all $\vec{\mathcal{H}}$ and $\vec{\mathfrak{X}} \in \Omega$.

Proof. The proof is given in [HMer98] where Ω is all matrices of a given size. It extends in a straight forward way to Ω which are convex sets. $\square$

6.5. Determining If Γ is Matrix Convex Everywhere. If someone gives us a symmetric noncommutative function Γ we can determine by computer algebra, if Γ is matrix convex everywhere, simply by computing its symbolic Hessian which is a quadratic $Q(\vec{Z})[\vec{H}]$ in $\vec{H}$ and then checking if the noncommutative function Q is matrix positive as we discussed in Section 3.

We may conclude that our theory of matrix positivity pays off not just in and of itself, but also for determining matrix convexity.

6.6. Regions of Matrix Convexity. Functions are seldom convex everywhere, so it behooves us to compute regions where a rational function Γ is "matrix convex". The paper [CHSYprept] contains quite a satisfying theory along these lines. It provides an algorithm, actually implemented in NCAlgebra. Given Γ the algorithm produces a set of "symbolic inequalities" which determines a domain $\mathcal{G}$ of matrix convexity for Γ. Moreover, the domain produced by the algorithm is the largest such domain (in a natural sense). The fact that the domain is the "largest" requires a substantial proof.

6.6.1. Some Examples. The rigorous setup takes a while to describe, so we shall begin with some formal examples of what our algorithm produces. In fact we introduce our method with an example of an NCAlgebra command (which embodies it). While this is a bit unusual, since it uses terms which have not been formally introduced, we find most people understand the example anyway and the example eases the many pages of definitions and constructions that the reader must endure before getting to the rewards of the method in [CHSYprept]. The command for finding the region of convexity is

$$\textbf{NCConvexityRegion}[\text{Function } \Gamma,\ \vec{X}\ ].$$

When we input a noncommutative rational function $\Gamma(\vec{Z})$ of $\vec{Z} = \{A_1, \ldots, A_m, X_1, \ldots, X_k\}$ this command outputs a family of inequalities which determine a domain $\mathcal{G}$ of $\vec{Z}$ on which Γ is "matrix convex" in $\vec{X} = \{X_1, \ldots, X_k\}$. This is illustrated by the next two examples.

Example 1

Suppose one wishes to determine the domain of convexity (concavity) with respect to X, Y of the following function on matrices $\vec{Z} = \{A, B, R, X, Y\}$:

$$F(\vec{Z}) = -(Y + A^T XB)(R + B^T XB)^{-1}(Y + B^T XA) + A^T XA,$$

where $X = X^T$ and $Y = Y^T$. We treat A, B, R, X, Y symbolically as noncommutative indeterminates and apply the command **NCConvexity-Region**[F, $\{X, Y\}$] which outputs the list

$$\{-2\,(R + B^T XB)^{-1},\ 0,\ 0,\ 0\}.$$

From this output, we conclude that whenever A, B, R, X, and Y are matrices of compatible dimension, the function F is "matrix concave" in X, Y on the domain $\mathcal{G}$ given by

$$\mathcal{G} := \{(X, Y) : (R + B^T XB)^{-1} > 0\}.$$

The command NCConvexityRegion also has an important feature which for this problem assures us that the "closure" of $\mathcal{G}$, in a certain sense, is the "biggest domain of matrix concavity" for F.

Example 2

Let $X = X^T$ and $Y = Y^T$, and define the function F as

$$F(X, Y) = (X - Y^{-1})^{-1}.$$

The output of **NCConvexityRegion**$[F, \{X, Y\}]$ is

$$\{(X - Y^{-1})^{-1}, \quad Y^{-1}, \quad 0\}.$$

Thus the function F is "matrix convex" on the region

$$\mathcal{G} := \{(X, Y) : Y^{-1} > 0 \quad \text{and} \quad (X - Y^{-1})^{-1} > 0\},$$

whenever the symbolic elements X and Y are substituted by any matrices of compatible dimension. Of course $\mathcal{G}$ is the same as $\{(X, Y) : Y > 0$ and $X - Y^{-1} > 0\}$. Again, our algorithm guarantees that the "closure" of $\mathcal{G}$ in a certain sense is the "biggest domain of matrix convexity" of F.

6.6.2. Representing Noncommutative Quadratics. We leave this exposition in an incomplete state rather than go into the fairly obvious but long definitions of what a "matrix convex function on a Symbolic Inequality Domain" means. Instead we indicate what type of mathematics is involved in proving that the domain we found is, up to a certain type of closure, the largest possible.

The core of the matter is a certain representation for noncommutative rational functions $\mathcal{Q}(Z_1, \ldots, Z_v, H_1, \ldots, H_k)$ which are quadratic functions of $\vec{H} = \{H_1, \ldots, H_k\}$. Our concern is describing the region $\mathcal{G}$ of $\vec{Z} = \{Z_1, \ldots, Z_v\}$ on which $\mathcal{Q}$ is "matrix positive" in $\vec{H}$. What we show under reasonable hypotheses is that $\mathcal{Q}$ *has a weighted sum of squares decomposition*

$$\mathcal{Q}(\vec{Z}, \vec{H}) := \sum_{j=1}^{r} L_j(\vec{Z}, \vec{H})^T D_j(\vec{Z}) L_j(\vec{Z}, \vec{H})$$

with L_j, D_j rational and L_j linear in $\vec{H}$, such that formal inequalities involving the D_j determine a set

$$\mathcal{G} := \{\vec{Z} : D_j(\vec{Z}) > 0, \ j = 1, \ldots, r\}$$

on which $\mathcal{Q}$ is "matrix positive" in $\vec{H}$. Moreover, a certain "closure" of $\mathcal{G}$ is the largest such set. The precise statement of this result is Theorem 8.2 of [CHSYprept] and a weaker more accessible result is Theorem 3.1 of [CHSYprept]. This is something of a Positivestellensatz for noncommutative quadratics Q, a very special class of functions, but the conclusions are more refined in that they give precisely the "set of positivity" of Q.

REFERENCES

[deAprept] J. D. D'Angelo. *Proper holomorphic mappings, positivity conditions, and isometric embedding.* preprint.

[BV96] J. A. Ball and V. Vinnikov, *Zero-pole interpolation for meromorphic matrix functions on an algebraic curve and transfer functions of 2D systems*, Acta Appl. Math. **45** (1996), 239–316.

[BV99] ———, *Zero-pole interpolation for meromorphic matrix functions on a compact Riemann surface and a matrix Fay trisecant identity*, Amer. J. Math. **121** (1999), 841–888.

[BEFB94] S. Boyd, L El Ghaoui, E. Feron, and V. Balakrishna, *Linear Matrix Inequalities In System and Control Theory* , SIAM, 1994, p.193.

[CHSY00] J. Camino, J.W. Helton, R.E. Skelton, and Jeiping Ye, *A Symbolic Algorithm For Determining Convexity of A Matrix Function: How To Get Schur Complements Out Of Your Life*, Conf. on Decision and Control, Sydney 2000.

[CHSYprept] J. Camino, J.W. Helton, R.E. Skelton, and Jeiping Ye, www.math.ucsd.edu/∼helton.

[ESLSL00] El Ghaoui, Laurent and Niculescu, Siliviu-Lilian eds. Advances in Linear Matrix Inequality Methods in Control, Adv. Des. and Control, SIAM, 2000

[HMer98] J. W. Helton and O. Merino, Sufficient Conditions for Optimization of Matrix Functions, pp. 1-5, Conf on Dec. and Control 98.

[HVprept] J. W. Helton and V.Vinnikov Linear Matrix Inequality Representation of Sets prept.

[Mprept] S. McCullough. *Factorization of operator valued polynomials in several non-commuting variables.* preprint, January 2000.

[P00] P. Parrilo, *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*, PhD Thesis Cal Tech may 2000 Available at http;//www.cds.caltech.edu/ pablo

[PSprept] Pablo A. Parrilo and Bernd Sturmfels, *Minimizing Polynomial Functions* preprint pp. 16

[PW98] V. Powers and T. Wormann, *An algorithm for sums of squares of real polynomials*, Journal of Pure and Applied Algebra 127, 1998, p. 99-104.

[R00] B. Reznick, *Some Concrete Aspects of Hilbert's 17th Problem.* Real Algebraic Geometry and Ordered Structures, C. Delzell and J. Madden, editors, Cont. Math 253, pp.251-272, 2000.

[S87] N. Z. Shor, Class of Global minimum bounds of polynomial functions, Cybernetics 23 (1987), no. 6, 731- 734 (Russian original Kibernetica, num 6 1987, 9-11)

[V89] V. Vinnikov, *Complete description of determinantal representations of smooth irreducible curves*, Linear Algebra Appl. **125** (1989), 103–140.

[V93] ———, *Self-adjoint determinantal represention of real plane curves*, Math. Ann. **296** (1993), 453–479.