Math 142B
Final Examination
June 15, 2012
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Instructions

1. **You may use any type of calculator, but no other electronic devices during this exam.**
2. **You may use one page of notes, but no books or other assistance during this exam.**
3. **Write your *Name*, *PID*, and *Section* on the front of your Blue Book.**
4. **Write your solutions clearly in your Blue Book**
   (a) **Carefully indicate the number and letter of each question and question part.**
   (b) **Present your answers in the same order they appear in the exam.**
   (c) **Start each question on a new side of a page.**
5. **Read each question carefully, and answer each question completely.**
6. **Show all of your work; no credit will be given for unsupported answers.**

1. (6 points) Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ -1 & \text{if } x \text{ is irrational.} \end{cases}$$

   (a) Find the lower integral $\underline{\int_0^1} f$.
   (b) Find the upper integral $\overline{\int_0^1} f$.
   (c) Is f integrable on $[0,1]$? If f is integrable, find $\int_0^1 f$; if f is not integrable, explain why it is not.

2. (6 points) Consider the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \sin\left(\frac{\pi}{x}\right) & \text{if } 0 < x \le 1. \end{cases}$
   Determine whether or not f is integrable on $[0,1]$ and justify your answer.

3. (8 points) Suppose that $g : [a,b] \to \mathbb{R}$ and $h : [a,b] \to \mathbb{R}$ are bounded functions with $g(x) \le h(x)$ for all x in $[a,b]$.
   (a) Show that $\underline{\int_a^b} g \le \underline{\int_a^b} h$.
   (b) Show that $\overline{\int_a^b} g \le \overline{\int_a^b} h$.
   (c) If $g : [a,b] \to \mathbb{R}$ and $h : [a,b] \to \mathbb{R}$ are also integrable, show that $\int_a^b g \le \int_a^b h$.

4. (8 points) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sin(x)$.
   (a) Determine a formula for the n^{th} Taylor polynomial $p_n(x)$ for $f(x)$ at $x_0 = \frac{\pi}{2}$.
   (b) Show that $\lim_{n\to\infty} p_n(x) = f(x)$; that is, show that $\lim_{n\to\infty} [f(x) - p_n(x)] = 0$.

5. (8 points) For each index n, define $f_n(x) = x^n$ for $0 \le x \le 1$.

   (a) Show that $\{f_n\}$ converges *pointwise* to

   $$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

   (b) Show that $\{f_n\}$ does *not* converge *uniformly* to $f(x)$.

6. (6 points) Let $\{f_n : [a,b] \to \mathbb{R}\}$ be a sequence of monotonically increasing functions that converge pointwise to a function $f : [a,b] \to \mathbb{R}$. Show that f is monotonically increasing on $[a,b]$.

7. (6 points) For each index n, let $f_n(x) = \sum_{k=0}^{n} c_k x^k$. Given $r > 0$, suppose that $\{f_n\}$ converges on $(-r,r)$ to $f : (-r,r) \to \mathbb{R}$. Show that for every interval $[a,b]$ contained in $(-r,r)$, $\int_a^b f = \sum_{k=0}^{\infty} \frac{c_k}{k+1}(b^{k+1} - a^{k+1})$. (Hint: Use the result that if $\{f_n\}$ is a sequence of integrable functions that converges uniformly on $[a,b]$ to a function f, then f is integrable and $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$.)

8. (8 points) For each index n, define $f_n(x) = \frac{1}{n}\tan^{-1}(n^2 x)$ for every real x.

   (a) Show that $\{f_n\}$ converges uniformly to a function $f : \mathbb{R} \to \mathbb{R}$.

   (b) Show that $f_n(x)$ is differentiable for every index n.

   (c) Show that $\{f_n'(0)\}$ is unbounded.