Spherical Coordinates

Cylindrical coordinates are related to rectangular coordinates as follows.

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2} & x &= r\sin\phi\cos\theta \\ \cos\phi &= \frac{z}{\sqrt{x^2+y^2+z^2}} & y &= r\sin\phi\sin\theta \\ \tan\theta &= \frac{y}{x} & z &= r\cos\phi \end{aligned}$$

The spherical coordinate vectors are defined as

$$\begin{aligned} \mathbf{e_r} &:= \frac{1}{|\nabla r|}\nabla r \\ \mathbf{e}_\phi &:= \frac{1}{|\nabla \phi|}\nabla \phi \\ \mathbf{e}_\theta &:= \frac{1}{|\nabla \theta|}\nabla \theta \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{e_r} &= \frac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k} \\ \mathbf{e}_\phi &= \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\,\mathbf{i} + \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\,\mathbf{j} - \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k} \\ \mathbf{e}_\theta &= -\frac{y}{\sqrt{x^2+y^2}}\,\mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\,\mathbf{j} \end{aligned}$$

In terms of r, ϕ, and θ, this becomes

$$\begin{aligned} \mathbf{e_r} &= \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k} \\ \mathbf{e}_\phi &= \cos\phi\cos\theta\,\mathbf{i} + \cos\phi\sin\theta\,\mathbf{j} - \sin\phi\,\mathbf{k} \\ \mathbf{e}_\theta &= -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} \end{aligned}$$

The inverse relationship is as follows.

$$\begin{aligned} \mathbf{i} &= \sin\phi\cos\theta\,\mathbf{e_r} + \cos\phi\cos\theta\,\mathbf{e}_\phi - \sin\theta\,\mathbf{e}_\theta \\ \mathbf{j} &= \sin\phi\sin\theta\,\mathbf{e_r} + \cos\phi\sin\theta\,\mathbf{e}_\phi + \cos\theta\,\mathbf{e}_\theta \\ \mathbf{k} &= \cos\phi\,\mathbf{e_r} - \sin\phi\,\mathbf{e}_\phi \end{aligned}$$

It is worth noting that the above computations also imply the following.

$$\frac{\partial r}{\partial x} = \sin\phi\cos\theta \qquad \frac{\partial \phi}{\partial x} = \frac{\cos\phi\cos\theta}{r} \qquad \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r\sin\phi}$$
$$\frac{\partial r}{\partial y} = \sin\phi\sin\theta \qquad \frac{\partial \phi}{\partial y} = \frac{\cos\phi\sin\theta}{r} \qquad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r\sin\phi}$$
$$\frac{\partial r}{\partial z} = \cos\phi \qquad \frac{\partial \phi}{\partial z} = -\frac{\sin\phi}{r}$$

The position vector $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is written

$$\mathbf{R} = r\,\mathbf{e_r}. \quad \text{(spherical coordinates)}$$

If $\mathbf{R} = \mathbf{R}(t)$ is a parameterized curve, then $\frac{d\mathbf{R}}{dt} = \frac{dr}{dt}\,\mathbf{e_r} + r\,\frac{d\mathbf{e_r}}{dt}$. Since $\mathbf{e_r} = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$, we have $\frac{d\mathbf{e_r}}{dt} = \frac{d\phi}{dt}\,(\cos\phi\cos\theta\,\mathbf{i} + \cos\phi\sin\theta\,\mathbf{j} - \sin\phi\,\mathbf{k}) + \sin\phi\frac{d\theta}{dt}\,(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}) = \frac{d\phi}{dt}\,\mathbf{e}_\phi + \sin\phi\frac{d\theta}{dt}\,\mathbf{e}_\theta$. Thus,

$$\frac{d\mathbf{R}}{dt} = \frac{dr}{dt}\,\mathbf{e_r} + r\frac{d\phi}{dt}\,\mathbf{e}_\phi + r\sin\phi\frac{d\theta}{dt}\,\mathbf{e}_\theta.$$

Hence, $d\mathbf{R} = dr\,\mathbf{e_r} + r\,d\phi\,\mathbf{e}_\phi + r\sin\phi\,d\theta\,\mathbf{e}_\theta$ and it follows that the element of volume in spherical coordinates is given by

$$dV = r^2\sin\phi\,dr\,d\phi\,d\theta$$

If $f = f(x, y, z)$ is a scalar field (that is, a real-valued function of three variables), then

$$\nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}.$$

If we view x, y, and z as functions of r, ϕ, and θ and apply the chain rule, we obtain

$$\nabla f = \left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}\right)\mathbf{i} + \left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}\right)\mathbf{j} + \left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial z}\right)\mathbf{k}$$

Writing this in terms of r, ϕ, θ, and the spherical coordinate vectors yields

$$\begin{aligned}\nabla f = &\left(\sin\phi\cos\theta\frac{\partial f}{\partial r} + \frac{\cos\phi\cos\theta}{r}\frac{\partial f}{\partial \phi} - \frac{\sin\theta}{r\sin\phi}\frac{\partial f}{\partial \theta}\right)(\sin\phi\cos\theta\,\mathbf{e_r} + \cos\phi\cos\theta\,\mathbf{e}_\phi - \sin\theta\,\mathbf{e}_\theta)\\ &+ \left(\sin\phi\sin\theta\frac{\partial f}{\partial r} + \frac{\cos\phi\sin\theta}{r}\frac{\partial f}{\partial \phi} + \frac{\cos\theta}{r\sin\phi}\frac{\partial f}{\partial \theta}\right)(\sin\phi\sin\theta\,\mathbf{e_r} + \cos\phi\sin\theta\,\mathbf{e}_\phi + \cos\theta\,\mathbf{e}_\theta)\\ &+ \left(\cos\phi\frac{\partial f}{\partial r} - \frac{\sin\phi}{r}\frac{\partial f}{\partial \phi}\right)(\cos\phi\,\mathbf{e_r} - \sin\phi\,\mathbf{e}_\phi)\end{aligned}$$

Simplifying, we obtain the result

$$\nabla f = \frac{\partial f}{\partial r}\,\mathbf{e_r} + \frac{1}{r}\frac{\partial f}{\partial \phi}\,\mathbf{e}_\phi + \frac{1}{r\sin\phi}\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta$$

If $\mathbf{F} = \mathbf{F}(x, y, z)$ is a vector field (that is, a vector-valued function of three variables), then we can write

$$\begin{aligned}\mathbf{F} &= F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}\\ &= (\sin\phi\cos\theta\,F_1 + \sin\phi\sin\theta\,F_2 + \cos\phi\,F_3)\,\mathbf{e_r} + (\cos\phi\cos\theta\,F_1 + \cos\phi\sin\theta\,F_2 - \sin\phi\,F_3)\,\mathbf{e}_\phi\\ &\qquad + (-\sin\theta\,F_1 + \cos\theta\,F_2)\,\mathbf{e}_\theta\end{aligned}$$

Thus, $\mathbf{F} = F_r\,\mathbf{e_r} + F_\phi\,\mathbf{e}_\phi + F_\theta\,\mathbf{e}_\theta$, where

$$\begin{aligned}F_r &= \sin\phi\cos\theta\,F_1 + \sin\phi\sin\theta\,F_2 + \cos\phi\,F_3 & F_1 &= \sin\phi\cos\theta\,F_r + \cos\phi\cos\theta\,F_\phi - \sin\theta\,F_\theta\\ F_\phi &= \cos\phi\cos\theta\,F_1 + \cos\phi\sin\theta\,F_2 - \sin\phi\,F_3 & F_2 &= \sin\phi\sin\theta\,F_r + \cos\phi\sin\theta\,F_\phi + \cos\theta\,F_\theta\\ F_\theta &= -\sin\theta\,F_1 + \cos\theta\,F_2 & F_3 &= \cos\phi\,F_r - \sin\phi\,F_\phi\end{aligned}$$

We compute the following.

$$\begin{aligned}\frac{\partial F_1}{\partial r} &= \sin\phi\cos\theta\,\frac{\partial F_r}{\partial r} + \cos\phi\cos\theta\,\frac{\partial F_\phi}{\partial r} - \sin\theta\,\frac{\partial F_\theta}{\partial r}\\ \frac{\partial F_2}{\partial r} &= \sin\phi\sin\theta\,\frac{\partial F_r}{\partial r} + \cos\phi\sin\theta\,\frac{\partial F_\phi}{\partial r} + \cos\theta\,\frac{\partial F_\theta}{\partial r}\\ \frac{\partial F_3}{\partial r} &= \cos\phi\,\frac{\partial F_r}{\partial r} - \sin\phi\,\frac{\partial F_\phi}{\partial r}\\ \frac{\partial F_1}{\partial \phi} &= \cos\phi\cos\theta\,F_r + \sin\phi\cos\theta\,\frac{\partial F_r}{\partial \phi} - \sin\phi\cos\theta\,F_\phi + \cos\phi\cos\theta\,\frac{\partial F_\phi}{\partial \phi} - \sin\theta\,\frac{\partial F_\theta}{\partial \phi}\\ \frac{\partial F_2}{\partial \phi} &= \cos\phi\sin\theta\,F_r + \sin\phi\sin\theta\,\frac{\partial F_r}{\partial \phi} - \sin\phi\sin\theta\,F_\phi + \cos\phi\sin\theta\,\frac{\partial F_\phi}{\partial \phi} + \cos\theta\,\frac{\partial F_\theta}{\partial \phi}\\ \frac{\partial F_3}{\partial \phi} &= -\sin\phi\,F_r + \cos\phi\,\frac{\partial F_r}{\partial \phi} - \cos\phi\cos\theta\,F_\phi - \sin\phi\,\frac{\partial F_\phi}{\partial \phi}\\ \frac{\partial F_1}{\partial \theta} &= -\sin\phi\sin\theta\,F_r + \sin\phi\cos\theta\,\frac{\partial F_r}{\partial \theta} - \cos\phi\sin\theta\,F_\phi + \cos\phi\cos\theta\,\frac{\partial F_\phi}{\partial \theta} - \cos\theta\,F_\theta - \sin\theta\,\frac{\partial F_\theta}{\partial \theta}\\ \frac{\partial F_2}{\partial \theta} &= \sin\phi\cos\theta\,F_r + \sin\phi\sin\theta\,\frac{\partial F_r}{\partial \theta} + \cos\phi\cos\theta\,F_\phi + \cos\phi\sin\theta\,\frac{\partial F_\phi}{\partial \theta} - \sin\theta\,F_\theta + \cos\theta\,\frac{\partial F_\theta}{\partial \theta}\\ \frac{\partial F_3}{\partial \theta} &= \cos\phi\,\frac{\partial F_r}{\partial \theta} - \sin\phi\,\frac{\partial F_\phi}{\partial \theta}\end{aligned}$$

Now we can transform $\nabla\cdot\mathbf{F}$ and $\nabla\times\mathbf{F}$ into spherical coordinates. To transform $\nabla\cdot\mathbf{F}$, we compute as follows.

$$\begin{aligned}\nabla\cdot\mathbf{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\\ &= \left(\frac{\partial F_1}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial F_1}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial F_1}{\partial \theta}\frac{\partial \theta}{\partial x}\right) + \left(\frac{\partial F_2}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial F_2}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial F_2}{\partial \theta}\frac{\partial \theta}{\partial y}\right) + \left(\frac{\partial F_3}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial F_3}{\partial \phi}\frac{\partial \phi}{\partial z}\right)\\ &= \left(\sin\phi\cos\theta\,\frac{\partial F_1}{\partial r} + \frac{\cos\phi\cos\theta}{r}\,\frac{\partial F_1}{\partial \phi} - \frac{\sin\theta}{r\sin\phi}\,\frac{\partial F_1}{\partial \theta}\right)\\ &\qquad + \left(\sin\phi\sin\theta\,\frac{\partial F_2}{\partial r} + \frac{\cos\phi\sin\theta}{r}\,\frac{\partial F_2}{\partial \phi} + \frac{\cos\theta}{r\sin\phi}\,\frac{\partial F_2}{\partial \theta}\right)\\ &\qquad\quad + \left(\cos\phi\,\frac{\partial F_3}{\partial r} - \frac{\sin\phi}{r}\,\frac{\partial F_3}{\partial \phi}\right)\end{aligned}$$

After writing the partial derivatives of F_1, F_2, and F_3 in terms of F_r, F_ϕ, F_θ, and their partial derivatives and simplifying, we obtain

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\phi}\frac{\partial}{\partial \phi}\left(F_\phi \sin\phi\right) + \frac{1}{r\sin\phi}\frac{\partial F_\theta}{\partial \theta}$$

$\nabla \times \mathbf{F}$ is handled similarly.

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k} \\
&= \left[\left(\frac{\partial F_3}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial F_3}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial F_3}{\partial \theta}\frac{\partial \theta}{\partial y}\right) - \left(\frac{\partial F_2}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial F_2}{\partial \phi}\frac{\partial \phi}{\partial z} + \frac{\partial F_2}{\partial \theta}\right)\right]\mathbf{i} \\
&\quad + \left[\left(\frac{\partial F_1}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial F_1}{\partial \phi}\frac{\partial \phi}{\partial z}\right) - \left(\frac{\partial F_3}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial F_3}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial F_3}{\partial \theta}\frac{\partial \theta}{\partial x}\right)\right]\mathbf{j} \\
&\quad\quad + \left[\left(\frac{\partial F_2}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial F_2}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial F_2}{\partial \theta}\frac{\partial \theta}{\partial x}\right) - \left(\frac{\partial F_1}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial F_1}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial F_1}{\partial \theta}\frac{\partial \theta}{\partial y}\right)\right]\mathbf{k} \\
&= \left[\left(\sin\phi\sin\theta\,\frac{\partial F_3}{\partial r} + \frac{\cos\phi\sin\theta}{r}\frac{\partial F_3}{\partial \phi} + \frac{\cos\theta}{r\sin\phi}\frac{\partial F_3}{\partial \theta}\right) - \left(\cos\phi\,\frac{\partial F_2}{\partial r} - \frac{\sin\phi}{r}\frac{\partial F_2}{\partial \phi}\right)\right]\mathbf{i} \\
&\quad + \left[\left(\cos\phi\,\frac{\partial F_1}{\partial r} - \frac{\sin\phi}{r}\frac{\partial F_1}{\partial \phi}\right) - \left(\sin\phi\cos\theta\,\frac{\partial F_3}{\partial r} + \frac{\cos\phi\cos\theta}{r}\frac{\partial F_3}{\partial \phi} - \frac{\sin\theta}{r\sin\phi}\frac{\partial F_3}{\partial \theta}\right)\right]\mathbf{j} \\
&\quad\quad + \left[\left(\sin\phi\cos\theta\,\frac{\partial F_2}{\partial r} + \frac{\cos\phi\cos\theta}{r}\frac{\partial F_2}{\partial \phi} - \frac{\sin\theta}{r\sin\phi}\frac{\partial F_2}{\partial \theta}\right)\right. \\
&\quad\quad\quad \left. - \left(\sin\phi\sin\theta\,\frac{\partial F_1}{\partial r} + \frac{\cos\phi\sin\theta}{r}\frac{\partial F_1}{\partial \phi} + \frac{\cos\theta}{r\sin\phi}\frac{\partial F_1}{\partial \theta}\right)\right]\mathbf{k}
\end{aligned}$$

After writing the partial derivatives of F_1, F_2, and F_3 in terms of F_r, F_ϕ, F_θ, and their partial derivatives and writing $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ in terms of $\mathbf{e_r}$, $\mathbf{e}_\phi$, and $\mathbf{e}_\theta$ and simplifying, we obtain

$$\nabla \times \mathbf{F} = \frac{1}{r^2\sin\phi}\begin{vmatrix} \mathbf{e_r} & r\mathbf{e}_\phi & r\sin\phi\mathbf{e}_\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ F_r & rF_\phi & r\sin\phi F_\theta \end{vmatrix}$$