**The Goat and Silo Problem**

**Problem.** A goat is tethered to a cylindrical silo of radius $r$ with a rope of length $l$, $l \leq \pi r$. (See Figure 1.) Find the area of the region the goat can graze.

**Solution.** The area the goat can graze is $\frac{\pi l^2}{2}$ plus twice the "evolute area" (see Figure 2). To find the evolute area, we evaluate the line integral $\frac{1}{2}\int_C -y\,dx + x\,dy$ along its boundary curve $C$. To do this, we observe that $C = C_1 \cup C_2 \cup C_3$, where $C_1$ is the circular arc along the silo, $C_2$ is the line tangent to the silo at the point where the goat is tethered, and $C_3$ is the evolute (Figure 2). The curves $C_1$, $C_2$, and $C_3$ can be parametrized as follows. (Note that the parametrizations describe a counterclockwise path.)

$$\begin{aligned} C_1 : \mathbf{r}(t) &= r\left(\cos\left(\frac{l}{r}-t\right), \sin\left(\frac{l}{r}-t\right)\right), \\ C_2 : \mathbf{r}(t) &= r\,(1,t)\ , \\ C_3 : \mathbf{r}(t) &= r\left(\cos t - \left(\frac{l}{r}-t\right)\sin t, \sin t + \left(\frac{l}{r}-t\right)\cos t\right), \end{aligned}$$

where $t \in [0, \frac{l}{r}]$. By additivity, the line integral around $C$ is the sum of the line integrals along $C_1$, $C_2$, and $C_3$.

$$\begin{aligned} \frac{1}{2}\int_{C_1} -y\,dx + x\,dy &= \frac{r^2}{2}\int_0^{\frac{l}{r}} \left[-\sin^2\left(\frac{l}{r}-t\right) - \cos^2\left(\frac{l}{r}-t\right)\right] dt \\ &= -\frac{r^2}{2}\int_0^{\frac{l}{r}} dt \\ &= -\frac{lr}{2} \\ \frac{1}{2}\int_{C_2} -y\,dx + x\,dy &= \frac{r^2}{2}\int_0^{\frac{l}{r}} dt \\ &= \frac{lr}{2} \\ \frac{1}{2}\int_{C_3} -y\,dx + x\,dy &= \frac{r^2}{2}\int_0^{\frac{l}{r}} \left\{\left[\sin t + \left(\frac{l}{r}-t\right)\cos t\right]\left(\frac{l}{r}-t\right)\cos t - \right. \\ &\qquad \left. \left[\cos t - \left(\frac{l}{r}-t\right)\sin t\right]\left(\frac{l}{r}-t\right)\sin t\right\} dt \\ &= \frac{r^2}{2}\int_0^{\frac{l}{r}} \left(\frac{l}{r}-t\right)^2 dt \\ &= \frac{l^3}{6r} \\ \frac{1}{2}\int_C -y\,dx + x\,dy &= -\frac{lr}{2} + \frac{lr}{2} + \frac{l^3}{6r} = \frac{l^3}{6r} \end{aligned}$$

Thus, the total area $A$ the goat can graze is given by

$$A = \frac{\pi l^2}{2} + 2\frac{l^3}{6r} = \frac{\pi l^2}{2} + \frac{l^3}{3r}\,.$$

The reason for the restriction $l \leq \pi r$ is illustrated in Figure 3. For another approach to this problem, see

M. E. Hoffman, The Bull and the Silo: An Application of Curvature, *American Mathematical Monthly*, January 1998, pages 55-58.

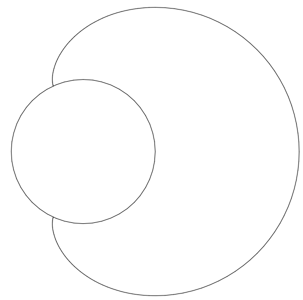

Figure 1: Goat's Grazing Area

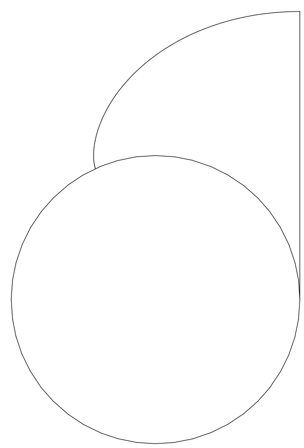

Figure 2: Evolute Area

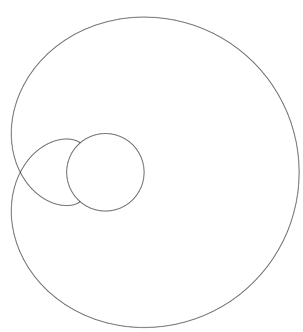

Figure 3: Goat's Grazing Area: $l > \pi r$