

Magnification relations in gravitational lensing via multidimensional residue integrals

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We investigate the so-called magnification relations of gravitational lensing models. We show that multidimensional residue integrals provide a simple explanation for the existence of these relations, and an effective method of computation. We illustrate the method with several examples, thereby deriving new magnification relations for galaxy lens models and microlensing (point mass lensing).

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I. INTRODUCTION

Gravitational lensing has proven to be not only important astrophysically, but intriguing mathematically as well. Mathematical investigations of gravitational lens theory have yielded important results and insights,^{1,2,3,4,5,6} employing techniques and results from such disparate areas as catastrophe theory, differential geometry and Morse theory. In this paper, we illustrate how another seemingly unrelated subject, multidimensional residue calculus, applies to gravitational lensing, and specifically we explain the origin of certain “magnification relations” that have been discussed in the lensing literature.^{7,8,9,10} We additionally demonstrate that calculations of these magnification relations are enormously simplified using residue techniques, and illustrate the method by deriving several new results.

This paper is organized as follows. In the remainder of this section, we introduce the relevant terminology of gravitational lensing, and describe the magnification relations. In Section II, we express the problem in terms of residue calculations at the image positions, and thereby relate it to a residue at infinity. Using one-dimensional residue calculus, we derive trace formulas for the magnification relations for a subset of lens models. In Section III, we consider the general class of lens models, and describe how to perform the necessary multidimensional residue integrals. In Section IV, we apply this formalism and derive previously known results, as well as new results. In Section V, we summarize our results and discuss implications. The material discussed in sections II-IV may be unfamiliar to astronomers, and so we provide a simple procedure which may be applied to models to obtain their magnification relations, without requiring a detailed understanding of the underlying mathematics.

A Gravitational lensing terminology

Numerous excellent introductions to gravitational lensing have been written, e.g. Refs.[1,2]; here we briefly summarize some of the results and terminology relevant to our discussion. The effects of gravitational lensing can perhaps best be understood by considering the time delay of trajectories connecting the lensed source to the observer.¹ The time delay is a simple sum of two terms, a “geometrical” piece, and a “gravitational” piece. Let ϕ_N be the 3-D Newtonian potential, and $\psi = (2/Dc^2) \int \phi_N dl$ the projected 2-D potential, where D is a function of cosmology and the source and lens redshifts. Just as $\nabla^2 \phi_N = 4\pi G\rho$, similarly $\nabla_{\perp}^2 \psi \propto 4\pi G\Sigma$, where $\Sigma = \int \rho dl$ is the surface density. In the thin-screen, small deflection limit, the time delay can be written as

$$\tau = \tau_0 \left(\frac{1}{2} |\vec{\Theta} - \vec{\beta}|^2 - \psi \right), \quad (1)$$

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where the normalization τ_0 depends upon cosmology and the lens and source redshifts, $\vec{\Theta}$ is the image location on the sky, and $\vec{\beta}$ is the location the source would have had, had not the lens intervened. Light rays follow null geodesics, which can be shown to obey Fermat's principle.² Plotting time delay as a function of the two angular coordinates on the sky, Fermat's principle demands that images arise at the stationary points of this time delay surface. Setting the gradient of the time delay to zero gives the so-called "lens equations", the (real) solutions of which are the image positions. Since the time delay surface can have multiple stationary points, multiple images of a single source can arise, and the image multiplicity can depend on the source position relative to the lens. The curves on the source plane separating regions of different image multiplicity are called caustics. As a source crosses over a caustic, its image multiplicity changes by two, as a pair of images either merge together and annihilate, or are created and move apart. The magnification μ , which relates differential area elements of the unlensed source to area elements of the lensed images, is simply the inverse of the Jacobian J of the mapping from image coordinates $\vec{\Theta}$ to source coordinates $\vec{\beta}$. Since the orientation of an image can be inverted relative to the unlensed source, the magnification can have either sign. Much of the lensing literature adopts the convention of positive magnifications (i.e. defining $\mu = 1/|J|$); in this paper we always take the magnification to be signed, i.e. $\mu = 1/J$.

B Lens models

Two astrophysically important types of lenses are compact objects (like stars or MACHOs) and galaxies. The former class are effectively point masses, and so have lensing potentials proportional to the Green's function for the 2-D Laplacian, i.e. $\psi = m \log r$ where m is proportional to the mass of the point lens. Only the weak-field regime is observationally relevant, so the potentials linearly superpose for multiple point masses (as long as they are not appreciably separated along the line of sight). Galaxies have more complicated, extended mass distributions with correspondingly complicated lens potentials. In principle, one may decompose the potential into eigenfunctions of the 2-D Laplacian¹¹

$$\psi = \sum_{m,n} (a_{mn} \cos m\theta + b_{mn} \sin m\theta) r^n. \quad (2)$$

This has the advantage that each term in the expansion relates to a corresponding multipole in the expansion of the surface density. In practice, it is necessary to truncate the series due to the limited observational constraints. Since galaxies are believed to have roughly "isothermal" $\rho \sim r^{-2}$ profiles, such truncated series generally consist of variations on the singular isothermal sphere (SIS) potential $\psi = br$. Two examples considered by Kochanek¹¹ are the SIS + elliptical potential $\psi = br(1 + \gamma \cos 2\theta)$ and the SIS + external shear $\psi = br + \frac{\gamma}{2} r^2 \cos 2\theta$. Another variation¹² is the singular isothermal ellipse (SIE) $\psi = bR = b\sqrt{x^2 + y^2/q^2}$ with axis ratio q . Other, more elaborate and more physically justified models have been employed in the lensing literature; here we will focus on simple models such as the above to avoid obfuscating the general method with heavy algebra. In table I we list the models considered in this paper.

Given a model for the lens, the lensed images of a given source may be found by solving the lens equations as described above. For simple potentials, the solutions are often analytic. For example, consider the the SIS + external shear model. The stationarity equations become

$$\frac{\partial \tau}{\partial r} = r - s \cos(\theta - \theta_s) - b - \gamma r \cos 2\theta = 0, \quad (3)$$

$$\frac{1}{r} \frac{\partial \tau}{\partial \theta} = s \sin(\theta - \theta_s) + \gamma r \sin 2\theta = 0, \quad (4)$$

where (r, θ) is the image position in polar coordinates, and (s, θ_s) is the source position. From this, it is easy to show that the quantity $u = e^{i\theta}$ satisfies the fourth degree polynomial equation

$$\gamma b u^4 + \left(\gamma s v + \frac{s}{v}\right) u^3 - \left(s v + \frac{\gamma s}{v}\right) u + \gamma b = 0, \quad (5)$$

with $v = e^{i\theta_s}$. As a quartic equation in u , this can be solved analytically, and from u the image coordinates (r, θ) follow simply. For many models, it is possible to eliminate all but one variable in a similar fashion and thereby obtain analytic solutions, but for more realistic models, the lens equations must be solved numerically.

C Magnification relations

As mentioned above, there are in general multiple lensed images for a given source. These images lie at the discrete positions satisfying the lens equations for a given source position, and the images have different magnifications and orientations. If we restrict ourselves to purely real solutions of the lens equations (i.e. the physically observable images) then the number of images changes when the source crosses over a caustic. The number of solutions, of course, does not change, but instead a pair of complex solutions become real (for a source crossing into a caustic) or a real pair become complex (for a source crossing out of a caustic).⁵ For certain lens models, such as the simple potentials described above, there may exist certain parameter ranges such that *all* solutions to the lens equations are real. In such cases, it has been shown that there exist interesting and surprising relations among the image positions and magnifications. First, Witt & Mao⁷ considered lensing by a binary microlensing system, involving two point masses, and derived the following result : when the image multiplicity is maximized, the sum of the signed magnifications of all the images is always 1. That is, $\sum_i \mu_i = 1$, independent of quantities such as the lens masses, separation, or source position (as long as the source is inside a caustic). This is quite an astonishing result, as the individual image magnifications can vary wildly as the source position changes, and even diverge as the source crosses a caustic. Witt & Mao derived this result by using resultants to obtain a monic polynomial equation satisfied by μ , and noted that the sum of the roots is given by the subleading coefficient. Rhie⁸, using similar reasoning, showed that a similar result is true for an arbitrary number of point mass lenses. Dalal⁹ extended this work to galaxy potentials like those described above. Again, the magnification relations were obtained by using elimination theory (e.g. Gröbner bases) to obtain monic polynomial equations in μ ; the “total magnification” was then given by the subleading coefficient. We summarize these results in table I.

Subsequently, Witt & Mao¹⁰ showed that for a particular class of power-law models, there exist additional magnification relations, involving not only μ but the image positions as well. They derived this result by separating the coupled lens equations into disjoint x and y equations, and then relating μ to the coefficients. Just as previous work had derived expressions for $\sum_i \mu_i$, Witt & Mao found expressions for $\sum_i \mu_i x_i^k$ and $\sum_i \mu_i y_i^k$. They called these the “ k^{th} moments”, and we adopt their terminology here. For example, they found that the first moment $\sum_i \mu_i \vec{\Theta}_i = 2\vec{\beta}$ for the SIE model.

The general pattern seen in previous work is that both the total magnification and the higher moments can be expressed in terms of the model parameters, and that progressively higher moments have progressively more complicated forms involving more of the parameters. The expressions’ independence of certain parameters suggests some sort of invariant, but clearly not a topological invariant since certain models seemed not to obey any magnification relations whatsoever. Indeed, the origin of the magnification relations and their absence in certain models has been a mystery. In this paper, we provide an explanation of these relations, and additionally find a method easier than elimination theory to derive them.

D Residue integrals

As noted above, the lens equations have multiple discrete solutions, both real and complex. While only the real solutions have physical meaning, it is instructive to consider the complex solutions as well. In this paper, we will henceforth treat the image coordinates $\vec{\Theta}$ as complex variables. We are interested in the sum over these discrete points of various quantities, such as the signed magnification, or magnification times position, etc. From complex analysis, we know that one may relate a sum over discrete points to an integral over a contour encircling those points, by choosing an integrand which has poles at those points. For lensed images, which are stationary points of the time delay, there is an obvious class of integrands, namely rational functions of the form

$$f(x, y) = \frac{g(x, y)}{\partial_x \tau \partial_y \tau}. \quad (6)$$

There are complications, which we discuss later, due to the fact that the integrals here are multidimensional; however, the analogy to the one dimensional case should be clear. We need only find the appropriate function g such that f will have a residue equaling the quantity we wish to sum over the images, and choose a contour large enough to enclose all the images. Now, converting a discrete sum to a contour integral wouldn’t seem to be much progress, however we can use another idea from one dimensional complex analysis. Recall that by inverting coordinates (mapping the origin to infinity and vice versa) one can see that the sum of the residues of poles inside the contour is equal to the sum of the residues of poles outside the contour, but with opposite sign. In our case, we are summing over all the finite solutions, so the only pole outside the contour is at infinity. This is the essence of the method described in this paper: we relate the sum over the images to the behavior of the time delay at infinity, and we simply evaluate the

residue at the point(s) at infinity. The validity of the resulting magnification relations does not depend on the image coordinates being real, although of course their physical applicability does.

In the following section, we further utilize ordinary complex analysis of one variable and derive trace formulas for those models for which it is easy to eliminate all but one variable. In Section III, we consider the general case, and describe the basics of multivariable residue calculus. In Section IV we illustrate with specific examples.

II. TRACE METHODS

In this paper, we shall focus upon models with polynomial lens equations. Algebraic functions, such as n th roots, can be accommodated by introducing an additional variable for each algebraic function along with the polynomial equation it satisfies. For example, an equation containing $\sqrt{x^3+1}$ is handled by introducing z satisfying $z^2 = x^3 + 1$. The method of this section is useful when all but one variable can be conveniently eliminated from the lens equations, e.g. Eq. (5). Of course, this is always possible in principle, but it may require computer implementation of Gröbner basis algorithms in practice. Thus, we assume that the x -coordinates of the images are the roots x_i of a polynomial equation of degree n ,

$$f(x) = \sum_{i=0}^n a_i x^i = 0. \quad (7)$$

We assume that the signed magnification $\mu(x)$ of an image at x is given by a rational function,

$$\mu = \frac{p(x)}{q(x)}, \quad (8)$$

where the denominator has no common roots with $f(x)$. This will necessarily be the case if the coordinates are chosen generically, since there will then be at most one image at a given x -coordinate, whose magnification must be a single-valued algebraic function of x . We wish to calculate the total magnification

$$M = \sum_i \mu(x_i) \equiv \text{Tr } \mu, \quad (9)$$

where the “trace” notation will be explained below. Generically, the roots x_i of $f(x)$ will be distinct; since M is determined by continuity when some roots coincide we will always consider the generic case.

Let $A = \mathbb{C}[x]$ be the ring of polynomials in x with complex coefficients. We call two polynomials g and h equivalent, writing $g \sim h$, if they differ by a polynomial multiple of $f(x)$. This sorts the polynomials into equivalence classes, and we denote the class containing g by $[g]$ and the set of all equivalence classes by A_f . This is of use for our problem because all polynomials in a given class take the same values at the x_i and therefore have the same trace. Addition and multiplication of classes are well-defined by $[g] + [h] = [g + h]$, $[g][h] = [gh]$, and A_f is itself a ring.

Now we observe that each class $[g] \in A_f$ contains a representative which has degree (at most) $n - 1$. Indeed, the relation $f \sim 0$ implies

$$a_n x^n \sim - \sum_{i=0}^{n-1} a_i x^i, \quad (10)$$

and this can be used to eliminate all terms of degree n or greater from a polynomial g . The resulting polynomial is unique, being determined by its values at the n roots x_i . It follows that A_f is in fact a vector space of dimension n over \mathbb{C} .

Now consider, instead of polynomials, the set R of rational functions of x which are defined at the zeros of $f(x)$, and call two elements equivalent if their difference is $f(x)$ times another element. Then all elements of a class have the same trace, and the set R_f of equivalence classes is again a ring. In fact, it is isomorphic to A_f . An isomorphism is obtained by associating to a class $[g]$ in A_f the obvious class $[g]$ in R_f . To see that this mapping is invertible, we must find a polynomial representative g of an arbitrary class $[h]$ in R_f . To do so, simply let $g(x)$ be the unique polynomial of degree $n - 1$ satisfying $g(x_i) = h(x_i)$ for all i . Then $g - h$ is a rational function vanishing at every x_i , so when expressed in lowest terms its numerator must be a multiple of $f(x)$. Therefore $g \in [h]$.

At this point we can explain the “trace” terminology for the sum over the roots x_i .¹³ The vector space A_f has a basis consisting of (the classes of) the n polynomials δ_i of degree $n - 1$ defined by $\delta_i(x_j) = \delta_{ij}$. Fix an element $g \in A_f$ and consider the linear operator on A_f given by multiplication by g . In the given basis, the matrix of this operator

is diagonal, with entries $g(x_i)$. Hence the trace of this matrix, which is independent of the basis and coincides with the trace of the operator, is simply $\text{Tr } g = \sum_i g(x_i)$. For example, consider $\text{Tr } x$, which just gives the sum of the roots. Choosing the basis $\{1, x, x^2, \dots, x^{n-1}\}$ for A_f , the matrix of the operator of multiplication by x has only one nonzero diagonal entry, $-a_{n-1}/a_n$, arising from the relation $x \cdot x^{n-1} \sim -(a_{n-1}/a_n)x^{n-1} + \dots$. This recovers the standard result for the sum of the roots of a polynomial and shows how our method generalizes others based on that result^{7,8,9,10}.

Returning to the problem of computing the total magnification, we see that $M = \text{Tr } \mu = \text{Tr } [\mu]$ can be computed using any element in its equivalence class. For example, we can choose the unique polynomial representative of degree $n - 1$. Of course, we do not determine this polynomial from its values $\mu(x_i)$, since we do not know the x_i explicitly. Instead, we seek a polynomial solution of degree $n - 1$ to the condition (8) defining $[\mu]$,

$$[\mu q(x)] = [p(x)]. \quad (11)$$

This is solved by using Eq. (10) to reduce the degree of each side to $n - 1$, and then equating coefficients.

To compute the trace we use a formula due to Euler, which we derive by means of the residue theorem for complex contour integrals. A purely algebraic proof is not difficult¹⁴, but our derivation shows the relevance of residue methods and motivates the multivariable generalization which we describe in the following section. Consider the contour integral,

$$\frac{1}{2\pi i} \oint \frac{f'(x)\mu}{f(x)} dx, \quad (12)$$

where the contour is a large circle in the complex plane enclosing all the zeros x_i of $f(x)$. The integrand has a pole of residue $\mu(x_i)$ at x_i , and consequently the integral is $\text{Tr } \mu$. However, we can also regard the contour as encircling the point at infinity and evaluate the integral in terms of the residue there, introducing if we wish the new variable $u = 1/x$ to move the point at infinity to the origin. Furthermore, the integral is unchanged if $f'(x)\mu$ is replaced by any member of its equivalence class. By choosing the polynomial representative of degree $n - 1$ we need only evaluate

$$\frac{1}{2\pi i} \oint \frac{x^k}{f(x)} dx = \frac{\delta_{k,n-1}}{a_n}, \quad 0 \leq k \leq n - 1. \quad (13)$$

This proves Euler's formula,

$$\text{Tr } \mu = \frac{\text{coefficient in degree } n - 1 \text{ of } f'(x)\mu}{\text{coefficient in degree } n \text{ of } f(x)}, \quad (14)$$

where it is understood that the polynomial representative of degree $n - 1$ is meant in the numerator. This makes it clear that the total magnification is determined by the leading behavior of $f(x)$ and $f'(x)\mu$ at infinity.

As an example, we consider a generalization of the SIS + elliptical potential ($n = 1$ multipole) with an arbitrary harmonic, $\psi = br + \gamma br \cos m\theta$. As an aid to clarity, we shall depart from conventional notation, and instead rewrite the lens equations so that the variables are x, y , and parameters are denoted a, b, c, \dots . In this case we define $x = e^{i\theta}, y = r, a = \gamma, b = b, c = s, d = e^{i\theta s}$. The lens equations then take the form,

$$f(x) = mabx^{2m} + cd^{-1}x^{m+1} - cdx^{m-1} - mab = 0, \quad (15)$$

$$2(y - b) - c(xd^{-1} + dx^{-1}) - ab(x^m + x^{-m}) = 0, \quad (16)$$

with the magnification satisfying

$$[m^2ab(x^m + x^{-m}) + c(xd^{-1} + dx^{-1})]\mu = 2y. \quad (17)$$

Eliminating y results in

$$[m^2abx^{2m} + cd^{-1}x^{m+1} + cdx^{m-1} + m^2ab]\mu = abx^{2m} + cd^{-1}x^{m+1} + 2bx^m + cdx^{m-1} + ab. \quad (18)$$

Dividing through by x and replacing x^{-1} in the resulting equation with a polynomial equivalent via the relation $x^{-1}f(x) \sim 0$ produces

$$[2m^2 abx^{2m-1} + (m+1)cd^{-1}x^m - (m-1)cdx^{m-2}]\mu = 2abx^{2m-1} + (1+m^{-1})cd^{-1}x^m + 2bx^{m-1} + (1-m^{-1})cdx^{m-2}, \quad (19)$$

where the left side is precisely in the Euler form $f'(x)\mu$. Then we immediately have the total magnification invariant as

$$M = \text{Tr } \mu = \frac{2ab}{mab} = \frac{2}{m}. \quad (20)$$

It is no harder to verify that the total magnification is the same for a potential containing an arbitrary finite sum of harmonics $\sum_{k=1}^m b_k r \cos k\theta$; the highest harmonic determines M . Other models are amenable to this method as well.

In principle one can compute moments by this method, by replacing μ with $x^k \mu$ in the relevant equations, but we have preferred the residue methods of the next section for moment computations.

III. RESIDUE METHODS: THEORY

Let us review the key steps in our contour integral derivation of Euler's formula in the previous section. First, we expressed the total magnification as a complex contour integral of a function having poles at the image locations. Second, we converted this to an integral around the point at infinity. This amounts to viewing the complex plane as a subset of the Riemann sphere, or complex projective space \mathbb{CP}^1 . Our change of variables $u = 1/x$ connects two coordinate charts on \mathbb{CP}^1 centered at the origin and at infinity. Finally, we evaluated the residue at infinity, making it clear that the total magnification depends on the behavior of the integrand at infinity.

These steps all have multivariable analogs. In fact, there is a well-developed, if little-known, residue theory for meromorphic differential forms in several complex variables. This makes it possible to compute directly the total magnification and moments for lens models without reducing to one-variable lens equations. The theory is particularly effective in the situation of two variables, and we describe it in this case. References include [15,16,17,18]. An application to chemical reaction rate equations appears in Ref. [19], but we are not aware of other physical applications in the literature.

We consider a meromorphic two-form

$$\omega = \frac{g(x, y) dx dy}{P_1(x, y)P_2(x, y)}, \quad (21)$$

on \mathbb{C}^2 , which we view as a subset of the compact complex projective space \mathbb{CP}^2 . Here P_1, P_2 are polynomials having finitely many common zeros (the image locations) of multiplicity one (as is generically the case), and g is also a polynomial. Such a form can be integrated over a 2-cycle, a compact two-dimensional real submanifold of \mathbb{CP}^2 . Since ω is closed ($d\omega = 0$), the integral depends only on the homology class of the cycle. For example, in a small neighborhood of a common zero of the P_i , we can integrate over the "torus" $T : \{|P_1| = |P_2| = \epsilon\}$, defining the residue of ω at this zero:

$$\text{Res } \omega = \left(\frac{1}{2\pi i} \right)^2 \int_T \omega. \quad (22)$$

[The standard orientation of T is that specified by the nonvanishing 2-form $d(\arg P_1) d(\arg P_2)$. That is, T is oriented so that $dP_1 dP_2 / (2\pi i)^2 P_1 P_2$ has positive integral.] We will always denote by J the naive Jacobian of the mapping $(x, y) \mapsto [P_1(x, y), P_2(x, y)]$, namely

$$J = \frac{\partial(P_1, P_2)}{\partial(x, y)} = \begin{vmatrix} \partial_x P_1 & \partial_y P_1 \\ \partial_x P_2 & \partial_y P_2 \end{vmatrix}. \quad (23)$$

This coincides with the physical Jacobian relating corresponding area elements in the source and image planes if x, y are rectangular coordinates, and P_1, P_2 are the corresponding lens equations, but requires a correction factor otherwise. The first key fact we need is that the residue at a nondegenerate zero (one where J does not vanish), located say at the origin, is given by¹⁸

$$\text{Res } \frac{g dx dy}{P_1 P_2} = \frac{g(0)}{J(0)}, \quad (24)$$

which is equal to the magnification of the corresponding image if g is chosen appropriately ($g = 1$ for rectangular coordinates). Moments of magnification can be computed by including additional monomial factors in $g(x, y)$.

Next we need the Global Residue Theorem¹⁸, which states that the sum of all the residues of a meromorphic form, such as ω , on any compact manifold, such as $\mathbb{C}P^2$, vanishes. Note that this sum is over the common zeros of the P_i only, so that points where only one polynomial vanishes do not contribute; also the points summed over may depend on the choice of the factorization $P_1 P_2$ of the denominator of ω . The theorem makes it possible to replace the sum over the residues at the common zeros in \mathbb{C}^2 by minus the sum of residues at points at infinity in $\mathbb{C}P^2$. This is the fundamental explanation for the existence of magnification relations in general: compactness relates the sum of finite residues to the behavior of the lens equations at infinity, indeed to a finite number of terms in an expansion around infinity. It remains to explain how to locate the common zeros at infinity and compute their residues.

$\mathbb{C}P^2$ is conveniently described by homogeneous coordinates $[X, Y, U] \neq [0, 0, 0]$, where $[\lambda X, \lambda Y, \lambda U]$ is identified with $[X, Y, U]$ for all complex $\lambda \neq 0$. The points with $U \neq 0$ can be represented in the form $[x, y, 1]$ and are viewed as the subset \mathbb{C}^2 of finite points. The polynomials $P_i(x, y)$ correspond to homogeneous polynomials $P_i^h(X, Y, U) \equiv U^{\deg P_i} P_i(X/U, Y/U)$. Their common zeros at infinity are those having $U = 0$. These also lie in coordinate charts diffeomorphic to \mathbb{C}^2 , described by $[1, y, u]$ or $[x, 1, u]$. In these charts they can be treated just as “finite” zeros are. The meromorphic form ω is homogenized so as to have total degree zero, that is,

$$\begin{aligned} \omega^h &= \frac{g(X/U, Y/U) d(X/U) d(Y/U)}{P_1(X/U, Y/U) P_2(X/U, Y/U)} \\ &= \frac{g^h U^{\deg P_1 + \deg P_2 - \deg g - 3} (U dX dY - X dU dY - Y dX dU)}{P_1^h P_2^h}. \end{aligned} \quad (25)$$

When $n = \deg g - \deg P_1 - \deg P_2 + 3 > 0$, the denominator of ω^h takes the form $U^n P_1^h P_2^h$. This creates a subtlety in that the above theory applies to a chosen factorization of the denominator into two factors. Thus, we may factor it as $(U^n P_1^h)(P_2^h)$ and treat these as the two factors in applying the residue theorem. The common zeros then consist of all (finite and infinite) common zeros of the P_i^h , together with any zeros of P_2^h alone at infinity. The latter might have been overlooked in a naive application of the theorem. In the examples we will consider in detail, no such additional zeros exist, but we will point out a case where they do.

Unfortunately, the zeros at infinity are rarely nondegenerate, and their residues cannot be computed using Eq. (24). Instead, they typically lie at singular points of the curves $P_i = 0$, that is, at least one curve has multiple branches meeting at this point.* There is a classical method, dating back to Newton, for finding the branches of an algebraic curve $P(x, y) = 0$ at a singular point, taken to be the origin. The branches are given as Puiseux series, or fractional power series, of the form

$$y = \sum_{i=0}^{\infty} a_i x^{\alpha_i}, \quad (26)$$

where the exponents α_i form an increasing sequence of rational numbers whose denominators eventually stabilize. The possibilities for the leading exponent α_0 are determined by requiring that at least two terms in the polynomial $P(x, a_0 x^{\alpha_0})$ have the same degree, while the remaining terms have higher degree. Then a_0 is found by demanding the vanishing of the terms of minimal degree. An elegant graphical method for identifying the possible exponents α_0 is provided by the Newton polygon, or diagram^{17,20}. For each monomial $x^a y^b$ appearing in $P(x, y)$, plot the point (a, b) in the coordinate plane. Begin at the lowest of the leftmost points (minimize a , then minimize b) and draw a polygonal path with vertices at a subset of the points, terminating at the leftmost of the lowest points (minimize b , then minimize a), and choosing each successive segment to have the smallest possible slope (steepest possible negative slope). This is the Newton diagram of the polynomial $P(x, y)$. The points lying on any segment of the Newton diagram represent terms in P which will have minimal degree if α_0 is chosen as the negative reciprocal of the slope of that segment. The next term in the Puiseux series can be found by applying the same procedure to $P(x, a_0 x^{\alpha_0} + \tilde{y})$, and so on. One term is often enough for computing the total magnification; higher moments require more terms in general. The denominators of the exponents stabilize at the stage where the Newton diagram has only a single segment. The curve $P(x, y) = 0$ has at least as many branches at the origin as there are segments in the Newton diagram, and has more if there are multiple solutions to the equations for the coefficients a_i .

Consider one particular branch X of the curve $P_1(x, y) = 0$ at a singular point, given by a Puiseux series $y = a_0 x^{m/n} + \dots$, with m, n relatively prime. On this branch draw a small circle C around the origin; its projection on

*Note that these curves have one complex, but two real dimensions, and so are real surfaces.

the x -plane must wind n times around the origin. Now construct a 2-torus δC by “thickening” C : at any point p of C take a plane transverse to X and a small circle in this plane with center p . As p moves around C this circle sweeps out the torus δC . We may construct such a torus for each branch of the curve. These are called Leray tori, and the “thickening” operator δ is the Leray coboundary.

Our objective is to compute the residue integral $\int_T \omega$ at the origin. We can work entirely within a small ball B around the origin. The residue depends on the homology class of T in $H_2(B - \{P_1 P_2 = 0\})$. As we explain in the Appendix, this class is, up to sign, the sum of the classes of the Leray tori constructed on the branches of *either* of the curves $P_1 = 0$ or $P_2 = 0$. Therefore the residue is the sum of the integrals of ω over either set of Leray tori, with appropriate orientation.

The integral over a Leray torus δC lying on a branch of $P_1(x, y) = 0$ given by a Puiseux series $y = p(x)$ is computed using the Leray residue formula. We give this formula under the assumption that $\partial_y P_1$ does not vanish on the given branch, which amounts to assuming that $y = p(x)$ is a branch of an irreducible factor of P_1 which appears to the first power only (the analog of a simple rather than multiple pole). The general formula can be found in Ref. [16]. Our case reads

$$\frac{1}{2\pi i} \int_{\delta C} \frac{g \, dx \, dy}{P_1 P_2} = - \int_C \frac{g \, dx}{P_2 \partial_y P_1} \Big|_{y=p(x)} \quad (27)$$

which is proved as follows. To integrate over δC , we can integrate first over the circles in the planes transverse to C , then over C itself. Since these circles are centered at $P_1 = 0$, we can change variables from x, y to x, P_1 by means of $dP_1 \, dx = -dx \, dy \, \partial_y P_1$ and then integrate over P_1 by means of the one-variable residue theorem. The evaluation of the residue at $P_1 = 0$ by substituting $y = p(x)$ leaves another one-variable residue integral to be performed. For this one must keep in mind that the cycle C may wind around the origin several times in the x -plane.

The Leray residue formula holds with obvious notational and possible sign changes if the Leray torus lies on a branch of $P_2(x, y) = 0$, or if we choose to eliminate x rather than y in favor of a P_i .

IV. RESIDUE METHODS: EXAMPLES

In this section we apply the residue methods just developed to various lens models. The first is a generalized $n = 2$ multipole model with potential $\psi = br + (\gamma/2)r^2 \cos m\theta$. We again redefine notation to clarify the computation, writing the variables as x, y and the parameters as a, b, c, \dots . Let $x = e^{i\theta}, y = r, a = \gamma, b = b, c = s, d = e^{i\theta s}$. The lens equations then take the form,

$$P_1 = cd^{-1}x^{m+1} - cdx^{m-1} + \frac{m}{2}ay(x^{2m} - 1) = 0, \quad (28)$$

$$P_2 = 2(y - b)x^m - cd^{-1}x^{m+1} - cdx^{m-1} - ay(x^{2m} + 1) = 0. \quad (29)$$

Because x, y are not rectangular coordinates, there is an extra Jacobian factor in the magnification, and we find

$$\mu = \frac{4yx^{2m-1}}{J}. \quad (30)$$

Consequently the total magnification is given by the residue sum,

$$M = \sum_{\text{images}} \text{Res} \frac{4yx^{2m-1} dx \, dy}{P_1 P_2}. \quad (31)$$

The residue theorem relates this to the sum of the residues at points at infinity in \mathbb{CP}^2 , which are found from the homogeneous forms of the P_i :

$$P_1^h = cd^{-1}X^{m+1}U^m - cdX^{m-1}U^{m+2} + \frac{m}{2}aY(X^{2m} - U^{2m}), \quad (32)$$

$$P_2^h = 2X^m(YU^m - bU^{m+1}) - cd^{-1}X^{m+1}U^m - cdX^{m-1}U^{m+2} - aY(X^{2m} + U^{2m}). \quad (33)$$

The common roots at infinity are those with $U = 0$, and there are two: $[X, Y, U] = [1, 0, 0]$, and $[0, 1, 0]$. The total magnification is minus the sum of the residues at these points of

$$\frac{4Y X^{2m-1} U^{2m+2} d(X/U) d(Y/U)}{P_1^h P_2^h}. \quad (34)$$

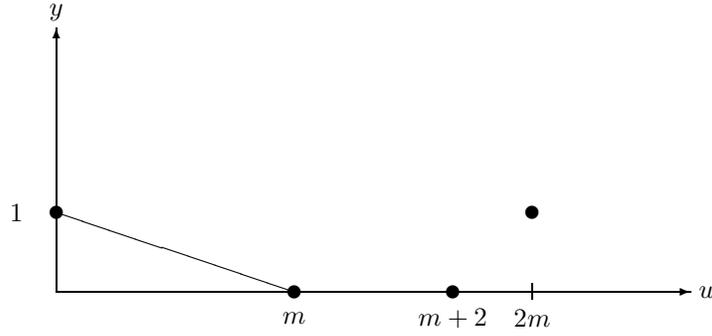


FIG. 1: The Newton diagram for $P_1^{X=1}$ in the $n = 2$ multipole model. The degree of a monomial in $y(u)$ is plotted vertically (horizontally). The case $m = 3$ is shown.

Consider first the point $[1, 0, 0]$, which we examine in the affine chart $[X, Y, U] = [1, y, u]$, where

$$P_1^{X=1} = cd^{-1}u^m - cdu^{m+2} + \frac{m}{2}ay(1 - u^{2m}), \quad (35)$$

$$P_2^{X=1} = 2u^m(y - bu) - cd^{-1}u^m - cdu^{m+2} - ay(1 + u^{2m}), \quad (36)$$

and we need the residue at the origin of $4yu^{2m-1} du dy / P_1^{X=1} P_2^{X=1}$.

The Newton diagram for P_1 is shown in Figure 1; there is a single branch on which to leading order $y \sim u^m$, as is easily verified by solving $P_1^{X=1} = 0$ for y . The Leray formula evaluates the residue as the one-variable residue of

$$-\frac{4yu^{2m-1} du}{P_2^{X=1} \partial_y P_1^{X=1}}, \quad (37)$$

where $y \sim u^m$. But it is easily seen that the leading behavior of this 1-form near $u = 0$ is $u^{2m-1} du$, so that there is no pole and no residue for $m > 0$.

We examine the remaining root $[0, 1, 0]$ in the chart $[X, Y, U] = [x, 1, u]$ where

$$P_1^{Y=1} = cd^{-1}x^{m+1}u^m - cdx^{m-1}u^{m+2} + \frac{m}{2}a(x^{2m} - u^{2m}), \quad (38)$$

$$P_2^{Y=1} = 2x^m u^m (1 - bu) - cd^{-1}x^{m+1}u^m - cdx^{m-1}u^{m+2} - a(x^{2m} + u^{2m}), \quad (39)$$

and we compute

$$\text{Res} \frac{4x^{2m-1}u^{2m-1} dx du}{P_1^{Y=1} P_2^{Y=1}}. \quad (40)$$

The Newton diagram for $P_1^{Y=1}$ is shown in Figure 2, and gives the leading behavior $u = lx + \dots$. With this behavior, the lowest-order terms of $P_1^{Y=1}$ vanish iff $l^{2m} = 1$, so that there are $2m$ branches with $l_p = \exp(ip\pi/m)$, $p = 1, \dots, 2m$.

With the Leray formula, the total magnification becomes

$$\sum_{u=l_p x + \dots} -\text{Res} \frac{4x^{2m-1}u^{2m-1} dx}{P_2^{Y=1} \partial_u P_1^{Y=1}}. \quad (41)$$

The leading terms are readily identified, and indeed the behavior is as dx/x , with residue

$$\sum_{p=1}^{2m} \frac{2}{m^2 a (l_p^m - a)} = \frac{2}{m^2 a} \sum_{p=1}^{2m} \frac{1}{(-1)^p - a} = \frac{4}{m(1 - a^2)}. \quad (42)$$

In terms of the original parameters of the model,

$$M = \frac{4}{m(1 - \gamma^2)}, \quad (43)$$

an attractive generalization of the known result⁹ for $m = 2$.

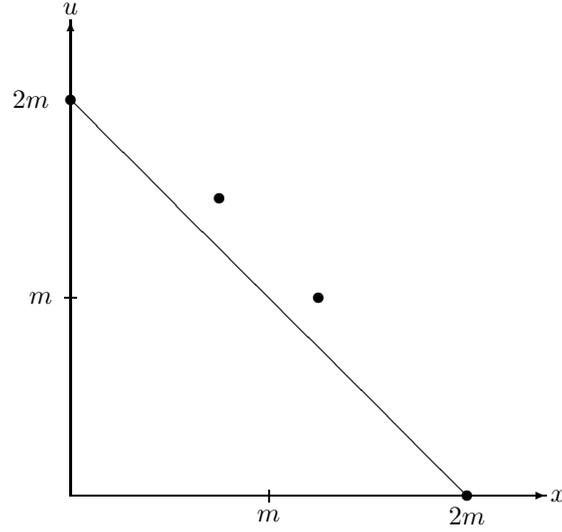


FIG. 2: The Newton diagram for $P_1^{Y=1}$ in the $n = 2$ multipole model.

Once the branches of the P_i have been identified, it is easy to modify the calculation to compute moments rather than total magnification. For example, let us compute the first x -moment, $\text{Tr } x\mu$. (We continue to use the Tr notation for the sum over the images.) Since $x = e^{i\theta}$ in terms of the physical variables of this model, from the real and imaginary parts of the result we can obtain the moments weighted by $\cos \theta$ and $\sin \theta$, in the case that all images are real. In homogeneous coordinates this simply gives an additional factor X/U in the residue in Eq. (34). At the point $[1, 0, 0]$ this adds a factor $1/u$ to the one-variable residue (37), changing the behavior to $u^{2m-2} du$. There is still no contribution for $m \geq 1$. At the point $[0, 1, 0]$, there is an extra factor $x/u = l^{-1} + \dots$ in the residue (41), changing the contribution to

$$\sum_p \frac{2}{m^2 a l_p (l_p^m - a)} = \frac{2}{m^2 a} \sum_{p=1}^{2m} \frac{e^{-ip\pi/m}}{(-1)^p - a}, \quad (44)$$

which can be evaluated in closed form if desired.

From the real part of $\text{Tr } yx\mu$ we can obtain the moment of $r \cos \theta$, the physical x -coordinate of the image. This leads to a factor YX/U^2 , which worsens the singularity at $[0, 1, 0]$ to a double pole, requiring an additional term in the Puiseux expansion to obtain the residue. The result, for the true external shear model $m = 2$, is

$$\text{Tr } \mu r \cos \theta = -\frac{2x_s}{(1-\gamma)(1-\gamma^2)}. \quad (45)$$

This example illustrates the general situation. Higher moments produce extra monomial factors in the residue expression. In general this will worsen the singular behavior at points at infinity, although this may not occur for certain branches, such as those at $[0, 1, 0]$ in the example of $\text{Tr } x\mu$. This will have two effects: points which do not contribute to the total magnification generally will contribute to higher moments, and more terms in the Puiseux expansions will be required for higher moments. Both effects will result in higher moments being given by more complex expressions, with more model parameters contributing.

We turn to a second example, the Singular Isothermal Ellipse (SIE) potential^{10,12}. For this model $\psi = bR = b\sqrt{x^2 + y^2 q^{-2}}$, where x, y are rectangular coordinates and b, q are parameters. The lens equations are

$$\tau_x = x - x_s - \frac{bx}{R} = 0, \quad (46)$$

$$\tau_y = y - y_s - \frac{by}{q^2 R} = 0, \quad (47)$$

and the magnification is given by

$$\mu^{-1} = \begin{vmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{xy} & \tau_{yy} \end{vmatrix}. \quad (48)$$

Algebraic manipulation leads to the polynomial equations,

$$p_1 = q^2 x \tau_y - y \tau_x = q^2 x (y - y_s) - y (x - x_s) = 0, \quad (49)$$

$$p_2 = R^2 \tau_x^2 - 2bRx\tau_x = (x - x_s)^2 (x^2 + y^2 q^{-2}) - b^2 x^2 = 0. \quad (50)$$

However, these equations have the extraneous solution $(x, y) = (0, 0)$, which does not satisfy the original lens equations. This can be eliminated by substituting $y = wx$, and adopting the modified equations

$$P_1 = p_1/q^2 x = wx - y_s - \eta wx + \eta wx_s, \quad (51)$$

$$P_2 = p_2/x^2 = b^2 - (x - x_s)^2 (1 + \eta w^2), \quad (52)$$

where we have set $\eta = q^{-2}$. Relating the naive Jacobian J of the P_i with respect to x, w to the Hessian of τ gives the magnification in the new variables,

$$\mu = \frac{2x(x - x_s)(1 + \eta w^2)}{J}. \quad (53)$$

The moments of the magnification with respect to x are given by

$$\text{Tr } x^k \mu = \sum \text{Res} \frac{2x^{k+1}(x - x_s)(1 + \eta w^2) dx dw}{P_1 P_2}. \quad (54)$$

In homogeneous coordinates we have

$$P_1^h = (1 - \eta)XW - y_s U^2 + \eta x_s WU, \quad (55)$$

$$P_2^h = b^2 U^4 - (X - x_s U)^2 (U^2 + \eta W^2), \quad (56)$$

and we need to compute

$$\text{Tr } \mu x^k = \sum \text{Res} \frac{2X^{k+1}U^{2-k}(X - x_s U)(U^2 + \eta W^2) d(X/U) d(W/U)}{P_1^h P_2^h}. \quad (57)$$

There are two common zeros of the P_i^h at infinity, namely $[X, W, U] = [1, 0, 0]$, $[0, 1, 0]$, and in the corresponding affine charts we have

$$P_1^{X=1} = (1 - \eta)w - y_s u^2 + \eta x_s wu, \quad (58)$$

$$P_2^{X=1} = b^2 u^4 - (1 - x_s u)^2 (u^2 + \eta w^2), \quad (59)$$

$$P_1^{W=1} = (1 - \eta)x - y_s u^2 + \eta x_s u, \quad (60)$$

$$P_2^{W=1} = b^2 u^4 - (x - x_s u)^2 (u^2 + \eta). \quad (61)$$

The residues at these zeros may be computed via the Lery formula applied to the branches of either P_1 or P_2 ; we have done both computations and the latter seems slightly simpler. In each case the branches can be determined directly without appealing to the Newton diagrams. At $[0, 1, 0]$, the equation $P_2^{W=1} = 0$ is solved by

$$x = x_s u \pm b \eta^{-1/2} u^2 \left(1 + \frac{u^2}{\eta}\right)^{-1/2}, \quad (62)$$

giving one branch for each choice of sign. The contribution to $\text{Tr } \mu x^k$ from one branch is

$$\text{Res} \frac{2x^{k+1}(x - x_s u)(u^2 + \eta) dx du}{u^{k+1} P_1^{W=1} P_2^{W=1}}. \quad (63)$$

Applying the Lery formula gives

$$\text{Res} \frac{2x^{k+1}(x - x_s u)(u^2 + \eta) du}{u^{k+1} P_1^{W=1} \partial_x P_2^{W=1}}, \quad (64)$$

which simplifies to

$$\text{Res} \frac{x^{k+1} du}{u^{k+1} [x - y_s u^2 - \eta(x - x_s u)]} = x_s^k, \quad (65)$$

where only the first term in the series expansion of x was required. The two branches at this point thus contribute $2x_s^k$ to the moment of x^k .

At the remaining point $[1, 0, 0]$, $P_2^{X=1} = 0$ is solved by

$$w = \pm \frac{i u}{\sqrt{\eta}} \sqrt{1 - \frac{b^2 u^2}{(1 - x_s u)^2}} = \pm \frac{i u}{\sqrt{\eta}} \left(1 - \frac{1}{2} b^2 u^2 + \dots\right), \quad (66)$$

and we need

$$\text{Res} \frac{2(1 - x_s u)(u^2 + \eta w^2) du dw}{u^{k+1} P_1^{X=1} P_2^{X=1}}, \quad (67)$$

summed over the two branches. Integrating over w using the Leray residue formula and simplifying yields

$$\text{Res} \frac{(u^2 + \eta w^2) du}{\eta w u^{k+1} (1 - x_s u) [(1 - \eta)w + \eta x_s w u - y_s u^2]}, \quad (68)$$

where $u^2 + \eta w^2 = b^2 u^4 + \dots$. Since $w \sim u$, there is no pole for $k < 2$: the total magnification and first moment are simply given by the contributions from $[0, 1, 0]$. There is an additional contribution to the second moment given by $-2b^2/(1 - \eta^2) = 2b^2 q^2/(1 - q^2)$, which agrees with the result of Ref. [10].

We have verified the results of Witt & Mao¹⁰ through the third moment. Those authors noted that the moments of x were independent of y_s to this order, and that ‘‘this seems remarkable’’ in view of the dependence of μ on this parameter. The explanation is that the term $y_s u^2$ in $P_1^{X=1}$ does not contribute to the residue for low moments; indeed it *does* contribute to the residues for the third moment but its contribution cancels between the two branches. The fourth and higher moments do depend on y_s .

Lastly, we note that it is entirely straightforward, and no more work, to generalize the calculation to include an arbitrarily oriented external shear term. The first moment, for example, takes the form

$$\text{Tr } \mu x = \frac{2}{(1 - \gamma^2)^2} (x_s + \gamma_1 x_s + \gamma_2 y_s), \quad \text{Tr } \mu y = \frac{2}{(1 - \gamma^2)^2} (y_s + \gamma_2 x_s - \gamma_1 y_s) \quad (69)$$

where $\gamma_1 \equiv \gamma \cos 2\theta_\gamma$, $\gamma_2 \equiv \gamma \sin 2\theta_\gamma$, and θ_γ is the orientation angle of the shear (see Table I).

As our final example we consider microlensing due to a collection of N coplanar point masses. We adopt Witt’s complex notation, writing $z = x + iy$ for the position of an image and w for the position of the source. The lenses have masses m_i and positions z_i . The lens equations are

$$z - w - \sum_{i=1}^N \frac{m_i}{\bar{z} - \bar{z}_i} = 0, \quad (70)$$

and its complex conjugate; when we complexify the coordinates x, y of an image, z and \bar{z} become independent variables and the conjugate equation becomes an independent condition as well. The observable (real) images are those for which \bar{z} is the conjugate of z . Clearing denominators, we set

$$P_1 = (z - w) \prod_i (\bar{z} - \bar{z}_i) - \sum_i m_i \prod_{j \neq i} (\bar{z} - \bar{z}_j), \quad (71)$$

$$P_2 = (\bar{z} - \bar{w}) \prod_i (z - z_i) - \sum_i m_i \prod_{j \neq i} (z - z_j). \quad (72)$$

For the magnification we find

$$\mu = J^{-1} \prod_i (z - z_i)(\bar{z} - \bar{z}_i), \quad (73)$$

where $J = \partial(P_1, P_2)/\partial(z, \bar{z})$. For the k th moment of magnification, we must compute

$$\text{Tr } \mu z^k = \text{Res} \frac{z^k \prod_i (z - z_i)(\bar{z} - \bar{z}_i) dz d\bar{z}}{P_1 P_2}. \quad (74)$$

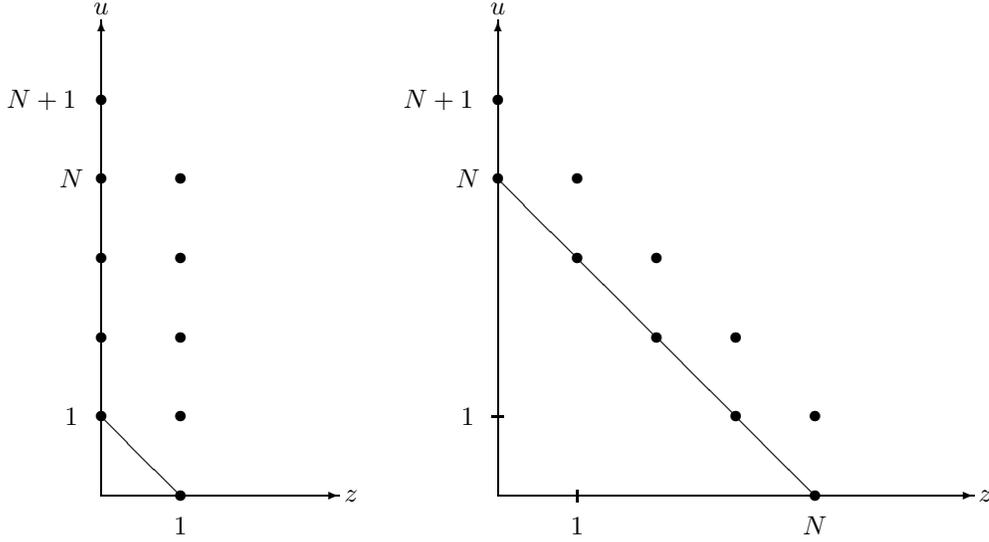


FIG. 3: The Newton diagrams for $P_1^{\bar{Z}=1}$ (left) and $P_2^{\bar{Z}=1}$ (right) for N point masses. The case $N = 4$ is shown.

The homogeneous polynomials are

$$P_1^h = (Z - wU) \prod_i (\bar{Z} - \bar{z}_i U) - \sum_i m_i U^2 \prod_{j \neq i} (\bar{Z} - \bar{z}_j U), \quad (75)$$

$$P_2^h = (\bar{Z} - \bar{w}U) \prod_i (Z - z_i U) - \sum_i m_i U^2 \prod_{j \neq i} (Z - z_j U). \quad (76)$$

Setting $U = 0$, we obtain $P_1^h = Z\bar{Z}^N$, $P_2^h = \bar{Z}Z^N$, so there are two common zeros at infinity, $[Z, \bar{Z}, U] = [1, 0, 0]$, $[0, 1, 0]$. However, we can say more: each of these zeros has multiplicity N . Since the homogeneous polynomials each have degree $N + 1$, the number of finite common zeros will be $(N + 1)^2 - 2N = N^2 + 1$, in agreement with previous results³.

Dehomogenizing the polynomials at these points, we find

$$P_1^{\bar{Z}=1} = (z - wu) \prod_i (1 - u\bar{z}_i) - u^2 \sum_i m_i \prod_{j \neq i} (1 - u\bar{z}_j), \quad (77)$$

$$P_2^{\bar{Z}=1} = (1 - \bar{w}u) \prod_i (z - uz_i) - u^2 \sum_i m_i \prod_{j \neq i} (z - uz_j), \quad (78)$$

$$P_1^{Z=1} = (1 - wu) \prod_i (\bar{z} - u\bar{z}_i) - u^2 \sum_i m_i \prod_{j \neq i} (\bar{z} - u\bar{z}_j), \quad (79)$$

$$P_2^{Z=1} = (\bar{z} - \bar{w}u) \prod_i (1 - uz_i) - u^2 \sum_i m_i \prod_{j \neq i} (1 - uz_j). \quad (80)$$

Because of the complex conjugation symmetry of these expressions, it suffices to examine the branches of, say, $P_i^{\bar{Z}=1}$ to deduce the others. The Newton diagrams of these are shown in Figure 3. In each case, the Puiseux series are ordinary power series. For P_1 there is a single branch $u = z/w + \dots$, while for P_2 there are N branches $u = z/z_i - (m_i/z_i^3)z^2 + \dots$.

It is now straightforward to compute the residues

$$\text{Res} \frac{Z^k U^{2-k} \prod_i (Z - z_i U) (\bar{Z} - \bar{z}_i U) d(Z/U) d(\bar{Z}/U)}{P_1^h P_2^h} \quad (81)$$

and determine the moments. The first two are $M = \text{Tr} \mu = 1$, and $\text{Tr} z\mu = w + \sum_i \frac{m_i}{\bar{w} - \bar{z}_i}$. The single branch contributes w^k to the k th moment, while the contribution of the N branches becomes progressively more complicated.

We have also considered a generalization of the model to include external shear. This brings the lens equations to the form³

$$z - w - \gamma \bar{z} - \sum_{i=1}^N \frac{m_i}{\bar{z} - \bar{z}_i} = 0 \quad (82)$$

Model	ψ	$\sum_i \mu_i$	$\sum_i \mu_i z_i$
point masses	$\sum_l m_l \log \vec{\Theta} - \vec{\Theta}_l $	1	$z_s + \sum_l \frac{m_l}{\bar{z}_s - \bar{z}_l}$
point masses + shear	$\psi_{\text{pm}} + \frac{\gamma}{2} r^2 \cos 2\theta$	$1/(1-\gamma^2)$	$\frac{1}{(1-\gamma^2)x} (z_s + \gamma \bar{z}_s)$
SIE	bR	2	$2z_s$
SIS + elliptical	$br + \gamma br \cos 2\theta$	1	$z_s + 2\gamma \bar{z}_s - \frac{\bar{z}_s^3}{32b^2\gamma^2}$
SIS + shear	$br + \frac{\gamma}{2} r^2 \cos 2\theta$	$2/(1-\gamma^2)$	$\frac{2}{(1-\gamma^2)x} (z_s + \gamma \bar{z}_s)$
SIE + shear	$bR + \frac{\gamma}{2} r^2 \cos 2(\theta - \theta_\gamma)$	$2/(1-\gamma^2)$	$\frac{2}{(1-\gamma^2)x} (z_s + \gamma e^{2i\theta_\gamma} \bar{z}_s)$

TABLE I: Model lens potentials and results. The l^{th} lens has mass m_l and position $\vec{\Theta}_l$. For the galaxy potentials, with only one lens, we choose coordinates centered on the lens and oriented along the ellipticity or shear axes. Here, the image position $\vec{\Theta} = (x, y) = (r \cos \theta, r \sin \theta)$ and source position $\vec{\beta} = (x_s, y_s) = (s \cos \theta_s, s \sin \theta_s)$. We use the complex notation³ $z = x + iy$, and similarly for z_s, z_l . Also, $R = \sqrt{x^2 + y^2/q^2}$ is an elliptical radial coordinate, γ is the strength of the shear, and q is the axis ratio.

and its complex conjugate. Defining P_i^h as before, their zeros at infinity are $[1, 0, 0]$ and $[\gamma, 1, 0]$ for P_1^h , and $[0, 1, 0]$ and $[1, \gamma, 0]$ for P_2^h . For $\gamma \neq 0, 1$ there are no common zeros at infinity. Factoring the denominator of ω^h as $(U^n P_1^h)(P_2^h)$, the residue theorem gives the sum over the finite common roots as minus the sum of residues at the infinite roots of P_2^h . We find, for example, the total magnification $1/(1-\gamma^2)$, and the first moment

$$\text{Tr } z\mu = \frac{w + \gamma \bar{w}}{(1-\gamma^2)^2}. \quad (83)$$

Note that the first moment's dependence upon the lens' positions has vanished.

V. DISCUSSION

We have introduced in this paper a new framework for analyzing gravitational lens models. The use of residue integrals makes clear the origin of the magnification relations, and facilitates their computation for a wide class of model potentials. We have also applied this method to a series of models, confirmed and extended previous results, and provided new magnification relations for several models. Although multidimensional residue calculus may be unfamiliar to readers from astronomy, one may follow a simple procedure to perform the necessary integrals. Basically, the procedure is as follows :

1. From the stationarity equations, construct two polynomials P_1, P_2 that simultaneously vanish at (and only at) the image positions.
2. Define the ‘‘Jacobian’’ $J_P = \det[\partial(P_1, P_2)/\partial(x, y)]$, and define $g(x, y) = J_P(x, y)\mu(x, y)$ where μ is the magnification.
3. Change to homogeneous coordinates: $(x, y) \rightarrow (X, Y, U)$ with $x = X/U, y = Y/U$ and homogenize the polynomials by multiplying each by the factor $U^{\deg P}$. Also, multiply g by $U^{\deg P_1 + \deg P_2}$.
4. From the homogenized g, P_1, P_2 , construct the 2-form $(g/P_1 P_2)d(\frac{X}{U})d(\frac{Y}{U})$, and factor the denominator into two groups; usually, the grouping P_1, P_2 suffices. (Henceforth P_1, P_2 shall refer to the two groups, not the original polynomials.) If the denominator contains explicit factors of U then redefine one of the polynomials, say P_2 , to contain these additional factors.
5. For $U = 0$, find the points (X, Y) where P_1, P_2 simultaneously vanish; these are the roots at infinity.
6. Pick one of the polynomials – say P_1 – and determine the behavior of $P_1 = 0$ in the vicinity of each common root. First define coordinates for the neighborhood of the root. For example, if the root is $Y = 0, U = 0$, then a good choice would be $(X, Y, U) = (1, y, u)$. As discussed, there are in general multiple branches of $P_1 = 0$ meeting at the root at infinity, each parametrized by a Puiseux series $y = \sum_i a_i u^{\alpha_i}$. The α_i 's can be determined from the Newton diagram, and by substituting in the specified α_i 's one may solve for the coefficients a_i . Usually, only the first one or two terms in the series are necessary.

7. Construct the quantity

$$\frac{g(u, a_1 u^{\alpha_1} + a_2 u^{\alpha_2} + \dots)}{\frac{\partial P_1}{\partial y}(u, a_1 u^{\alpha_1} + \dots) P_2(u, a_1 u^{\alpha_1} + \dots)}$$

and pick out the term $\sim u^{-1}$. The coefficient of u^{-1} is the contribution for this branch; summing over all the branches gives the residue for each root.

8. Repeat this procedure for all the roots at infinity, and sum their residues. Negating this quantity gives the sum of the residues at finite poles (the images).

The above is of course just a rough outline; section III describes the method in full detail. Following this procedure, the results listed in Table I can be reproduced with minimal effort. As mentioned earlier, the Newton diagram method of determining the branches near each pole is effective only if the lens equations can be brought into polynomial form, such as for these simple potentials. For more complicated models, introduction of additional variables may be necessary to handle fractional powers. Our method may not be applicable to models involving transcendental functions, in particular functions with essential singularities at infinity.

Dalal⁹ and Witt & Mao¹⁰ have considered the applicability of such magnification relations to real gravitational lenses. For galaxies, Dalal⁹ has shown that these relations can be an aid in fitting models to lensed objects, or can be used to rule out models *a priori*. Witt & Mao¹⁰ have shown, however, that reliance upon simple galaxy models can be misleading, when applied to realistic galaxy potentials. This limits the applicability of magnification relations to making statements about models, as opposed to statements about the lenses themselves. For microlensing, however, there is no doubt about the accuracy of the point mass approximation, and as such our derived magnification relations may be considered exact. Witt & Mao⁷ have already shown how the total signed magnification (“zerth moment”) can be used for binary microlensing to set lower limits on the overall unsigned magnification, useful for example for detecting source blending. Although the multiple images of a microlensing event cannot as yet be resolved, precluding the present-day experimental verification of our prediction regarding the first moment, we are hopeful that the future advent of space-based interferometers will allow our microlensing formulae to be tested observationally.

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APPENDIX : THE HOMOLOGY CLASS OF THE TORUS T

Here we explain why the torus T defining the local residue at a singular point, taken to be the origin, is homologous to the sum of the Leray tori constructed from the branches of *either* of the polynomials $P_i(x, y)$, and how to determine the correct orientations. We are grateful to Eduard Looijenga and Peter Teichner for explaining the topology to us. See also section 2.2 of Dimca²¹.

We are working locally in a closed ball B around the origin, and we denote by X the zero locus $P_1 P_2 = 0$ within B . The various tori define homology classes in $H_2(B - X)$. It is known that X is topologically a cone with vertex the origin and base $X \cap \partial B$, which is a linked collection of topological circles^{17,21}. Denote the several branches of X as X_i , and the Leray torus built on a given branch as δC_i . Let σ_{ij} be a path running from the origin to ∂B along X_i , and returning to the origin along branch X_j . A subset of these paths forms a basis for the relative homology group $H_1(X, \partial B)$. Furthermore, by Alexander duality¹⁶, an element of $H_2(B - X)$ is uniquely determined by its linking numbers with these paths (indeed, with those in a basis alone). (The linking number $l(c^1, c^2)$ of a 1-cycle with a 2-cycle in a 4-ball is the intersection number of c^1 with any 3-manifold having boundary c^2 .)

The Leray tori have linking numbers

$$l(\sigma_{ij}, \delta C_k) = \delta_{ik} - \delta_{jk}. \tag{A1}$$

Indeed, the intersection of σ_{ij} with a solid torus bounded by δC_k is the intersection of σ_{ij} with C_k , which is one point if the outward segment of σ_{ij} lies on branch X_k , and one point (with opposite orientation) if the returning segment does.

The sum $\sum_{P_1} \delta C_k$ of Leray tori built on branches of $P_1 = 0$ therefore has linking number with σ_{ij} equal to +1 if X_i is a branch of P_1 and X_j is a branch of P_2 , -1 if vice-versa, and 0 if X_i and X_j are branches of the same polynomial. The sum $\sum_{P_2} \delta C_k$ has the negatives of these linking numbers and therefore represents the same homology class but with opposite orientation.

It remains to show that T has the linking numbers of $\sum_{P_1} \delta C_k$. A solid torus bounded by T is given by $\{|P_1| \leq \epsilon, |P_2| = \epsilon\}$. This meets any branch X_k of $P_1 = 0$ given by a Puiseux series $y = p(x)$ in the locus $|P_2(x, p(x))| = \epsilon$ on X_k . This set is topologically a circle around the origin, having intersection number +1 with a path radially outward from the origin. The solid torus meets no branch of $P_2 = 0$. Therefore, radially outward (inward) paths on any branch of P_1 contribute +1 (-1) to intersection numbers with this solid torus, while paths on branches of P_2 contribute nothing. This duplicates the linking numbers of $\sum_{P_1} \delta C_k$.

The orientation for T used in this argument is indeed the standard one prescribed in section III. Using the Leray residue formula to evaluate

$$\left(\frac{1}{2\pi i}\right)^2 \int_T \frac{dP_1 dP_2}{P_1 P_2} \quad (\text{A2})$$

produces a positive contribution from every branch of P_1 .

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