The genealogy of branching Brownian motion with absorption

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Outline

1. A population model and conjectures
2. Background information
   - a. Coalescent processes
   - b. Continuous-state branching processes
   - c. Branching Brownian motion
3. Our model and main results
4. Ideas behind proofs

A population model with selection

Brunet-Derrida-Mueller-Munier (2006, 2007)

- Population has fixed size N.
- Each individual has $k \geq 2$ offspring.
- The fitness of each offspring is the parent's fitness plus an independent random variable with distribution μ.
- Of the kN offspring, the N with the highest fitness survive to form the next generation.

Durrett-Mayberry (2009) studied related model in context of predator-prey systems.

Related work: Bérard-Gouéré (2009), Durrett-Remenik (2009).

Three conjectures

Brunet-Derrida-Mueller-Munier (2006, 2007)

1. (Brunet-Derrida, 1997) Let L_m be the maximum of the fitnesses of the individuals in generation m. Then $L_m/m \to v_N$ a.s. Let $v_\infty = \lim_{N\to\infty} v_N$. Then
$$v_\infty - v_N \sim \frac{C}{(\log N)^2}.$$

2. If two individuals are sampled from the population at random in some generation, the number of generations back to their most recent common ancestor is $O((\log N)^3)$.

3. If n individuals are sampled from some generation and their ancestral lines are followed backwards in time, the coalescence of these lineages is be described by the Bolthausen-Sznitman coalescent.

The first conjecture

Brunet-Derrida-Mueller-Munier (2006, 2007) studied solutions $u(x,t)$ to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^2 + \sqrt{\frac{u(1-u)}{N}}\, W(x,t), \qquad (1)$$

where $W(x,t)$ is space-time white noise.

Without the noise term, this is FKPP equation (Fisher (1937), Kolmogorov-Petrovsky-Piscounov (1937)).

FKPP equation has traveling-wave solutions $u(t,x) = w(x - vt)$.

Theorem (Mueller-Mytnik-Quastel, 2009): The velocity of solutions to (1) differs from v by $O(1/(\log N)^2)$.

Theorem (Bérard-Gouéré, 2009): Conjecture 1 holds in the case $k = 2$ under suitable regularity conditions on μ.

Proof builds on Gantert-Hu-Shi (2008) and Pemantle (2009).

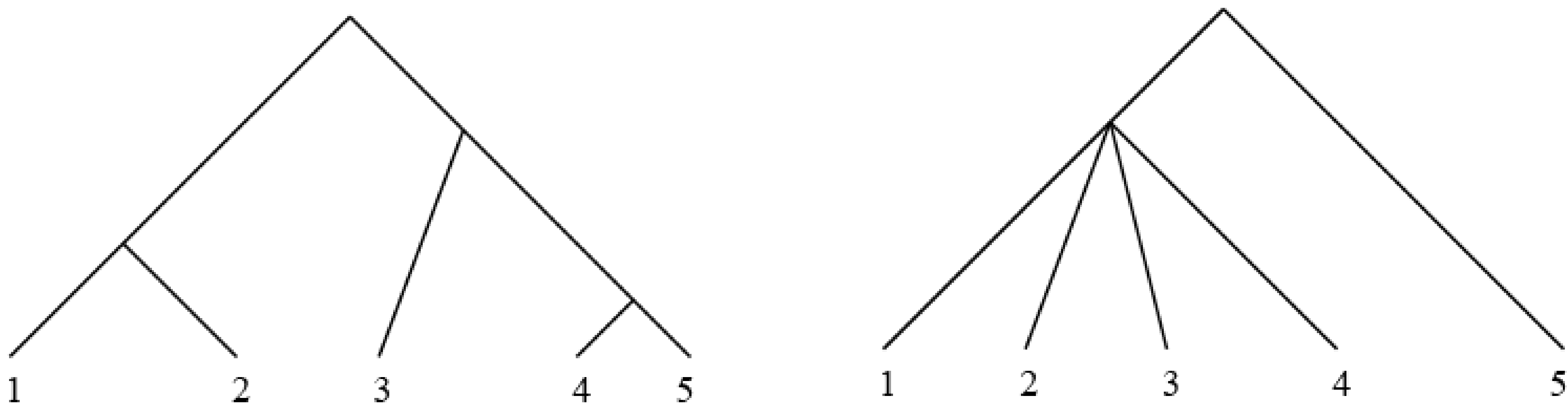
Our goal: prove rigorous versions of Conjectures 2 and 3.

Coalescent Processes

Sample n individuals at random from a population. Follow their ancestral lines backwards in time. The lineages coalesce, until they are all traced back to a common ancestor.

Represent by a stochastic process $(\Pi(t), t \geq 0)$ taking its values in the set of partitions of $\{1, \ldots, n\}$.

Kingman's Coalescent (Kingman, 1982): Only two lineages merge at a time. Each pair of lineages merges at rate one.



Λ-coalescents (Pitman (1999), Sagitov (1999)): Many lineages can merge at a time.

Bolthausen-Sznitman coalescent

Let Λ be a finite measure on $[0,1]$. The Λ-coalescent is the coalescent process such that when there are b blocks, each k-tuple ($2 \leq k \leq b$) of blocks merges at rate

$$\lambda_{b,k} = \int_0^1 p^{k-2}(1-p)^{b-k} \, \Lambda(dp).$$

When $\Lambda(\{0\}) = 0$, obtain a Λ-coalescent from a Poisson point process on $[0,\infty) \times (0,1]$ with intensity $dt \times p^{-2}\Lambda(dp)$. If (t,p) is a point of this Poisson process, a p-merger occurs at time t. That is, we flip a coin for each lineage having probability p of heads, and merge all lineages whose coin is a head.

Bolthausen-Sznitman (1998) coalescent: Λ is uniform on $[0,1]$. Linked to random recursive trees (Goldschmidt-Martin, 2005), Derrida's GREM (Bovier-Kurkova, 2007).

Continuous-state branching processes (Lamperti, 1967)

A continuous-state branching process (CSBP) is a $[0, \infty)$-valued Markov process $(X(t), t \geq 0)$ whose transition functions satisfy

$$p_t(a+b, \cdot) = p_t(a, \cdot) * p_t(b, \cdot).$$

CSBPs arise as scaling limits of Galton-Watson processes.

Let $(Y(s), s \geq 0)$ be a Lévy process with no negative jumps with $Y(0) > 0$, stopped when it hits zero. Let

$$S(t) = \inf\{u : \int_0^u Y(s)^{-1}\, ds > t\}.$$

The process $(X(t), t \geq 0)$ defined by $X(t) = Y(S(t))$ is a CSBP. Every CSBP can be obtained this way.

If $Y(0) = a$, then $E[e^{-uY(t)}] = e^{au + t\Psi(u)}$, where

$$\Psi(u) = \alpha u + \beta u^2 + \int_0^\infty (e^{-ux} - 1 + ux 1_{\{x \leq 1\}})\, \nu(dx).$$

The function Ψ is the branching mechanism of the CSBP.

The genealogy of Neveu's CSBP

Neveu (1992) considered the CSBP with branching mechanism

$$\Psi(u) = au + bu \log u = cu + \int_0^\infty (e^{-ux} - 1 + ux\mathbf{1}_{\{x \leq 1\}})\, bx^{-2}\, dx.$$

Bertoin-Le Gall (2000): the genealogy of Neveu's CSBP is given by the Bolthausen-Sznitman coalescent.

- Let A be the current population size.
- After a jump of size x, a fraction $p = x/(A+x)$ of the population was born at the time of the jump. Tracing ancestral lines backwards in time, a p-merger occurs at this time.
- Since $x/(A+x) \geq r$ if and only if $x \geq Ar/(1-r)$, we get p-mergers with $p \geq r$ at rate

$$A \int_{Ar/(1-r)}^\infty bx^{-2}\, dx = \frac{b(1-r)}{r}.$$

- For the Bolthausen-Sznitman coalescent, rate of p-mergers with $p \geq r$ is

$$\int_r^1 p^{-2} \Lambda(dp) = \frac{1-r}{r}.$$

Branching Brownian motion

Begin with some configuration of particles at time zero.

Each particle independently moves according to standard one-dimensional Brownian motion.

Each particle splits into two at rate 1.

Early work on position $M(t)$ of right-most particle (starting with one particle at origin):

- McKean (1975) showed that $u(t,x) = P(M(t) \leq x)$ solves the FKPP equation.
- Bramson (1978) showed that if $m(t)$ is the median of $M(t)$,
$$m(t) = \sqrt{2}t - (3/2\sqrt{2}) \log t + O(1).$$
- Sharper results by Bramson (1983), Lalley-Sellke (1987).

We think of particles as representing individuals in a population, and their position as being the fitness of the individual.

Branching Brownian motion with absorption

To model selection, we kill particles that get too far to the left.

Let $\mu > 0$, and give the Brownian particles a drift of $-\mu$. Kill particles that reach the origin.

Theorem (Kesten, 1978): Starting with one particle at $x > 0$, this process dies out almost surely if $\mu \geq \sqrt{2}$. If $\mu < \sqrt{2}$, the number of particles grows exponentially with positive probability.

Our model: We consider branching Brownian motion with particles killed at the origin and

$$\mu = \mu_N = \sqrt{2 - \frac{2\pi^2}{(\log N + 3 \log \log N)^2}}.$$

The $O((\log N)^{-2})$ correction is related to Conjecture 1.

Other related work: Harris-Harris-Kyprianou (2006), Harris-Harris (2007), Derrida-Simon (2007, 2008).

Branching random walks: Gantert-Hu-Shi (2008), Jaffuel (2009).

Notation

Let $M_N(t)$ be the number of particles at time t.

Let $X_{1,N}(t) \geq X_{2,N}(t) \geq \cdots \geq X_{M_N(t),N}(t)$ be the positions of the particles at time t.

Let $L = \frac{1}{\sqrt{2}}\bigg(\log N + 3 \log \log N\bigg)$.

Let $Y_N(t) = \displaystyle\sum_{i=1}^{M_N(t)} e^{\mu X_{i,N}(t)}$.

Let $Z_N(t) = \displaystyle\sum_{i=1}^{M_N(t)} e^{\mu X_{i,N}(t)} \sin\left(\frac{\pi X_{i,N}(t)}{L}\right) \mathbf{1}_{\{X_{i,N}(t) \leq L\}}$.

$Z_N(t)$ will be a useful measure of the "size" of the process at time t, disregarding particles to the right of L.

Main results

Theorem (Berestycki-Berestycki-Schweinsberg, 2009): Suppose $Z_N(0)/[N(\log N)^2] \Rightarrow \nu$ and $Y_N(0)/[N(\log N)^3] \Rightarrow 0$. Then for some $a \in \mathbb{R}$, the finite-dimensional distributions of

$$\left(\frac{1}{2\pi N} M_N((\log N)^3 t), t > 0\right)$$

converge to those of the CSBP with initial distribution ν and branching mechanism $\Psi(u) = au + 2\pi^2 u \log u$.

Theorem (Berestycki-Berestycki-Schweinsberg, 2009): Fix $t > 0$, and choose n particles at random from the $M_N((\log N)^3 t)$ particles at time $t(\log N)^3$. Let $\Pi_N(s)$ be the partition of $\{1, \dots, n\}$ such that i and j are in the same block if and only if the ith and jth sampled particles have the same ancestor at time $(t - s/2\pi)(\log N)^3$. Under the above assumptions and $\nu(\{0\}) = 0$, the finite-dimensional distributions of $(\Pi_N(s), 0 \leq s \leq 2\pi t)$ converge to those of the Bolthausen-Sznitman coalescent.

The key heuristic

Brunet-Derrida-Mueller-Munier (2006, 2007)

Occasionally, a particle gets very far to the right.

This particle has a large number of surviving descendants, as the descendants avoid the barrier at zero.

This leads to sudden jumps in the number of particles, and multiple mergers of ancestral lines.

The proof strategy

Find the level L such that a particle must reach L to give rise to a jump in the number of particles.

Show that the behavior of branching Brownian motion with particles killed at 0 and L is approximately deterministic.
(This is a "Law of Large Numbers" or "fluid limit" result that is proved by calculating first and second moments.)

Separately determine the (random) contribution of the particles that reach L.

Important questions

1. Why is the drift that keeps the population size stable

$$\mu = \sqrt{2 - \frac{2\pi^2}{(\log N + 3 \log \log N)^2}}\ ?$$

2. Why must a particle get near

$$L = \frac{1}{\sqrt{2}} \left(\log N + 3 \log \log N \right)$$

   to produce a large jump in the population size?

3. Why does $Z_N(t)$ measure of the "size" of the population?

4. Why is the characteristic time scale $(\log N)^3$?

5. Why does the Bolthausen-Sznitman coalescent arise?

   (a) Why is jump rate proportional to the number of particles?

   (b) Why is the rate of jumps of size greater than x proportional to $\int_x^\infty y^{-2}\, dy = x^{-1}$?

Branching Brownian motion in a strip

Consider Brownian motion killed at 0 and L. If there is initially one particle at x, the "density" of the position at time t is:

$$q_t(x,y) = \frac{2}{L} \sum_{n=1}^{\infty} e^{-\pi^2 n^2 t/2L^2} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right).$$

Add branching and drift of $-\mu$, "density" becomes:

$$p_t(x,y) = q_t(x,y) \cdot e^{\mu(x-y)-\mu^2 t/2} \cdot e^t,$$

meaning that if $B \subset (0,L)$, the expected number of particles in B at time t is

$$\int_B p_t(x,y)\, dy.$$

For $t \gg L^2$,

$$p_t(x,y) \approx \frac{2}{L} e^{(1-\mu^2/2-\pi^2/2L^2)t} e^{\mu x} \sin\left(\frac{\pi x}{L}\right) e^{-\mu y} \sin\left(\frac{\pi y}{L}\right).$$

Obervations related to density formula

$$p_t(x, y) \approx \frac{2}{L} e^{(1-\mu^2/2-\pi^2/2L^2)t} e^{\mu x} \sin\left(\frac{\pi x}{L}\right) e^{-\mu y} \sin\left(\frac{\pi y}{L}\right).$$

- When $1-\mu^2/2-\pi^2/2L^2 = 0$, the formula does not depend on t. We choose μ to satisfy this equation, so keep the number of particles relatively stable.

- Formula is proportional to $e^{\mu x} \sin(\pi x/L)$. Summing over multiple particles at time zero, this becomes $Z_N(0)$. Thus, $Z_N(0)$ determines how many particles will be in a given set at future times.

- With μ chosen as above, $(Z_N(t), t \geq 0)$ is a martingale.

- Formula is proportional to $e^{-\mu y} \sin(\pi y/L)$. For $t \gg (\log N)^2$, particles settle into a fairly stable limiting configuration.

The choice of L

Initially place N particles on $(0, L)$ at random with density

$$g(y) = CLe^{-\mu y} \sin\left(\frac{\pi y}{L}\right).$$

Then

$$Z_N(0) \approx N \int_0^L e^{\mu y} \sin\left(\frac{\pi y}{L}\right) g(y)\, dy = O(NL^2).$$

By the martingale property, $Z_N(t) = O(NL^2)$ for larger t.

If we start with a single particle at L, usually the right-most descendant will only get to $L + \alpha$, where α is a constant. Therefore

$$Z_N(t) \approx e^{\mu L} \sin\left(\frac{L}{L+\alpha}\right) = O(L^{-1} e^{\mu L}).$$

Note that NL^2 and $L^{-1}e^{\mu L}$ are of the same order when

$$L = \frac{1}{\sqrt{2}}\left(\log N + 3 \log\log N\right) + O(1).$$

The $(\log N)^3$ time scale

We need to determine how often particles hit L.

Fix $\beta > 0$. A particle between $L-\beta$ and L at time t has positive probability of hitting L by time $t+1$.

Configuration at time t is like that of N particles with density

$$g(y) = CLe^{-\mu y} \sin\left(\frac{\pi y}{L}\right).$$

Expected number of particles between $L - \beta$ and L is approximately

$$N \int_{L-\beta}^{L} CLe^{-\mu y} \sin\left(\frac{\pi y}{L}\right) dy = O((\log N)^{-3}).$$

We have to wait $O((\log N)^3)$ time for a particle to reach L.

A continuous-time branching process

Consider branching Brownian motion with drift $-\sqrt{2}$ started with one particle at L.

Let $M(y)$ be the number of particles that reach $L-y$, if particles are killed upon reaching $L-y$.

Conditional on $M(x)$, $M(x+y)$ is the sum of $M(x)$ independent random variables with the same distribution as $M(y)$. Therefore, $(M(y), y \geq 0)$ is a continuous-time branching process.

Offspring distribution has finite mean, not in the $L \log L$ class.

Theorem (Neveu, 1987): There exists a random variable W such that almost surely

$$\lim_{y \to \infty} y e^{-\sqrt{2}y} M(y) = W.$$

Furthermore, for all $u \in \mathbb{R}$,

$$E[e^{-e^{\sqrt{2}u}W}] = \psi(u),$$

where ψ satisfies Kolmogorov's equation $\frac{1}{2}\psi'' - \sqrt{2}\psi' = \psi(1-\psi)$.

The tail behavior of W

Lemma: As $x \to \infty$, we have $P(W > x) \sim \dfrac{1}{x\sqrt{2}}$.

Proof Outline:

- Use a Tauberian theorem to reduce this to a problem about $E[e^{-\lambda W}]$ for small values of λ.
- Since $E[e^{-e^{\sqrt{2}u}W}] = \psi(u)$, this reduces to a problem about the asymptotic behavior of $\psi(u)$ as $u \to -\infty$.
- Following an idea of Harris (1999), we obtain this asymptotic result probabilistically, reducing it to a property of the three-dimensional Bessel process.

Connection with Bolthausen-Sznitman coalescent

Fix $\theta > 0$ small. Expected number of particles that reach

$$L_A = \frac{1}{\sqrt{2}}\bigg(\log N + 3\log\log N - A\bigg)$$

by time $t = \theta(\log N)^3$ is roughly $2\pi e^A \theta \cdot Z_N(0)/(N(\log N)^2)$.

Choose a large constant y. If particle hits L_A before time t, it has approximately $y^{-1}e^{\sqrt{2}y}W$ descendants that reach $L_A - y$.

Each such descendant contributes $\sqrt{2}\pi y e^{-\sqrt{2}y} N(\log N)^2$ to $Z_N(t)$.

Contribution to $Z_N(t)$ of the particle that hits L_A is roughly $\pi\sqrt{2}e^{-A}N(\log N)^2 W$. Exceeds $xN(\log N)^2$ if $W \geq xe^A/(\pi\sqrt{2})$.

When A is large, the probability that this occurs is approximately

$$\frac{1}{\sqrt{2}} \cdot \frac{\pi\sqrt{2}}{xe^A} = \pi x^{-1} e^{-A},$$

so

$$P(Z_N(t) - Z_N(0) > xN(\log N)^2) \approx 2\pi^2 x^{-1}\theta \cdot \frac{Z_N(0)}{N(\log N)^2}.$$