The genealogy of branching Brownian motion with absorption

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A population model with selection

Brunet-Derrida-Mueller-Munier (2006, 2007)

- Population has fixed size N.
- Each individual has $k \geq 2$ offspring.
- The fitness of each offspring is the parent's fitness plus an independent random variable with distribution μ.
- Of the kN offspring, the N with the highest fitness survive to form the next generation.

Durrett-Mayberry (2009) studied related model in context of predator-prey systems.

Related work: Bérard-Gouéré (2009), Durrett-Remenik (2009).

Brunet-Derrida-Mueller-Munier (2006, 2007) studied solutions $u(x,t)$ to noisy FKPP equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^2 + \sqrt{\frac{u(1-u)}{N}}\, W(x,t),$$

where $W(x,t)$ is space-time white noise. Arrived at 3 conjectures:

1. Let L_m be the maximum of the fitnesses of the individuals in generation m. Then $L_m/m \to v_N$ a.s. Let $v_\infty = \lim_{N\to\infty} v_N$. Then $v_\infty - v_N \sim C/(\log N)^2$.

2. If two individuals are sampled from the population at random in some generation, the number of generations back to their most recent common ancestor is $O((\log N)^3)$.

3. If n individuals are sampled from some generation and their ancestral lines are followed backwards in time, the coalescence of these lineages is be described by the Bolthausen-Sznitman coalescent.

Theorem (Bérard-Gouéré, 2009): Conjecture 1 holds in the case $k = 2$ under suitable regularity conditions on μ.

Bolthausen-Sznitman coalescent

Let Λ be a finite measure on $[0,1]$. The Λ-coalescent (Pitman (1999), Sagitov (1999)) has the property that when there are b lineages, each k-tuple ($2 \leq k \leq b$) of blocks merges at rate

$$\lambda_{b,k} = \int_0^1 p^{k-2}(1-p)^{b-k}\,\Lambda(dp).$$

When $\Lambda(\{0\}) = 0$, obtain a Λ-coalescent from a Poisson point process on $[0,\infty) \times (0,1]$ with intensity $dt \times p^{-2}\Lambda(dp)$. If (t,p) is a point of this Poisson process, a p-merger occurs at time t. That is, we flip a coin for each lineage having probability p of heads, and merge all lineages whose coin is a head.

Bolthausen-Sznitman (1998) coalescent: Λ is uniform on $[0,1]$. Linked to Derrida's GREM (Bovier-Kurkova, 2007).

Our model: branching Brownian motion with absorption

- Begin with some configuration of particles in $(0, \infty)$.
- Each particle independently moves according to standard one-dimensional Brownian motion with drift $-\mu$.
- Each particle splits into two at rate 1.
- Particles are killed if they reach the origin.

Particles represent individuals in a population, and their position represents the fitness of the individual.

Theorem (Kesten, 1978): Starting with one particle at $x > 0$, this process dies out almost surely if $\mu \geq \sqrt{2}$. If $\mu < \sqrt{2}$, the number of particles grows exponentially with positive probability.

We take $\mu = \mu_N = \sqrt{2 - \frac{2\pi^2}{(\log N + 3 \log \log N)^2}}$.

The $O((\log N)^{-2})$ correction is related to Conjecture 1.

Notation

Let $M_N(t)$ be the number of particles at time t.

Let $X_{1,N}(t) \geq X_{2,N}(t) \geq \cdots \geq X_{M_N(t),N}(t)$ be the positions of the particles at time t.

Let $L = \frac{1}{\sqrt{2}}\bigg(\log N + 3 \log\log N\bigg)$.

Let $Y_N(t) = \sum_{i=1}^{M_N(t)} e^{\mu X_{i,N}(t)}$.

Let $Z_N(t) = \sum_{i=1}^{M_N(t)} e^{\mu X_{i,N}(t)} \sin\left(\frac{\pi X_{i,N}(t)}{L}\right) \mathbf{1}_{\{X_{i,N}(t) \leq L\}}$.

$Z_N(t)$ will be a useful measure of the "size" of the process at time t, disregarding particles to the right of L.

Main result

Theorem (Berestycki-Berestycki-Schweinsberg, 2009): Suppose $Z_N(0)/[N(\log N)^2] \Rightarrow \nu$ and $Y_N(0)/[N(\log N)^3] \Rightarrow 0$. Fix $t > 0$. Choose n particles at random from the $M_N((\log N)^3 t)$ particles at time $t(\log N)^3$. Let $\Pi_N(s)$ be the partition of $\{1, \dots, n\}$ such that i and j are in the same block if and only if the ith and jth sampled particles have the same ancestor at time $(t - s/2\pi)(\log N)^3$. Then the finite-dimensional distributions of $(\Pi_N(s), 0 \leq s \leq 2\pi t)$ converge to those of the Bolthausen-Sznitman coalescent.

Note: The initial conditions will be satisfied of $O(N)$ particles are put down in a relatively "stable" configuration, with no particles too far to the right.

The key heuristic

Brunet-Derrida-Mueller-Munier (2006, 2007)

Occasionally, a particle gets very far to the right.

This particle has a large number of surviving descendants, as the descendants avoid the barrier at zero.

This leads to multiple mergers of ancestral lines.

The proof strategy

Find the level L such that a particle must reach L to give rise to a jump in the number of particles.

Show that the behavior of branching Brownian motion with particles killed at 0 and L is approximately deterministic (Law of Large Numbers).

Separately consider the particles that reach L.

Branching Brownian motion in a strip

Consider Brownian motion killed at 0 and L. If there is initially one particle at x, the "density" of the position at time t is:

$$q_t(x,y) = \frac{2}{L} \sum_{n=1}^{\infty} e^{-\pi^2 n^2 t/2L^2} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right).$$

Add branching and drift of $-\mu$, "density" becomes:

$$p_t(x,y) = q_t(x,y) \cdot e^{\mu(x-y)-\mu^2 t/2} \cdot e^t.$$

For $t \gg L^2$,

$$p_t(x,y) \approx \frac{2}{L} e^{(1-\mu^2/2-\pi^2/2L^2)t} e^{\mu x} \sin\left(\frac{\pi x}{L}\right) e^{-\mu y} \sin\left(\frac{\pi y}{L}\right).$$

Key points:

- The "density" of particles at time t is proportional to $Z_N(0)$.
- For $t \gg (\log N)^2$, particles settle into a limiting configuration.
- For μ and L as chosen, $(Z_N(t), t \geq 0)$ is a martingale.

A continuous-time branching process

Consider branching Brownian motion with drift $-\sqrt{2}$ started with one particle at L.

Let $M(y)$ be the number of particles that reach $L-y$, if particles are killed upon reaching $L-y$.

Conditional on $M(x)$, the distribution of $M(x+y)$ independent random variables with the same distribution as $M(y)$. Therefore, $(M(y), y \geq 0)$ is a continuous-time branching process.

Offspring distribution has finite mean, not in the $L \log L$ class.

Theorem (Neveu, 1987): There exists a random variable W such that almost surely

$$\lim_{y \to \infty} y e^{-\sqrt{2}y} M(y) = W.$$

Furthermore, for all $u \in \mathbb{R}$,

$$E[e^{-e^{\sqrt{2}u} W}] = \psi(u),$$

where ψ satisfies Kolmogorov's equation $\frac{1}{2}\psi'' - \sqrt{2}\psi' = \psi(1-\psi)$.

The tail behavior of W

Lemma: As $x \to \infty$, we have $P(W > x) \sim \dfrac{1}{x\sqrt{2}}$.

Proof Outline:

- Use a Tauberian theorem to reduce this to a problem about $E[e^{-\lambda W}]$ for small values of λ.

- Since $E[e^{-e^{\sqrt{2}u}W}] = \psi(u)$, this reduces to a problem about the asymptotic behavior of $\psi(u)$ for large u.

- Following an idea of Harris (1999), we obtain this asymptotic result probabilistically, reducing it to a property of the three-dimensional Bessel process.

Connection with Bolthausen-Sznitman coalescent

Waiting time for a particle to hit L is typically $O((\log N)^3)$.

Rate at which particles reach L is proportional to $Z_N(t)$.

If a particle hits L, its contribution is proportional to the number of descendants that hit $L - y$ for large y. The probability that $Z = Z_N(t)/(N(\log N)^2)$ jumps by at least r is roughly Cr^{-1}.

After a jump of size r, a fraction $p = r/(Z + r)$ of the population is descended from the particle that hit L. Tracing lineages backwards in time, a p-merger occurs at this time.

Since $r/(Z+r) \geq x$ if and only if $r \geq Zx/(1-x)$, we get p-mergers with $p \geq x$ at rate

$$CZ \cdot \frac{1-x}{Zx} = \frac{C(1-x)}{x}.$$

For the Bolthausen-Sznitman coalescent, rate of p-mergers with $p \geq x$ is

$$\int_x^1 p^{-2} \Lambda(dp) = \frac{1-x}{x}.$$

Continuous-state branching processes

Because the rate of jumps of size at least r is proportional to $Z_N(t)$ and r^{-1}, we get convergence to a CSBP.

Theorem (Berestycki-Berestycki-Schweinsberg, 2009): Under the above initial conditions, there is an $a \in \mathbb{R}$ such that the finite-dimensional distributions of

$$\left(\frac{1}{N(\log N)^2} Z_N((\log N)^3 t), t \geq 0\right)$$

and those of

$$\left(\frac{1}{2\pi N} M_N((\log N)^3 t), t > 0\right)$$

converge to the finite-dimensional distrbutions of the CSBP with branching mechanism $\Psi(u) = au + 2\pi^2 u \log u$.

Note: Neveu (1992) had studied the CSBP with $\Psi(u) = u \log u$. Bertoin and Le Gall (2000) showed that the genealogy of Neveu's CSBP is given by the Bolthausen-Sznitman coalescent.