

The loop-erased random walk and the uniform spanning tree on the four-dimensional discrete torus

by Jason Schweinsberg

University of California at San Diego

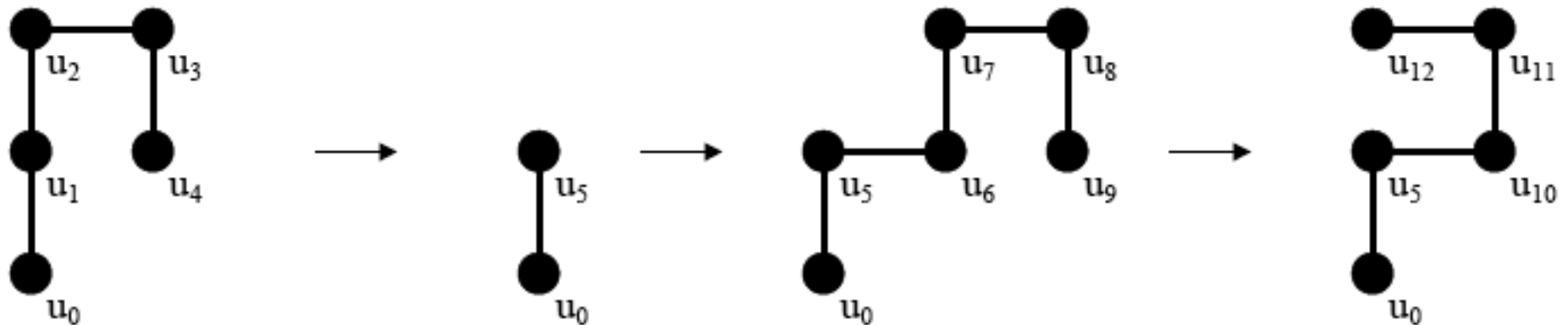
Outline of Talk

1. Loop-erased random walk
2. Uniform spanning tree
3. Continuum random tree
4. Main result
5. Outline of proof

Loop-erased random walk

Let $\lambda = (u_0, u_1, \dots, u_j)$ be a path in a graph $G = (V, E)$.

Define the loop-erasure $LE(\lambda)$ by erasing loops in the order in which they appear.



A random walk $(X_t)_{t=0}^{\infty}$ on a graph G is a V -valued Markov chain. At each step, moves to a randomly chosen neighboring vertex.

The loop-erased random walk (LERW) is the path $LE((X_t)_{t=0}^{\infty})$.

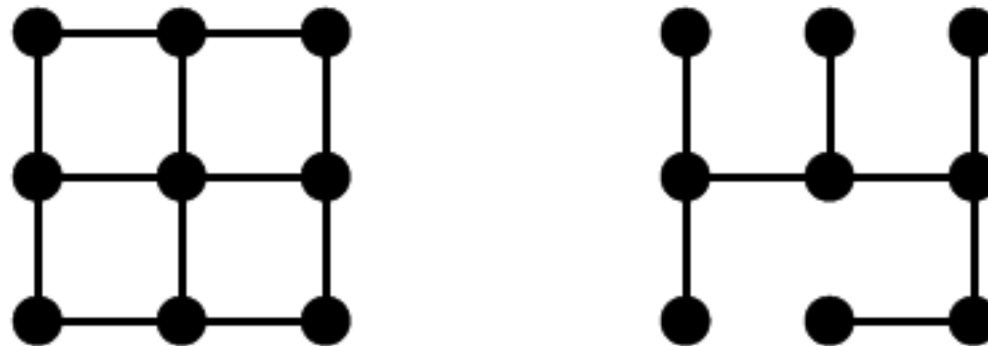
Well-defined when $(X_t)_{t=0}^{\infty}$ is transient.

Behavior of LERW on \mathbb{Z}^d :

- $d \geq 5$ (Lawler, 1980): All loops are short, length of the path $LE((X_t)_{t=0}^n)$ is $O(n)$, process converges to Brownian motion.
- $d = 4$ (Lawler, 1986, 1995): Length of the path $LE((X_t)_{t=0}^n)$ is $O(n/(\log n)^{1/3})$, process converges to Brownian motion.
- $d = 2$ (Lawler-Schramm-Werner, 2004): LERW converges to SLE(2).
- $d = 3$ (Kozma, 2005): Scaling limit exists.

Uniform spanning tree

A spanning tree of a finite connected graph G is a connected subgraph of G containing every vertex and no cycles.

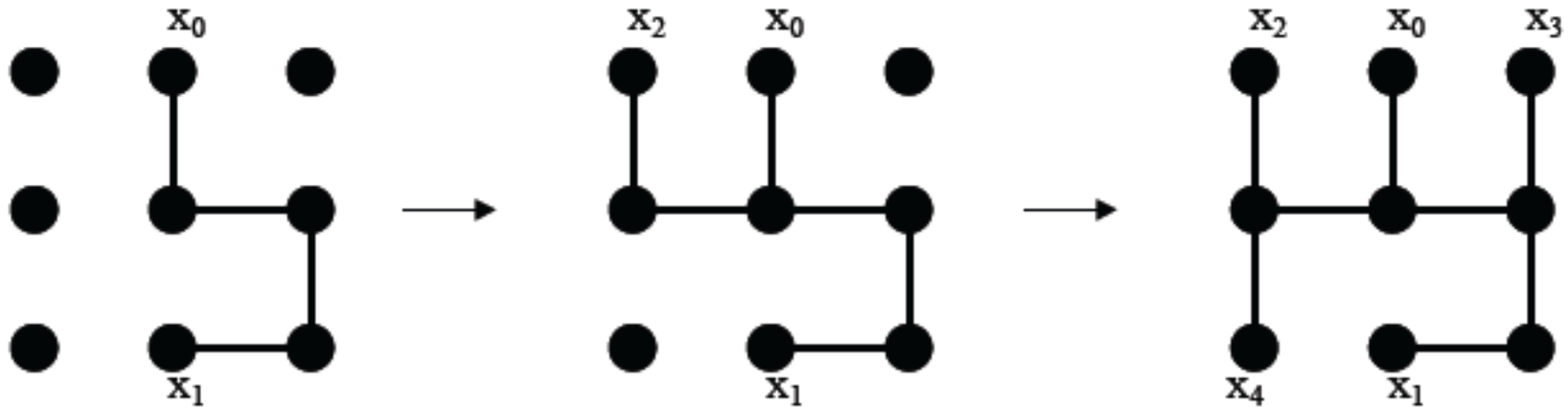


A uniform spanning tree (UST) is a spanning tree chosen uniformly at random.

Wilson's Algorithm (Wilson, 1996)

One can construct a UST of G as follows:

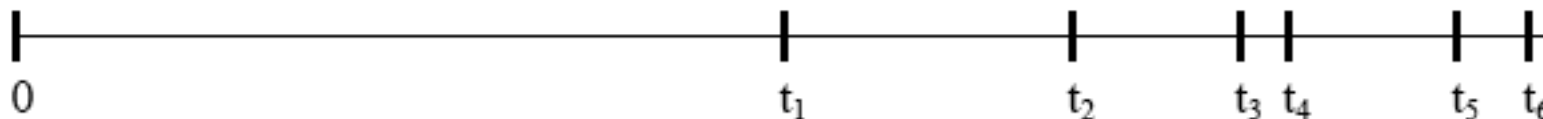
- Pick vertices x_0 and x_1 , run LERW from x_0 to x_1 to get \mathcal{T}_1 .
- Given \mathcal{T}_k , pick a vertex x_{k+1} , get \mathcal{T}_{k+1} by adjoining to \mathcal{T}_k an LERW from x_{k+1} to \mathcal{T}_k .
- Continue until all vertices are in the tree.



Can choose vertices in any order, can depend on current tree.

Continuum Random Tree (Aldous, 1991, 1993)

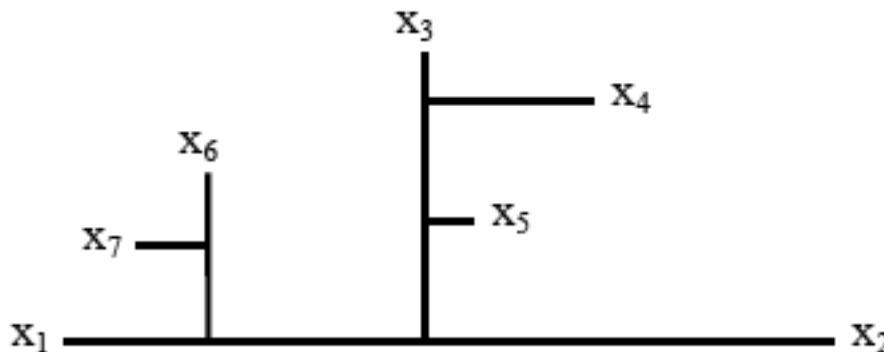
Consider Poisson process on $[0, \infty)$, intensity $r(t) = t$.



Begin with a segment of length t_1 , call the endpoints x_1, x_2 .

Attach segment of length $t_2 - t_1$ to a uniform point on the initial segment, label the endpoint x_3 .

Continue, each segment orthogonal to all previous segments.



Limiting random metric space is continuum random tree (CRT).

Denote by μ_k the distribution of $(d(x_i, x_j))_{1 \leq i < j \leq k}$.

Scaling limits of UST on finite graphs

Theorem (Aldous, 1991): Consider the UST on the complete graph K_m with m vertices (equivalently, a uniform random tree on m labeled vertices). Let y_1, \dots, y_k be vertices chosen uniformly a random. Let $d(y_i, y_j)$ be the number of vertices on the path from y_i to y_j in the UST. Then

$$\left(\frac{d(y_i, y_j)}{\sqrt{m}} \right)_{1 \leq i < j \leq k} \xrightarrow{d} \mu_k.$$

Theorem (Peres-Revelle, 2004): Consider the UST on the torus \mathbb{Z}_n^d for $d \geq 5$. Let x_1, \dots, x_k be vertices chosen uniformly at random. Then there is a constant β such that

$$\left(\frac{d(x_i, x_j)}{\beta n^{d/2}} \right)_{1 \leq i < j \leq k} \xrightarrow{d} \mu_k.$$

The CRT scaling limit also holds for UST on larger class of graphs including hypercubes \mathbb{Z}_2^n , expander graphs.

Corollary (Peres-Revelle, 2004): Let x and y be uniformly chosen from \mathbb{Z}_n^d , $d \geq 5$. Let $(X_t)_{t=0}^T$ be a random walk from x to y . Then

$$\lim_{n \rightarrow \infty} P(|LE((X_t)_{t=0}^T)| > \beta n^{d/2} x) = e^{-x^2/2}.$$

Limiting distribution called Rayleigh distribution.

Note: Benjamini-Kozma (2005) had proved that for $d \geq 5$ the length of LERW from x to y on \mathbb{Z}_n^d is $O(n^{d/2})$. They conjectured the length in \mathbb{Z}_n^4 is $O(n^2(\log n)^{1/6})$.

Theorem (Schweinsberg, 2006): Consider the UST on \mathbb{Z}_n^4 . Let x_1, \dots, x_k be vertices chosen uniformly at random. There is a sequence of constants $(\gamma_n)_{n=1}^\infty$ bounded away from 0 and ∞ such that

$$\left(\frac{d(x_i, x_j)}{\gamma_n n^2 (\log n)^{1/6}} \right)_{1 \leq i < j \leq k} \xrightarrow{d} \mu_k.$$

In particular, if $(X_t)_{t=0}^T$ is a random walk from x_1 to x_2 , then

$$\lim_{n \rightarrow \infty} P(|LE((X_t)_{t=0}^T)| > \gamma_n n^2 (\log n)^{1/6} x) = e^{-x^2/2}.$$

Coupling idea (Peres-Revelle, 2004)

- Choose y_1, \dots, y_k uniformly from K_m .
- Choose x_1, \dots, x_k uniformly from \mathbb{Z}_n^4 .
- Construct partial UST $\tilde{\mathcal{T}}_k$ on K_m using Wilson's algorithm, starting random walks from y_1, \dots, y_k .
- Construct partial UST \mathcal{T}_k on \mathbb{Z}_n^4 using Wilson's algorithm, starting random walks from x_1, \dots, x_k .
- Segments in random walks of length $r = \lfloor n^2(\log n)^{9/22} \rfloor$ on \mathbb{Z}_n^4 correspond to individual vertices in walks on K_m .
- Obtain tree \mathcal{T}_k^* from \mathcal{T}_k by collapsing random walk segments of length r into a single vertex.
- Find coupling such that $\tilde{\mathcal{T}}_k = \mathcal{T}_k^*$ with high probability.
- Deduce CRT limit for UST on \mathbb{Z}_n^4 from Aldous' result on K_m .

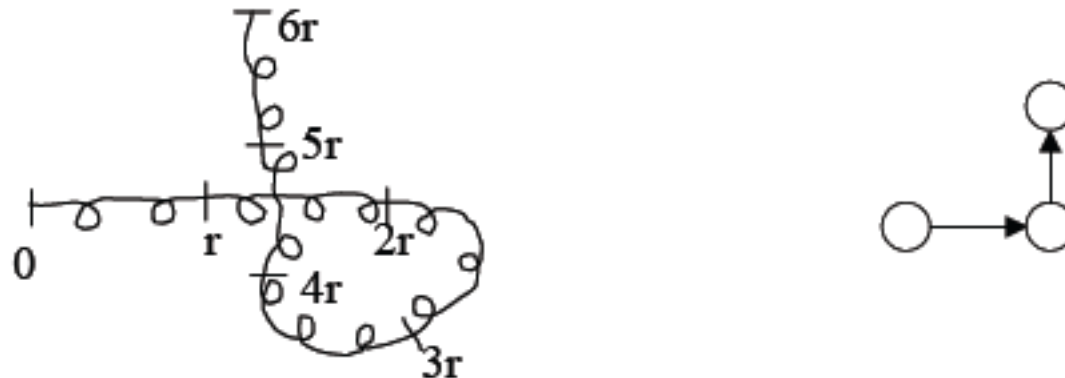
Coupling of random walks

Random walk on \mathbb{Z}_n^4 makes short loops (occur within segment of length r) and long loops (occur on \mathbb{Z}_n^4 but not on \mathbb{Z}^4). Long loops correspond to loops of walk on K_m .

Example: random walk on K_6 begins (3, 4, 6, 2, 4, 1).



Random walk $(X_t)_{t=0}^{6r-1}$ on \mathbb{Z}_n^4 . Suppose that $X_s = X_t$ for some $s \in [r, 2r)$ and $t \in [4r, 5r)$.



Coupling k random walks gives coupling of $\tilde{\mathcal{T}}_k$ and \mathcal{T}_k^* .

Intersection probabilities for loop-erased segments

Let $(V_t)_{t=0}^{\infty}$ and $(W_t)_{t=0}^{\infty}$ be random walks on \mathbb{Z}^d started at origin. Let R_n be the cardinality of $\{s, t \in \{1, \dots, n\} : V_s = W_t\}$. If $d = 4$, then $E[R_n] = O(\log n)$.

Mixing time for a random walk on \mathbb{Z}_n^4 is $O(n^2)$.

To find the probability of intersection between two segments of length r , assume they both start from the uniform distribution.

Let $(X_t)_{t=0}^{r-1}$ and $(Y_t)_{t=0}^{r-1}$ be random walks on \mathbb{Z}_n^4 started from the uniform distribution.

- $P(X_s = Y_t) = 1/n^4$ for all s, t .
- Expected number of intersections is r^2/n^4 .
- If there is one intersection, there are $O(\log r)$ intersections.
- Probability of intersection is $O\left(\frac{r^2}{n^4 \log r}\right) = O((\log n)^{-2/11})$.

Given two independent transient Markov chains with the same transition probabilities, loop-erasing one path reduces the probability that the paths intersect by at most a factor of 2^8 (Lyons-Peres-Schramm, 2003).

Probability $LE((X_t)_{t=0}^{r-1})$ and $(Y_t)_{t=0}^{r-1}$ intersect is $O((\log n)^{-2/11})$.

For $U \subset \mathbb{Z}_n^4$, let $\text{Cap}_r(U) = P(Y_t \in U \text{ for some } t < r)$.

We have $E[\text{Cap}_r(LE((X_t)_{t=0}^{r-1}))] = a_n(\log n)^{-2/11}$ for a sequence of constants $(a_n)_{n=1}^{\infty}$ bounded away from 0 and ∞ .

Distribution of $\text{Cap}_r(LE((X_t)_{t=0}^{r-1}))$ is highly concentrated around its mean (break walk into pieces, apply LLN to the probabilities of hitting individual pieces).

Let $m = \lfloor a_n^{-1}(\log n)^{2/11} \rfloor$. The next segment intersects each previous segment with probability approximately $1/m$.

Trees $\tilde{\mathcal{T}}_k$ and \mathcal{T}_k^* coupled with high probability.

Length of loop-erased segments

Couple $(X_t)_{t=0}^{r-1}$ with walk $(Z_t)_{t=0}^{r-1}$ on \mathbb{Z}^4 so that $X_t = Z_t \pmod{n}$.

$$P(X_s = X_t \text{ and } Z_s \neq Z_t \text{ for some } s, t) \leq \frac{Cr^2}{n^4 \log r} = o(1).$$

Lengths of $LE((X_t)_{t=0}^{r-1})$ and $LE((Z_t)_{t=0}^{r-1})$ are $O\left(\frac{r}{(\log r)^{1/3}}\right)$.

We have $E[|LE((X_t)_{t=0}^{r-1})|] = b_n n^2 (\log n)^{5/66}$ for a sequence of constants $(b_n)_{n=1}^{\infty}$ bounded away from 0 and ∞ . Distribution of length is concentrated around mean.

Approximate $d(x_i, x_j)$ by multiplying number of vertices on path in \mathcal{T}_k^* by $b_n n^2 (\log n)^{5/66}$.

Distances $d(x_i, x_j)$ now coupled with $d(y_i, y_j)$ in K_m .

Adding a root vertex

For the first step of Wilson's algorithm, we need to run LERW from x to y , but it takes $O(n^4)$ steps of a random walk to hit y .

Walks on \mathbb{Z}_n^4 of length L intersect with probability $O\left(\frac{L^2}{n^4 \log L}\right)$.

Intersections first occur when $L = O(n^2(\log n)^{1/2})$.

Add root vertex ρ to \mathbb{Z}_n^4 , connected to all vertices. Random walk goes to ρ after a geometric number of steps with mean $\beta n^2(\log n)^{1/2}$.

To apply Wilson's algorithm, first run LERW from x_1 to ρ , then start next walks at x_2, \dots, x_k . This gives weighted spanning tree.

Removing edges leading to ρ gives spanning forest on \mathbb{Z}_n^4 .

Stochastic domination

(Peres-Revelle, 2004): If $d'(x_i, x_j)$ denote distances in the spanning forest, total variation distance between $(d'(x_i, x_j))_{1 \leq i < j \leq k}$ and $(d(x_i, x_j))_{1 \leq i < j \leq k}$ is at most the probability that x_1, \dots, x_k are in different tree components. Choose β large to reduce this below ϵ .

Add root to K_m , so the probability of going to the root in one step is same as probability that r -step walk on \mathbb{Z}_n^4 visits the root.

Where does the 1/6 come from?

Need walks of length $O(n^2(\log n)^{1/2})$ to get intersections.

After loop-erasure, length multiplied by $(\log n)^{-1/3}$.

Dynamics of LERW

Let $(X_t)_{t=0}^{\infty}$ be a random walk on \mathbb{Z}_n^d , $d \geq 4$.

Let $Y_t = |LE((X_s)_{s=0}^t)|$ be length of the loop-erasure at time t .

- Y_t increases linearly when there are no long loops.
- Long loops happen at rate proportional to Y_t .
- Long loops hit a uniform point on path.

Definition (Evans-Pitman-Winter, 2006): The Rayleigh process $(R(t), t \geq 0)$ is a $[0, \infty)$ -valued Markov process such that:

- $R(t)$ increases linearly at unit speed between jumps.
- At time t , jump rate is $R(t-)$. At jump times, process gets multiplied by an independent $\text{Uniform}(0, 1)$ random variable.

Rayleigh distribution is stationary distribution.

Theorem (Schweinsberg, 2006): For some constants a_n and b_n , the processes $(b_n Y_{\lfloor a_n t \rfloor}, t \geq 0)$ converge to $(R(t), t \geq 0)$.

Note: This result was conjectured by Jim Pitman. Result for the complete graph was proved by Evans, Pitman, Winter (2006).