Math 10C Calculus       Practice Final Exam       June 4, 2007

Name: ______________________       ID: ______________________

Section Hour: ______________       TA: ______________________

Guidelines for the test:

• No books, notes, or calculators are allowed.
• Use the space provided. If necessary, write “see other side” and continue working on the back of the same sheet.
• **Circle your final answers** when relevant.
• Show all steps in your solutions and make your reasoning clear. Answers with no explanation will receive no credit, even if they are correct.
• You have 180 minutes.

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1. In 1950 an experiment was done observing the time gaps between successive cars on the Arroyo Seco Freeway. The data showed that the density function of these time gaps was given approximately by

\[ p(x) = 0.122e^{-0.122x} \]

where \( x \) is the time in seconds.

(a) (5 points) Compute the probability that the gap between any particular pair of successive cars is between 2 and 4 seconds.

(b) (5 points) Compute the mean time gap.

(c) (5 points) Compute the median time gap.

(d) (5 points) Sketch rough graphs of \( p \) and its associated cumulative distribution function \( P \).
2. Determine whether or not each of the following series is geometric. For those that are, compute the value of the series or explain why it does not converge.

(a) (5 points) $3 + 9 + 27 + 81 + 243 + \cdots + 3^{21} + 3^{25}$

(b) (5 points) $3 + 9 + 27 + 81 + 243 + \cdots$

(c) (5 points) $\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \frac{1}{5^5} + \cdots$

(d) (5 points) $x - x^3 + x^5 - x^7 + x^9 - \cdots$
3. Suppose $f(x)$ is a function whose 6th-degree Taylor polynomial about $x = 1$ is

$$P_6(x) = 4 - 2(x - 1)^2 + (x - 1)^4 + 8(x - 1)^6.$$ 

(a) (7 points) Explain why $f$ must have a local maximum at $x = 1$.

(b) (7 points) What is the value of $f^{(6)}(1)$?

(c) (6 points) Use $P_6(x)$ to approximate $f(1.1)$. 

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4. Let \( f(x, y) = 8 - x^2 - 2y^2 \).
(a) (5 points) Sketch a graph of the trace of \( f \) in the \( xz \)-plane.

(b) (5 points) Sketch a graph of the trace of \( f \) in the \( yz \)-plane.

(c) (5 points) Sketch a graph of the level curve \( f(x, y) = 0 \).

(d) (5 points) Sketch a graph of the portion of the surface \( z = f(x, y) \) that lies in the first octant.
5. A ship is heading due west at 50 km/hr relative to the water. The current is moving towards the southeast at 10 km/hr.

(a) (5 points) Give the vector (in component form) representing the movement of the current.

(b) (5 points) Give the vector representing the actual movement of the boat, relative to dry land.

(c) (5 points) How fast is the boat going, relative to dry land?

(d) (5 points) By what angle does the current push the boat off of its due west course?
6. An experiment to test the memory of rats is conducted by repeatedly putting rats through the same maze and measuring the time it takes them to find the cheese at the end of the maze. The values in the table below show the time \( t = f(d, n) \), measured in minutes, it takes an average rat to complete a maze as a function of the length \( d \) of the maze, measured in meters, and the number \( n \) of times the rat has previously completed the same maze.

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(a) (10 points) Approximate \( f_d(6, 20) \) and explain in words what this partial derivative represents in the context of the problem.

(b) (10 points) Approximate \( f_n(6, 20) \) and explain in words what this partial derivative represents in the context of the problem.
7. Let \( f(x, y) = xe^{xy} \).

(a) (10 points) Find the rate of change of \( f \) at the point \((3, 0)\) in the direction of the vector \( 2\mathbf{i} - \mathbf{j} \).

(b) (10 points) Find the equation of the tangent plane to the surface \( z = f(x, y) \) at the point \((3, 0)\).
8. Let \( f(x, y) = g(x)h(y) \), where \( g \) (left) and \( h \) (right) are differentiable functions whose graphs are given below.

\[
\begin{array}{cc}
\text{z} & 4 \\
3 & \\
2 & \\
1 & 1
\end{array}
\]
\[
\begin{array}{cc}
\text{z} & 4 \\
3 & \\
2 & \\
1 & 1
\end{array}
\]

(a) (10 points) Find all critical points of \( f \) in the region
\[
R = \{(x, y)\mid -2 \leq x \leq 2, -2 \leq y \leq 2\}.
\]
(Hint: There are exactly two.)

(b) (10 points) Classify each critical point as a local maximum, local minimum, or saddle. Justify your answer.
9. A firm manufactures a commodity at two different factories. The total cost of manufacturing depends on the quantities, $q_1$ and $q_2$, supplied by each factory, and is expressed by the joint cost function,

$$C = f(q_1, q_2) = 3q_1^2 + 2q_1q_2 + 4q_2^2 + 100.$$ 

The company’s objective is to produce 1200 units, while minimizing production costs. How many units should be supplied by each factory?
10. Let \( f(x, y) = 2x(y + 1) \).

(a) (10 points) Find the critical points of \( f \) and determine if they are local maxima, minima, or saddle points.

(b) (10 points) Use the method of Lagrange Multipliers to find the maximum and minimum values of \( f \) subject to the constraint

\[
x^2 + y^2 = 1,
\]

and indicate all points \((x, y)\) at which these values occur.
**Median and Mean**

Let \( p(x) \) be a probability density function.

- The median value of \( x \) is the value \( T \) satisfying
  \[
  \int_{-\infty}^{T} p(x)dx = 0.5.
  \]

- The mean value \( \mu \) of \( x \) is given by
  \[
  \mu = \int_{-\infty}^{\infty} xp(x)dx.
  \]

**Geometric Series**

\[
a + ax + ax^2 + \ldots + ax^{n-1} = a \left( \frac{1-x^n}{1-x} \right), \text{ if } x \neq 1
\]
\[
a + ax + ax^2 + ax^3 + \ldots = \frac{a}{1-x}, \text{ if } |x| < 1
\]

(If \( |x| \geq 1 \), then the infinite series diverges.)

**Taylor Polynomial**

The \( n \textsuperscript{th} \)-degree Taylor polynomial for \( f(x) \) based at \( x = a \) is given by

\[
P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n,
\]

where \( f^{(n)}(a) \) is the \( n \textsuperscript{th} \) derivative of \( f \), evaluated at \( a \), and \( n! = n(n-1)(n-2)\ldots(3)(2)(1) \).

**Vector Projection**

If \( \vec{v} \) is a vector and \( \vec{u} \) is a unit vector, then the projection of \( \vec{v} \) onto the line determined by \( \vec{u} \) is given by

\[
\text{proj}_a \vec{v} = \text{parallel} = (\vec{v} \cdot \vec{u}) \vec{u}.
\]

The *orthogonal* projection of \( \vec{v} \) (i.e., the component of \( \vec{v} \) which is orthogonal to \( \vec{u} \)) is given by

\[
\text{orth}_a \vec{v} = \vec{v}_{\text{perp}} = \vec{v} - \vec{v}_{\text{parallel}}.
\]

**Differential**

The differential, \( df \) (or \( dz \)), of a differentiable function \( f \) at a point \((a,b)\) is the linear function of \( dx \) and \( dy \) given by the formula

\[
df = f_x(a,b)dx + f_y(a,b)dy.
\]
Directional Derivative

If $f$ is differentiable at $(a, b)$ and $\vec{u} = \vec{u}_1 \vec{i} + \vec{u}_2 \vec{j}$ is a unit vector, then the rate of change of $f$ at $(a, b)$ in the direction of $\vec{u}$ is given by

$$f_{\vec{u}}(a, b) = f_x(a, b)\vec{u}_1 + f_y(a, b)\vec{u}_2 = \nabla f \cdot \vec{u},$$

where $\nabla f(a, b)$ (or $\text{grad } f(a, b)$) is the gradient vector of $f$ at $(a, b)$.

First-Degree Taylor Polynomial

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Second-Degree Taylor Polynomial

$$Q(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + \frac{f_{xx}(a,b)}{2}(x - a)^2 + f_{xy}(a, b)(x - a)(y - b) + \frac{f_{yy}(a,b)}{2}(y - b)^2$$

Second Derivative Test

Suppose that $(a, b)$ is a critical point of $f$, and let

$$D = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix}.$$  

Then

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

(d) If $D = 0$ or is undefined, or if $D > 0$ and $f_{xx}(a, b) = 0$, then the test is inconclusive.

Lagrange Multipliers

To find the absolute (global) maximum and minimum values of $f(x, y)$ subject to the constraint $g(x, y) = k$, compare the values of $f$ at all points $(a, b)$ where the gradient vector of $f$ is parallel to the gradient vector of $g$, i.e., at all points $(a, b)$ satisfying $\nabla f(a, b) = \lambda \nabla g(a, b)$, for some scalar $\lambda$. The largest value is the maximum, and the smallest is the minimum.