14. Homotopy theory

Constructions

1. Define the smash product of based spaces \( X \land Y \) as the quotient space \((X \times Y)/(X \lor Y)\), where \(X \lor Y\) is the subspace \((X \times \{y_0\}) \cup (\{x_0\} \times Y)\). Prove that this is the correct definition of coproduct in the category of based spaces. Show that \(S^1 \land X\) is another way of defining the reduced suspension \(\Sigma X\) of a based space, and that \(S^n \land S^m \cong S^{m+n}\).

2. Use the homotopy extension principle to prove that the cone \(CK\) on a CW-complex \(K\) is contractible.

3. Show that the homotopy extension principle for \((X, A)\) is satisfied if and only if there is a retraction (it doesn’t have to be a deformation retraction) \(r : X \times I \to X \times \{0\} \cup A \times I\).

Homotopy groups

4. Suppose that \(x_0 \in A \subseteq X\), and that there is a retraction \(r : X \to A\). Prove that
   \[
   \pi_n(X, x_0) \cong \pi_n(X, A, x_0) \oplus \pi_n(A, x_0).
   \]

5. For any based space \((X, x_0)\) we can consider the set of free homotopy classes \([S^n, X]\), defined without reference to the basepoint, as well as the usual set of based homotopy classes \([S^n, X]_0 = \pi_n(X, x_0)\). There is a natural map \([S^n, X]_0 \to [S^n, X]\); show that if \(X\) is path-connected, this map identifies \([S^n, X]\) as the set of orbits in \(\pi_n(X, x_0)\) under the action of \(\pi_1(X, x_0)\).

6. Show that for a path-connected \(H\)-space \(Y\), the action of \(\pi_1(Y, y_0)\) on \(\pi_{n \geq 1}(Y, y_0)\) is trivial.

7. One proof that a topological group \(G\) has abelian fundamental group \(\pi_1(G, 1)\) goes as follows. Define operations \(\ast\) (path-composition) and \(\cdot\) (pointwise multiplication) on the set \(\Omega_1 Y\) of loops based at the identity. They satisfy \((a \ast b) \cdot (c \ast d) = (a \cdot c) \ast (b \cdot d)\), and suitable constant loop substitutions give the result. Use this idea to show that
   (a) for an \(H\)-space \(Y\), \(\pi_1(Y)\) is abelian (and the two group operations coincide).
   (b) the natural group structures on the homotopy sets \([\Sigma \Sigma X, Y]_0\) and \([X, \Omega \Omega Y]_0\) are abelian, for any (based spaces) \(X\) and \(Y\).

Higher homotopy

8. If \(f : X \to Y\) is a map, define the mapping cone of \(f\) to be \(CX \cup_f Y\), where \(CX\) is the cone on \(X\), and the gluing is the obvious one. Suppose \(f\) maps \(S^{2n-1}\) to \(\mathbb{C}P^{n-1}\); what does the mapping cone look like? Use this to show that \(\pi_{2n-1}(\mathbb{C}P^{n-1}) \neq 0\).

9. Give an example of a CW-pair \((K, L)\) such that \(\pi_* (K, L) \neq \pi_*(K/L)\).

10. Let \((X, A)\) be an arbitrary pair of topological spaces with \(\pi_i(X, A) = 0\) for \(i \leq n\). Suppose \(K\) is a CW-complex of dimension less than \(n\). Show that \([K, X] = [K, A]\). Use this lemma to prove that a weak homotopy equivalence \(f : X \to Y\) between arbitrary spaces induces a bijection \(f_* : [K, X] \to [K, Y]\), whenever \(K\) is a CW-complex.

11. A CW-complex \(X\) which has only one non-vanishing homotopy group \(\pi_n(X) \cong \pi\) is called an Eilenberg-MacLane space \(K(\pi, n)\).
(a). Recall how to build, for any group $\pi$, a path-connected CW-complex with fundamental group $\pi$. Show how to attach cells of dimension $\geq 3$ so as to kill all the higher homotopy groups and thereby construct a $K(\pi, 1)$.

(c). Use a similar procedure to prove that for any $n \geq 1$ and group $\pi$, there exists a space of type $K(\pi, n)$, provided only that the obvious restriction holds: $\pi$ must be abelian if $n \geq 2$.

Whitehead

12. Let $X$ be a path-connected CW-complex with $\pi_{\geq 2}(X) = 0$ and $\pi_1(X)$ being a free group on a set $S$. Show that there is a homotopy equivalence between $X$ and a bouquet of circles indexed by $S$.

13. These example show the necessity of hypotheses in various forms of Whitehead’s theorem.

(a). Show that $S^2 \times \mathbb{R}P^n$ and $\mathbb{R}P^2 \times S^n$ have the same homotopy groups, but (for $n \geq 2$) different homotopy type.

(b). Show that the subspace of the plane $\{x = 0\} \cup \{(x, \sin(1/x)) : x \neq 0\}$ is weakly homotopy-equivalent to, but not homotopy-equivalent to, the three-point space.

(c). Find a pair of 1-connected spaces with the same homology but different homotopy type.

(d). Let $X = (S^2 \vee S^1) \cup B^3$, where the 3-cell attaches via the element $2t - 1 \in \pi_2(S^2 \vee S^1) \cong \mathbb{Z}[t, t^{-1}]$. Show that inclusion of the circle into $X$ induces isomorphisms of fundamental groups and homology but that it is not a homotopy equivalence.

Hurewicz

14. Use the universal cover to calculate the second homotopy groups $\pi_2(S^2 \vee S^1)$ and $\pi_2(S^2 \vee \mathbb{R}P^2)$ and their $\mathbb{Z}\pi_1$-module structures.

15. Let $X$ be a 1-connected CW-complex with $H_2(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H_{\geq 3}(X) = 0$. Prove that $X$ is homotopy-equivalent to the “bouquet of two spheres” $S^2 \vee S^2$.

16. Let $M = S^3 - N(K)$ be a knot complement, where $K$ is a (“tamely” i.e. polygonally) embedded circle in the 3-sphere, and $N(K)$ is an open solid torus neighbourhood; thus $M$ is a compact 3-manifold with boundary $S^1 \times S^1$. It can be shown that $\pi_2(M) = 0$; use this to prove that the only non-vanishing homotopy group of $M$ is $\pi_1$, so that “$M$ is a $K(\pi, 1)$”.

391