1. (a). Calculate the 3-colouring invariant $\tau_3$ of the following pretzel knot $P_{3,3,3}$.

You can use any method you want, but it might help if you first think about colourings of the following piece of diagram:

(b). Explain why the calculation shows that the unknotting number of $P_{3,3,3}$ is greater than or equal to 2.

2. (a) By considering the following fragment of diagram,

evaluate the Kauffman bracket of a “necklace” diagram of the type shown below, consisting of a chain of $n$ components.

(b) Suppose that instead of a linear chain of $n$ unknots, we form a network of $n$ interlinked unknots according to the pattern defined by a planar tree with $n$ vertices, as in the picture shown below.

What is the bracket of this diagram?
3. For \( n \geq 0 \), let \( L_n \) be the oriented link shown in the following diagram with \( n \) crossings.

Let \( V_n(t) \) be the Jones polynomial of \( L_n \) and let \( P_n(t) = -t^{(1-n)/2}(1 + t)V_n(t) \).

(a) Use the Jones skein relation to prove that \( P_n = (t - 1)P_{n-1} + tP_{n-2} \).

(b) What are the values of \( P_0 \) and \( P_1 \)?

(c) Use induction to prove that \( P_n(t) = t^{n+1} + (-1)^n(1 + t + t^2) \) and hence that when \( n \) is odd, so that \( L_n \) is the torus knot \( T_{2,n} \), the Jones polynomial is

\[
V_n(t) = t^{(n-1)/2} \frac{1 - t^3 - t^{n+1} + t^{n+2}}{1 - t^2}
\]

4. Use the classification of surfaces to identify each of the following surfaces as one of the standard ones \( M_g^n \) (orientable, genus \( g \), \( n \) boundary circles) or \( N_h^n \) (non-orientable, genus \( h \), \( n \) boundary circles). Explain your reasoning.

(a). The surface obtained by gluing three solid squares as shown below:

(b). The surface obtained from a chessboard shading of a diagram of the Borromean rings:

(c). The surface obtained by applying Seifert’s algorithm to the standard diagram of the \((p, q)\) torus link \( T_{p,q} \). (The picture below shows the standard picture for \( T_{3,4} \): in general there are \( p \) strings running from left to right through the twisted part, and \( q(p - 1) \) crossings altogether.)
5. True or false? You don’t have to give any explanation, just write “T” or “F”.

(1). No connected surface with Euler characteristic 1 can contain a 2-sided curve.

(2). There exists a knot $K$ whose Jones polynomial is the zero polynomial: $V(K) = 0$

(3). For any knot $K$, the crossing number $c(K)$ is less than or equal to half of the unknotting number $u(K)$.

(4). The number of crossings $c(D)$ in an alternating knot diagram $D$ equals the span of the Jones polynomial of the knot represented by $D$.

(5). For any knot $K$, the number of 2-colourings $\tau_2(K)$ is 2.

(6). Any surface with Euler characteristic $\chi \geq 3$ must be disconnected.

(7). There are exactly four connected surfaces (with or without boundary, and considered up to equivalence) with Euler characteristic zero.

(8). For any two knots, the unknotting number satisfies $u(K_1 \# K_2) \leq u(K_1) + u(K_2)$.

(9). If $F$ and $G$ are surfaces whose connect-sum $F \# G$ is orientable, then both $F$ and $G$ are orientable.

(10). The crossing number $c(K)$ of an alternating knot equals the span of its Jones polynomial.

(11). Any closed connected surface with all its curves separating is equivalent to the sphere.

(12). The number of 3-colourings $\tau$ satisfies the connect-sum formula $\tau(K_1 \# K_2) = \tau(K_1)\tau(K_2)$. 