Generators and Relations

Groups can often be conveniently described in terms of generators and relations. A group $G$ is generated by a set of elements $S=\{x_1, \ldots, x_k\}$ if $G$ is the smallest subgroup which contains the $x_i$. We write $G = \langle x_1, \ldots, x_k \rangle$ and call $S$ a set of generators for $G$. $G$ must contain inverses of the elements in $S$ and also all products that can be formed of elements in $S$ and their inverses. This set of products is the subgroup generated by $S$.

The Generators usually satisfy relations. Consider groups that have two generators $x$ and $y$. A relation is given as an equality: $x^3 = e$, $y^2 = e$ are relations. The SEARCH command in Groups32 can be used to find the groups of orders 1-32 which have a given set of generators satisfying given relations:

```
G1>> SEARCH
Enter distinct generators as a string
e.g.  RS means two generators R and S
Generators:  xy
Do you want these to generate the entire group? (y or n) Y

Enter the exact order for each generator.
Press Enter for no order specified
X is of order    3
Y is of order    2

A relation is of the form  LHS = RHS
Put in LHS RHS or  LHS  ( if RHS is e )
<Press ENTER to quit>

Generators:
  XY
Orders:
  X=  3
  Y=  2

RELATIONS:

-- Pressing ESC will abort the search --

7  group order =  6  X = C  Y = D
8  group order =  6  X = B  Y = D
23  group order = 12  X = B  Y = D
46  group order = 18  X = C  Y = B
63  group order = 24  X = B  Y = D
70  group order = 24  X = F  Y = E
```
We have asked for groups having a generator x of order 3 and a generator y of order 2. We have not imposed any additional relations on these generators. We obtain 6 groups. Let's look at the two groups of order 6:

<table>
<thead>
<tr>
<th>G1 &gt;&gt; CHART</th>
<th>Order of Groups (1-32 or 0)</th>
<th>Number 6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>7</td>
<td>8*</td>
</tr>
</tbody>
</table>

There are 2 Groups of order 6
1 abelian and 1 non-abelian

We have seen these groups many times before. Group 7 is isomorphic to $\mathbb{Z}_6$ (the abelian group of order 6) and group 8 is isomorphic to $S_3$ (the non-abelian group of order 6).

$\mathbb{Z}_6$

Since G is abelian, there is an additional relation between x and y namely $yx = xy$. We see that every element of G can be written as a product $x^ay^b$ where a is 0,1,2 and b is 0,1. We get 6 distinct products this way. The multiplication of two such products is determined by the additional relation: $x^ay^bx^cy^d = x^{a+c}y^{b+d}$ where a+c is taken mod 3 and b+d is taken mod 2. We can make a Cayley table using this multiplication. This group is cyclic and xy is an element of order 6.

$S_3$

(or $yx$ In this case there is also an additional relation between x and y: $yx = x^2y = x^3y$). We see, again, that every element of G can be written as a product $x^ay^b$ where a is 0,1,2 and b is 0,1. We get 6 distinct products this way. The multiplication of two such products is determined by the additional relation. The relation tells us how to move an x to the left past a y. We leave it as an exercise to the reader to show that:

$x^ay^bx^cy^d = x^{a+2c}y^{b+d}$ where the exponent of x is taken mod 3 and the exponent of y is taken mod 2. Again we can take find the Cayley table for the multiplication.

Why only two groups of order 6?

Cauchy's Theorem assures us that a group of order 6 must have an element, x, of order 3 and an element, y, of order 2. The subgroup H generated by x is of order 3 and so of index 2. Subgroups of index 2 are always normal. Thus $z = yxy^{-1}$ must be an element of H and it must have order 3. The only possibilities are $z = x$ and $z = x^2$. From this we find that either $yx = xy$ or $yx = x^{-1}y$. The analysis above shows that x and y generate an abelian group of order 6 in the first case and a non-abelian group of order 6 in the second case.
How can we get a group of order > 6?

Here is one of the other groups listed which have two generators, one of order 3 and one of order 2. Notice that the generators are given as elements B and D respectively.

\[
\begin{align*}
23 & \text{ group order } = 12 & X = B & Y = D \\
\end{align*}
\]

The matter is mysterious only if you assume that everything in the group can be written as a product \( x^a y^b \). This would only give 6 elements.

G23>> EVALUATE (use ' for inverse) a= A
G23>> EVALUATE (use ' for inverse) b= B
G23>> EVALUATE (use ' for inverse) bb= C
G23>> EVALUATE (use ' for inverse) ad= D
G23>> EVALUATE (use ' for inverse) bd= G
G23>> EVALUATE (use ' for inverse) bbd= J

The only elements obtained by putting a product of “B” in front and a product of “D” in back are the 6 elements A,B,C,D,G,J.

Now \( yx \) is DB = E which is not one of the 6 elements. We do not have a relation which allows us to “switch an y past an x”. Neither the subgroup generated by B nor the subgroup generated by D is normal. G is not the product of these two subgroups.

G23>> SUBGROUPS of Group Number 23
... wait

* = Normal subgroup

<table>
<thead>
<tr>
<th>Generators</th>
<th>Subgroup</th>
</tr>
</thead>
<tbody>
<tr>
<td>0  { }</td>
<td>*( A )</td>
</tr>
<tr>
<td>1  { D }</td>
<td>( A D )</td>
</tr>
<tr>
<td>2  { I }</td>
<td>( A I )</td>
</tr>
<tr>
<td>3  { K }</td>
<td>( A K )</td>
</tr>
<tr>
<td>4  { B }</td>
<td>( A B C )</td>
</tr>
<tr>
<td>5  { F }</td>
<td>( A F G )</td>
</tr>
<tr>
<td>6  { E }</td>
<td>( A E J )</td>
</tr>
<tr>
<td>7  { H }</td>
<td>( A H L )</td>
</tr>
<tr>
<td>8  { D I }</td>
<td>*( A D I K )</td>
</tr>
<tr>
<td>9  { B D }</td>
<td>*( A B C D E F G H I J K L )</td>
</tr>
</tbody>
</table>

However, DB = E is an element of order 3. So we do have a relation \((yx)^3 = e\). This group is, in fact, the only one that has generators x of order 3, y of order 2 and satisfies \((yx)^3 = e\):

G23>> SEARCH
Enter distinct generators as a string
e.g. RS means two generators R and S
Generators: xy
Do you want these to generate the entire group? (y or n) Y
Enter the exact order for each generator.
Press Enter for no order specified
X is of order 3
Y is of order 2

A relation is of the form LHS = RHS
Put in LHS RHS or LHS (if RHS is e)

LHS RHS >> yxyxyx
Generators:
XY
Orders:
X= 3
Y= 2

RELATIONS:
YXYXYX= e

-- Pressing ESC will abort the search --

23 group order = 12 X = B Y = D

The elements of this group can be written as products of B and D, but not with all the B’s on the left.

G23>> EVALUATE (use ' for inverse) a= A
G23>> EVALUATE (use ' for inverse) d= D
G23>> EVALUATE (use ' for inverse) b= B
G23>> EVALUATE (use ' for inverse) bd= G
G23>> EVALUATE (use ' for inverse) bb= C
G23>> EVALUATE (use ' for inverse) bbd= J
G23>> EVALUATE (use ' for inverse) db= E
G23>> EVALUATE (use ' for inverse) dbb= F
G23>> EVALUATE (use ' for inverse) dbd= L
G23>> EVALUATE (use ' for inverse) bdb= H
G23>> EVALUATE (use ' for inverse) bbdb= K
G23>> EVALUATE (use ' for inverse) bdbb= I

[It might be interesting to point out that this group is isomorphic to A₄ which can be generated by x = (1 2 3) and y = (1 2)(3 4) and for which yx = (2 4 3) does have order 3.]