Math 103A – Winter,2001
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Normal Subgroups

Take a look at the list of subgroups of various groups generated by the Groups32 `SUBGROUPS` command:

```
G1>> SUBGROUPS   of Group Number 7
  ... wait

   * = Normal subgroup
   Generators             Subgroup
  0  { }                  *{ A }
  1  { D }                *{ A D }
  2  { C }                *{ A C E }
  3  { B }                *{ A B C D E F }
```

```
G7>> SUBGROUPS   of Group Number 8
  ... wait

   * = Normal subgroup
   Generators              Subgroup
  0  { }                   *{ A }
  1  { D }                  { A D }
  2  { E }                  { A E }
  3  { F }                  { A F }
  4  { B }                 *{ A B C }
  5  { B D }               *{ A B C D E F }

G8>> CHART   Order of Groups (1-32 or 0) Number 6
       7    8*
        There are 2 Groups of order 6
        1 abelian and 1 non-abelian
```

Notice that some of these are marked as "normal". There are several equivalent ways to define what it means for a subgroup to be normal:

Definition: A subgroup H of a group G is called normal if any one of the following conditions holds:

(1) $\forall\, g \in G, h \in H$ we have $ghg^{-1} \in H$
(2) $\forall\, g \in G, gHg^{-1} \subseteq H$
(3) $\forall\, g \in G, gH = Hg$
(4) Every right coset of H is a left coset
(5) H is the kernel of a homomorphism of G to some other group

It is easy to see from condition (1) that:

Proposition: If G is an abelian group, then any subgroup is normal.

Group 7, above, is an abelian group -- all of its subgroups are normal. In group 8, above, the subgroup <F> is not normal, while the subgroup <B> is. Lets check the cosets of these two subgroups.

```
G8>> COSETS   of subg generated by set: { f }

 Left Cosets                 Right Cosets
{ A F }                     { A F }
{ B E }                     { B D }
{ C D }                     { C E }

The subgroup { A F }  is NOT a NORMAL subgroup

G8>> COSETS   of subg generated by set: { b }

 Left Cosets                 Right Cosets
{ A B C }                   { A B C }
{ D E F }                   { D E F }

The subgroup { A B C }  is a NORMAL subgroup
```

(6) If H=<F> we see that the right coset HB = {B D} does not coincide with any of the left cosets. H is not a normal subgroup for this reason. We also can see that condition (1) is not satisfied. We must find a $g \in G$ and $h \in H$ so that ghg^{-1} lies outside H.

```
G8>> EVALUATE   (use ' for inverse) bfb'= D
```

We can also see this using permutations. Group 8 is isomorphic to S_3. The elements B and C are of order 3, so must be 3-cycles. The elements D, E, F are of order 2 so are 2-cycles. Let H be the subgroup <(1 2)>, Let g = (1 2 3). $(1\ 2\ 3)(1\ 2)(1\ 2\ 3)^{-1} = (2\ 3)$ which is not in H.

Please note, to show H is not normal we only have to find one g and one h for which $ghg^{-1} \notin H$.

The earlier conditions are usually the easiest to verify. The importance of normal subgroups, however, come from the fact that they are kernels of homomorphisms and that the set of cosets can be made into a group:

Let G be a group and H a subgroup. The cosets are the equivalence classes for congruence mod H. We would like to imitate what was done for integers mod n and define [a]*[b] = [a*b] to make the equivalence classes a group. We will see that this only works when H is normal.