Math 103A – Winter,2001
Professor John J Wavrik

Quotient Groups

We use Group32 to generate $G=S_4$. We notice that S_4 has a normal subgroup N of order 4. We determine the structure of the quotient group G/N.

STEP 1: S_4 is generated by (1 2 3 4) and (1 2):

```
  CENTER        CENTRALIZER   CHART         CONJ-CLS
  COSETS        EVALUATE      EXAMPLES      GENERATE
  GROUP         HELP          INFO          ISOMORPHISM
  LEFT          NORMALIZER    ORDERS        PERMGRPS
  POWERS        QUIT          RESULT        RIGHT
  SEARCH        STOP          SUBGROUPS     TABLE
  X

G1>> PERMGRPS

  CREATE        ELEMENTS      HELP          INFO
  INSTALL       MAIN          MULTIPLY      QUIT
  X
```

```
PERM>> CREATE
Subgroup of Sn -- what is n?  Number 4
     Put in generators as product of cycles.
     End with a blank line
Generator (1 2 3 4)
Generator (1 2)
Generator
Group is of order 24
A  ()                        B  (3 4 )                  C  (2 3 )
D  (2 3 4 )                  E  (2 4 3 )                F  (2 4 )
G  (1 2 )                    H  (1 2 )(3 4 )            I  (1 2 3 )
J  (1 2 3 4 )                K  (1 2 4 3 )              L  (1 2 4 )
M  (1 3 2 )                  N  (1 3 4 2 )              O  (1 3 )
P  (1 3 4 )                  Q  (1 3 )(2 4 )            R  (1 3 2 4 )
S  (1 4 3 2 )                T  (1 4 2 )                U  (1 4 3 )
V  (1 4 )                    W  (1 4 2 3 )              X  (1 4 )(2 3 )

PERM>> INSTALL
    Install as table k (1..5)  Number 1
```

```
PERM>> MAIN

  CENTER       CENTRALIZER  CHART        CONJ-CLS
  COSETS       EVALUATE     EXAMPLES     GENERATE
  GROUP        HELP         INFO         ISOMORPHISM
  LEFT         NORMALIZER   ORDERS       PERMGRPS
  POWERS       QUIT         RESULT       RIGHT
  SEARCH       STOP         SUBGROUPS    TABLE
  X
```

Step 2: Look at the subgroups of S_4

```
G1>> SUBGROUPS   of Group Number 1
  ... wait

   * = Normal subgroup
   Generators            Subgroup
  0  { }                *{ A }
  1  { B }               { A B }
  2  { C }               { A C }
  3  { F }               { A F }
  4  { G }               { A G }
  5  { H }               { A H }
  6  { O }               { A O }
  7  { Q }               { A Q }
  8  { V }               { A V }
  9  { X }               { A X }
 10  { D }               { A D E }
 11  { I }               { A I M }
 12  { L }               { A L T }
 13  { P }               { A P U }
 14  { B G }             { A B G H }
 15  { F O }             { A F O Q }
 16  { J }               { A J Q S }
 17  { R }               { A H R W }
 18  { K }               { A K N X }
 19  { H Q }            *{ A H Q X }
 20  { C V }             { A C V X }
 21  { B C }             { A B C D E F }
 22  { C G }             { A C G I M O }
 23  { F G }             { A F G L T V }
 24  { B O }             { A B O P U V }
 25  { F H }             { A F H J O Q S X }
 26  { C H }             { A C H K N Q V X }
 27  { B Q }             { A B G H Q R W X }
 28  { D H }            *{ A D E H I L M P Q T U X }
 29  { B I }            *{ A B C D E F G H I J K L M N O P Q R S T U V W X }
```

Step 3: Look at cosets of subgroup 19, generated by H and Q

```
G1>> COSETS   of subg generated by set: { hq }

 Left Cosets                 Right Cosets
{ A H Q X }                 { A H Q X }
{ B G R W }                 { B G R W }
{ C K N V }                 { C K N V }
{ D L M U }                 { D L M U }
{ E I P T }                 { E I P T }
{ F J O S }                 { F J O S }

The subgroup { A H Q X }  is a NORMAL subgroup
```

Step 4: Select representatives and do some arithmetic

```
G1>> EVALUATE   (use ' for inverse) bb= A
G1>> EVALUATE   (use ' for inverse) cc= A
G1>> EVALUATE   (use ' for inverse) dd= E
G1>> EVALUATE   (use ' for inverse) ee= D
G1>> EVALUATE   (use ' for inverse) ff= A
G1>> EVALUATE   (use ' for inverse) ddd= A
G1>> EVALUATE   (use ' for inverse) eee= A
```

Step 5: Translate to notation used in class

```
[A] = NA = { A H Q X }
[B] = NB = { B G R W }
[C] = NC = { C K N V }
[D] = ND = { D L M U }
[E] = NE = { E I P T }
[F] = NF = { F J O S }

[B], [C], and [F] are of order 2
[D] and [E] are of order 3
```

We can easily write a Cayley table for this group. However, since it is a group of order 6 we know it is either isomorphic to S_3 or to $\mathbf{Z}_6$. There is no element of order 6, so it must be S_3. Thus $G/\mathbf{N} \approx S_3$.

Step 6: Determine N

Notice that **N** = { A H Q X }. A is the identity while H, Q and X have order 2. This means that **N** ≈ $\mathbf{Z}_2 \times \mathbf{Z}_2$. It might be interesting to look at this in terms of permutations. Go back to where we generated G:

```
Group is of order 24
A  ()                      B  (3 4 )                   C  (2 3 )
D  (2 3 4 )                E  (2 4 3 )                 F  (2 4 )
G  (1 2 )                  H  (1 2 )(3 4 )             I  (1 2 3 )
J  (1 2 3 4 )              K  (1 2 4 3 )               L  (1 2 4 )
M  (1 3 2 )                N  (1 3 4 2 )               O  (1 3 )
P  (1 3 4 )                Q  (1 3 )(2 4 )             R  (1 3 2 4 )
S  (1 4 3 2 )              T  (1 4 2 )                 U  (1 4 3 )
V  (1 4 )                  W  (1 4 2 3 )               X  (1 4 )(2 3 )
```

The subgroup **N** consists of the permutations of the form (a b)(c d) together with the identity.

S_4 is an extension of $\mathbf{Z}_2 \times \mathbf{Z}_2$ by S_3.

The importance of being Normal:

We defined the multiplication of cosets by picking representatives: [x][y] = [xy]. It is true (but we didn't check) that we could pick any element in [B] and multiply by any other element in [B] and we would always get an element of [A]. This justifies our assertion that [B] is of order 2.

Let's see what would happen if we picked a subgroup, also of order 4, which is not normal.

```
G1>> SUBGROUPS   of Group Number 1
  ... wait

   * = Normal subgroup
   Generators            Subgroup
  0  { }                 *{ A }
  1  { B }                { A B }
  2  { C }                { A C }
  3  { F }                { A F }
  4  { G }                { A G }
  5  { H }                { A H }
  6  { O }                { A O }
  7  { Q }                { A Q }
  8  { V }                { A V }
  9  { X }                { A X }
 10  { D }                { A D E }
 11  { I }                { A I M }
 12  { L }                { A L T }
 13  { P }                { A P U }
 14  { B G }              { A B G H }
 15  { F O }              { A F O Q }
 16  { J }                { A J Q S }
 17  { R }                { A H R W }
 18  { K }                { A K N X }
 19  { H Q }             *{ A H Q X }
 20  { C V }              { A C V X }
 21  { B C }              { A B C D E F }
 22  { C G }              { A C G I M O }
 23  { F G }              { A F G L T V }
 24  { B O }              { A B O P U V }
 25  { F H }              { A F H J O Q S X }
 26  { C H }              { A C H K N Q V X }
 27  { B Q }              { A B G H Q R W X }
 28  { D H }             *{ A D E H I L M P Q T U X }
 29  { B I }             *{ A B C D E F G H I J K L M N O P Q R S T U V W X }
```

The subgroups of order 4 which have one generator are cyclic, they are isomorphic to $\mathbf{Z}_4$.
The subgroups of order 4 which have two generators are isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$.

Let me pick the subgroup 14 which has two generators but is not normal.

```
G1>> COSETS   of subg generated by set: { bg }

 Left Cosets              Right Cosets
{ A B G H }             { A B G H }
{ C D M N }             { C E I K }
{ E F S T }             { D F J L }
{ I J O P }             { M O S U }
{ K L U V }             { N P T V }
{ Q R W X }             { Q R W X }

The subgroup { A B G H }  is NOT a NORMAL subgroup
```

The element C is in the second right coset. The square of C is in the first coset.
The element E is also in the second right coset. The square of E is in the third right coset.

```
G1>> EVALUATE   (use ' for inverse) cc= A
G1>> EVALUATE   (use ' for inverse) ee= D
```

This is what happens when the subgroup is not normal: we cannot define multiplication of cosets consistently just by choosing representatives. We get different ideas of what the square of the second coset should be: C tells us it it should be the first coset – but its companion, E, disagrees.